

INVERSE HEAT CONDUCTION PROBLEM IN A THIN CIRCULAR PLATE AND ITS THERMAL DEFLECTION

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ABSTRACT An inverse problem of transient heat conduction in a thin finite circular plate with the given temperature distribution on the interior surface of a thin circular plate being a function of both time and position has been solved with the help of integral transform technique and also determine the thermal deflection on the outer curved surface of a thin circular plate defined as $0 \leq r \leq a$, $0 \leq z \leq h$. The results, obtained in the series form in terms of Bessel's functions, are illustrated numerically.

1. INTRODUCTION

An inverse thermoelastic problem consists of determination of the temperature of the heating medium, the heat flux on the boundary surfaces of the solid when the conditions of the displacement and stresses are known at the some points of the solid under consideration. Grysa and Cialkowski [1], Grysa and Kozlowski [2] investigated one dimensional transient thermoelastic problems and derived the heating temperature and the heat flux on the surface of an isotropic infinite slab. Noda N. [3] studied inverse problem of coupled thermal stress fields in a thick plate. Ashida F. et.al. [4] studied the inverse problem of two-dimensional Piezothermoelasticity in an orthotropic plate exhibiting crystal class mm^2 . Noda N. et al. [5] attempted an inverse thermoelastic problem in an isotropic plate associated with a Piezoelectric ceramic plate. The direct Problems of normal deflection of an axisymmetrically heated circular plate in the case of fixed and simply supported edges have been considered by Boley and Weiner [6]. Further, Roy Choudhuri [7] has succeeded in determining the normal deflection of a thin clamped circular palte due to ramp-type heating of a concentric circular region of the upper face. In this paper, we reconstruct the problem studied by Roy Choudhuri [7] and deals with the inverse thermoelastic problem of a thin clamped circular plate. The temperature distribution, the unknown temperature gradient and the thermal deflection on the outer curved surface of a thin circular plate of thickness h defined as $D: 0 \leq r \leq a$, $0 \leq z \leq h$ are discuss. No one has attempted this problem so far. The results, obtained in a series form involving Bessel's functions, are illustrated numerically.

2. STATEMENT OF THE INVERSE HEAT CONDUCTION PROBLEM

Consider a thin circular plate of thickness h occupying the space $D: 0 \leq r \leq a$, $0 \leq z \leq h$. The plate is initially at zero temperature with temperature distribution on the interior surface of a thin circular plate and the faces ($z = 0$, $z = h$) of a thin circular plate are kept at zero temperature.

Key Words : *Quasi-static, Transient problem, Thermal deflection, Inverse problem.*

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The differential equation governing the temperature function $T(r, z, t)$, is as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{K} \frac{\partial T}{\partial t}, \quad (1)$$

$$0 \leq r \leq a, \quad 0 \leq z \leq h, \quad t \geq 0$$

subject to the initial condition, boundary conditions and interior condition, respectively as

$$T(r, z, t) \Big|_{t=0} = 0, \quad (2)$$

$$T(r, z, t) \Big|_{z=0} = 0, \quad (3)$$

$$T(r, z, t) \Big|_{z=h} = 0, \quad (4)$$

$$T(r, z, t) \Big|_{r=a} = g(z, t) \text{ (unknown)} \quad (5)$$

and

$$T(r, z, t) \Big|_{r=\xi} = f(z, t); \quad 0 < \xi < a, \quad 0 < z < h \text{ (known)} \quad (6)$$

where K is thermal diffusivity of the material of the circular plate. The equations (1) to (6) constitute the mathematical formulation of the inverse heat conduction problem.

3. SOLUTION OF THE INVERSE HEAT CONDUCTION PROBLEM

Determination of the Temperature Function $T(r, z, t)$ and the Unknown Temperature Gradient $g(z, t)$

On applying the finite Fourier sine transform and Laplace transform to the equations (1) to (6) and then applying their inversions to the resultant equations, one obtains, the expressions for temperature distribution and unknown temperature gradient, respectively as

$$T(r, z, t) = \frac{4K}{h\xi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sin\left(\frac{m\pi z}{h}\right) \cdot \frac{\beta_n J_0(\beta_n r)}{J_1(\beta_n \xi)} \cdot \phi_m(n, t) \right] \quad (7)$$

$$\text{and} \quad g(z, t) = \frac{4K}{h\xi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\sin\left(\frac{m\pi z}{h}\right) \cdot \frac{\beta_n J_0(\beta_n a)}{J_1(\beta_n \xi)} \cdot \phi_m(n, t) \right] \quad (8)$$

where m and n are positive integers and β_n is the n^{th} positive root of the transcendental equation

$$J_0(\beta a) = 0 \quad (9)$$

and

$$\phi_m(n, t) = \int_0^t \bar{f}_s(m, t') \cdot \exp\{-K(\beta_n^2 + p^2)(t - t')\} dt' \quad (10)$$

