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EXISTENCE OF SOLUTIONS OF FUZZY DELAY INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITION

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ABSTRACT. In this paper we prove the existence of solutions of fuzzy delay integrodifferential equations with nonlocal condition. The results are obtained by using the fixed point principles.

1. INTRODUCTION

Several authors [3-7,11,12] have studied the fuzzy differential equations by using the H-differentiability for the fuzzy valued mappings of a real variable whose values are normal,convex, upper semi continuous and compactly supported fuzzy sets in R^n. Seikkala [10] defined the fuzzy derivative which is generalization of the Hukuhara derivative in [8]. For the Cauchy problem $x' = f(t,x)$, $x(t_0) = x_0$, the local existence theorems are proved in [11], and the existence theorems under compactness-type conditions are investigated in [12] when the fuzzy valued mapping f satisfies the generalized Lipschitz condition. Park et al [7] studied the fuzzy differential equation with nonlocal condition. Nieto [6] proved an existence theorem for fuzzy differential equations on the metric space (E^n, D). Balachandran and Prakash [2] proved the existence of solutions of fuzzy delay differential equations with nonlocal condition of the form

$$\begin{aligned} x'(t) &= f(t, x(\sigma_1(t)), x(\sigma_2(t)), \cdots, x(\sigma_n(t))), \quad t \in J = [0,a], \\ x(0) &- g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0. \end{aligned}$$

In this paper we study the existence of solutions of fuzzy delay integrodifferential equations with nonlocal condition of the form

$$(1) \qquad x'(t) = f\left(t, x(\sigma_1(t)), \int_0^t h\left(t, s, x(\sigma_2(s)), \int_0^s k(s,\tau,x(\sigma_3(\tau)))d\tau\right) ds\right),$$

$$(2) \qquad x(0) - g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0,$$

where $f : J \times E^n \times E^n \to E^n$, $h : J \times J \times E^n \times E^n \to E^n$ and $k : J \times J \times E^n \to E^n$ are levelwise continuous functions, $g : J^p \times E^n \to E^n$ satisfies the Lipschitz condition

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and $\sigma_i : J \to J$, $i = 1,2,3$ are continuous functions, $\sigma_i(t) \leq t$ for all $t \in J$. The existence of solutions for non fuzzy case of the problem (1)-(2) has been discussed in [5]. The symbol $g(t_1, t_2, \cdots t_p, x(\cdot))$ is used in the sense that in the place of '$\cdot$', we can substitute only elements of the set $\{t_1, t_2, \cdots, t_p\}$. For example, $g(t_1, t_2, \cdots, t_p, x(\cdot))$ can be defined by the formula

$$g(t_1, t_2, \cdots, t_p, x(\cdot)) = c_1 x(t_1) + c_2 x(t_2) + \cdots + c_p x(t_p),$$

where $c_i (i = 1, 2, \cdots, p)$ are given constants.

2. Preliminaries

Let $P_K(R^n)$ denote the family of all nonempty, compact, convex subsets of R^n. Addition and scalar multiplication in $P_K(R^n)$ are defined as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} ||a-b||, \sup_{b\in B}\inf_{a\in A} ||a-b||\right\},$$

where $||\cdot||$ denote the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n), d)$ becomes a metric space. Let $I = [t_0, t_0 + a] \subset R$ $(a > 0)$ be a compact interval and let E^n be the set of all $u : R^n \to [0, 1]$ such that u satisfies the following conditions:

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u, v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[\tilde{g}(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n$, $0 \leq \alpha \leq 1$ and continuous function g. Further, we have $[u + v]^\alpha = [u]^\alpha + [v]^\alpha, [ku]^\alpha = k[u]^\alpha$, where $k \in R$. Define $D : E^n \times E^n \to [0, \infty)$ by the relation $D(u, v) = \sup_{0\leq\alpha\leq1} d([u]^\alpha, [v]^\alpha)$, where d is the Hausdorff metric defined in $P_K(R^n)$. Then D is a metric in E^n.

Further we know that [9]

(i) (E^n, D) is a complete metric space,
(ii) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$,
(iii) $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v \in E^n$ and $\lambda \in R$.

It can be proved that $D(u+v, w+z) \leq D(u,w) + D(v,z)$ for u, v, w and $z \in E^n$.

Definition 2.1. [3] *A mapping $F : I \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : I \to P_K(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(R^n)$ has the topology induced by the Hausdorff metric d.*

Definition 2.2. [3] *A mapping $F : I \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x(t) \in F_0(t)$.*

Definition 2.3. *The integral of a fuzzy mapping $F : I \to E^n$ is defined levelwise by $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt$ = The set of all $\int_I f(t)dt$ such that $f : I \to R^n$ is a measurable selection for F_α for all $\alpha \in [0,1]$.*

Definition 2.4. [1] *A strongly measurable and integrably bounded mapping $F : I \to E^n$ is said to be integrable over I if $\int_I F(t)dt \in E^n$.*

Note that if $F : I \to E^n$ is strongly measurable and integrably bounded, then F is integrable. Further if $F : I \to E^n$ is continuous, then it is integrable.

Proposition 2.1. Let $F, G : I \to E^n$ be integrable and $c \in I, \lambda \in R$. Then

(i) $\displaystyle\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt,$

(ii) $\displaystyle\int_I (F(t)+G(t))dt = \int_I F(t)dt + \int_I G(t)dt,$

(iii) $\displaystyle\int_I \lambda F(t)dt = \lambda \int_I F(t)dt,$

(iv) $D(F,G)$ is integrable,

(v) $\displaystyle D\left(\int_I F(t)dt, \int_I G(t)dt\right) \leq \int_I D(F(t),G(t))dt.$

Definition 2.5 *A mapping $F : I \to E^n$ is Hukuhara differentiable at $t_0 \in I$ if for some $h_0 > 0$ the Hukuhara differences*

$$F(t_0 + \Delta t) -_h F(t_0), \quad F(t_0) -_h F(t_0 - \Delta t)$$

exist in E^n for all $0 < \Delta t < h_0$ and there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0+} D((F(t_0 + \Delta t) -_h F(t_0))/\Delta t, F'(t_0)) = 0$$

and

$$\lim_{\Delta t \to 0+} D((F(t_0) -_h F(t_0 - \