Determination Of Quasi-Static Thermal Deflection

Differential equation satisfied by deflection $\omega(r, t)$ as in [7] is

$$D \nabla^2 \omega = -(\nabla^2 M_T)/(1 - \nu) \quad (11)$$

where M_T is the thermal moment of the plate, ν is the Poisson's ratio of the plate material and D is the flexural rigidity of the plate,

$$D = \frac{E h^3}{12(1 - \nu^2)}$$

$$\text{and} \quad \nabla^2 = \frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (12)$$

Since, the edge of the circular plate is fixed and clamped

$$\omega = \frac{\partial \omega}{\partial r} = 0 \quad \text{at } r = a \quad (13)$$

Assume the solution of (11) satisfying conditions (13) as

$$\omega(r, t) = \sum_{n=1}^{\infty} C_n(t) \{ 2aJ_0(\beta_n r) - 2aJ_0(\beta_n a) \} + \beta_n (r^2 - a^2) J_1(\beta_n a) \} \quad (14)$$

where $C_n(t)$ are selected in such a way that $\omega(r, t)$ satisfies equation (11).

The thermal moment M_T is defined as

$$M_T(r, t) = \alpha E \int_0^h z T(r, z, t) dz \quad (15)$$

where α and E are the coefficient of linear thermal expansion and Young's modulus, respectively.

On substituting the value of the temperature (7) in (15), one obtains

$$M_T(r, t) = -\frac{4K\alpha E h}{\pi \xi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{m} \cdot \cos(m\pi) \cdot \frac{\beta_n}{J_1(\beta_n \xi)} \phi_m(m, t) \right\} J_0(\beta_n r) \} \quad (16)$$

Using (14), (16) and the well known result

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) J_0(\beta_n r) = -\beta_n^2 J_0(\beta_n r) \quad (17)$$

in equation (11), one obtains the constant $C_n(t)$ as

$$C_n(t) = \frac{-2K\alpha E h}{Da\pi\xi(1-\nu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{m} \cdot \cos(m\pi) \cdot \frac{1}{\beta_n J_1(\beta_n \xi)} \phi_m(n, t) \right] \quad (18)$$

Finally, substituting equation (18) in equation (14), one obtains the expression for the quasi-static thermal deflection $\omega(r, t)$ as

$$\omega(r, t) = \frac{-2K\alpha E h}{Da\pi\xi(1-\nu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{m} \cdot \cos(m\pi) \cdot \frac{1}{\beta_n J_1(\beta_n \xi)} \phi_m(n, t) \right] \cdot [2a\{J_0(\beta_n r) - J_0(\beta_n a)\} + \beta_n (r^2 - a^2) J_1(\beta_n a)] \quad (19)$$

Special Case

$$\text{Set } f(z, t) = T_0(1 - e^{-At}) z(z - h) \quad (20)$$

where $T_0 > 0$, $A > 0$ are constants.

On applying finite Fourier sine transform, one obtains

$$\bar{f}_s(m, t') = -4T_0 \frac{h^3}{m^3 \pi^3} (1 - e^{-At'}) \quad , \text{ if } m \text{ is odd} \quad (21)$$

Substitute (21) in (10), one obtains

$$\phi_m(n, t) = -\frac{4T_0 h^3}{\pi^3} \left[\frac{1}{m^3} \right] \cdot \left[\frac{1 - e^{-K(\beta_n^2 + p^2)t}}{K(\beta_n^2 + p^2)} + \frac{e^{-At} - e^{-K(\beta_n^2 + p^2)t}}{A - K(\beta_n^2 + p^2)} \right] \quad (22)$$

On using (22) in (7), (8) and (19), one obtains the expressions for the temperature distribution, the unknown temperature gradient and the thermal deflection, respectively as

$$T(r, z, t) = \frac{-16T_0Kh^2}{\pi^3\xi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{m^3} \sin\left(\frac{m\pi z}{h}\right) \cdot \frac{\beta_n J_0(\beta_n r)}{J_1(\beta_n \xi)} \right] \cdot \left[\frac{1 - e^{-K(\beta_n^2 + p^2)t}}{K(\beta_n^2 + p^2)} + \frac{e^{-At} - e^{-K(\beta_n^2 + p^2)t}}{A - K(\beta_n^2 + p^2)} \right], \quad (23)$$

$$g(z, t) = \frac{-16T_0Kh^2}{\pi^3\xi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{m^3} \sin\left(\frac{m\pi z}{h}\right) \cdot \frac{\beta_n J_0(\beta_n a)}{J_1(\beta_n \xi)} \right] \cdot \left[\frac{1 - e^{-K(\beta_n^2 + p^2)t}}{K(\beta_n^2 + p^2)} + \frac{e^{-At} - e^{-K(\beta_n^2 + p^2)t}}{A - K(\beta_n^2 + p^2)} \right] \quad (24)$$

and

$$w(r, t) = \frac{-8T_0KaEh^4}{Da\pi^4\xi(1-\nu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{m^4} \frac{[2a\{J_0(\beta_n r) - J_0(\beta_n a)\} + \beta_n(r^2 - a^2)J_1(\beta_n a)]}{\beta_n J_1(\beta_n \xi)} \right] \cdot \left[\frac{1 - e^{-K(\beta_n^2 + p^2)t}}{K(\beta_n^2 + p^2)} + \frac{e^{-At} - e^{-K(\beta_n^2 + p^2)t}}{A - K(\beta_n^2 + p^2)} \right] \quad (25)$$

4. NUMERICAL RESULTS AND DISCUSSIONS

For computational work, we take a steel material (SN50C) as the example for which the material constants are as follows :

$$\alpha = 11.6 \times 10^{-6} [K^{-1}], \quad K = 15.9 \times 10^{-6} [m^2 s^{-1}], \quad E = 215 [GPa],$$

$$\nu = 0.281, \quad a = 5 \text{ m}, \quad h = 1 \text{ m}, \quad A = 0.3, \quad \xi = 1 \text{ m}, \quad \pi = 3.14, \quad t \text{ in second}$$

The plate is thin due to one fifth thickness of the largest dimension in the work of Nowinski [8] and the first five roots of the transcendental equation $J_0(\beta a) = 0$ as in [9]

are $\beta_1 = 0.4809$, $\beta_2 = 1.1040$, $\beta_3 = 1.7307$, $\beta_4 = 2.3583$, $\beta_5 = 2.9861$. Using the numerical values of the above material constants and roots of the transcendental equation, temperature distribution, unknown heating temperature and thermal deflection are evaluated. The variations are shown in the figures (1) to (6).

Figure 1 shows that the temperature oscillates for different times.

Figure 2 shows that fluctuation in temperature for the different radii.

Figure 3 shows that temperature increases upto certain limit and then start decline for different radii.

Figure 4 shows that the unknown temperature distribution is goes on increasing upto a certain limit and then starts decline and reaches to the boundary for the different times.

Figure 5 shows that the thermal deflection goes on increasing as the time increases for the radii of the plate.

Figure 6 shows that initially the thermal deflection goes on increasing for small radii and when the radii increases the deflection starts decreasing for different times.

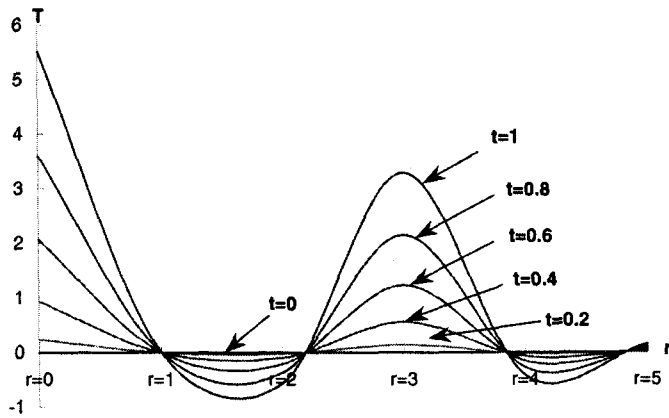


FIGURE 1 : Variation of T versus r ($r = 0, 1, 2, \dots, 5$) for different time $t = 0.02, 0.04, \dots, 0.1$

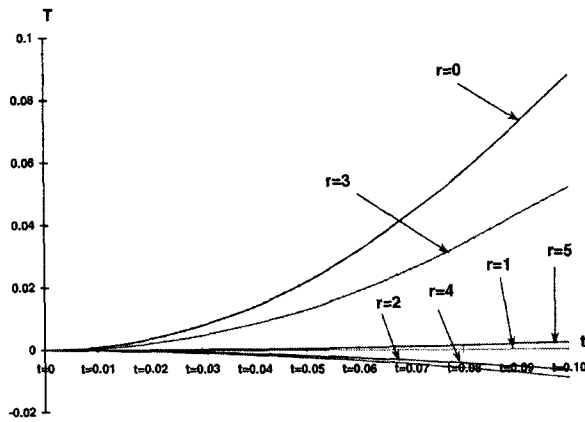


FIGURE 2 : Variation of T versus t ($t = 0, 0.01, 0.02, \dots, 0.1$) for different radii $r = 0, 1, \dots, 5$

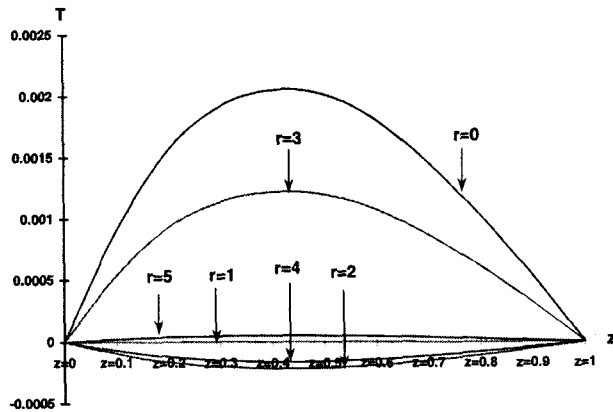


FIGURE 3 : Variation of T versus z ($z = 0, 0.1, 0.2, \dots, 1$) for different values of $r = 0, 1, \dots, 5$

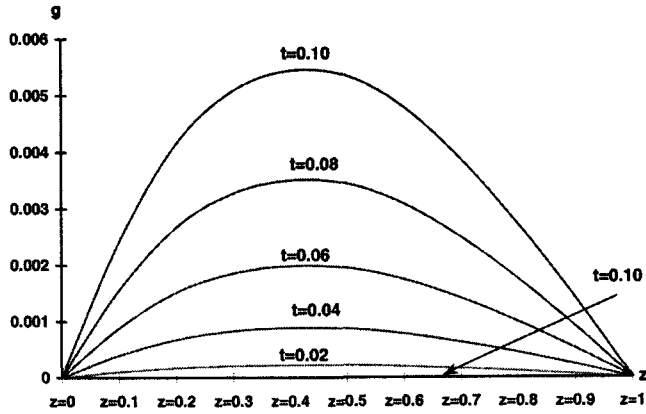


FIGURE 4 : Variation of g versus $z = (0, 0.1, 0.2, \dots, 1)$ for different values of $t = 0, 0.02, 0.04, \dots, 0.1$

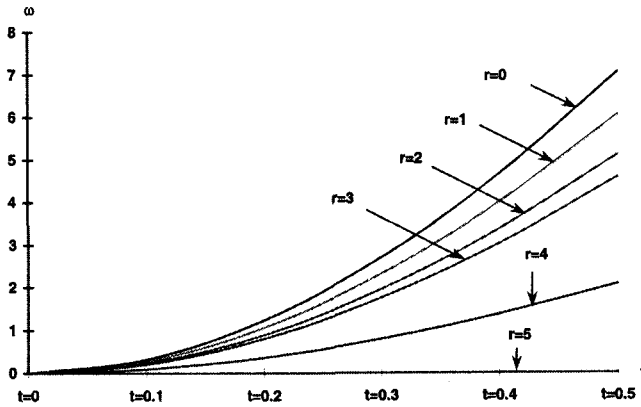


FIGURE 5 : Variation of ω versus $t (t = 0, 0.1, 0.2, \dots, 0.5)$ for different values of $r = 0, 1, \dots, 5$

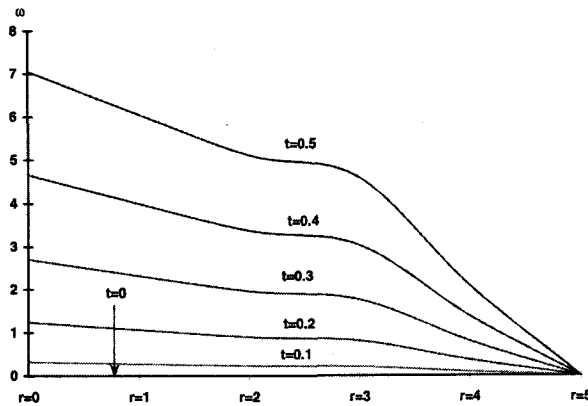


FIGURE 6 : Variation of ω versus $r (r = 0, 1, \dots, 5)$ for different values of $t = 0, 0.1, \dots, 0.5$

5. CONCLUDING REMARKS

Roy Choudhuri [7] who studied the direct problem and deals with quasi-static thermal deflection of a thin clamped circular plate due to ramp-type of heating of a concentric circular region of the upper face, while we modify the work of Roy Choudhuri [7] and deals with the inverse heat conduction problem on the outer curved surface of a thin clamped circular plate. As a special case, mathematical model is constructed and determined the expressions for temperature distribution, unknown heating temperature and thermal deflection on the outer curved surface of a thin circular plate and illustrated numerically. The results, obtained here mainly applicable in engineering problems, particularly for industrial machines subjected to the heating such as the main shaft of a lathe, turbines and the roll of rolling mill. Any particular case of the special interest can be derived by assigning suitable values to the parameters and functions in the expressions (7), (8) and (19).

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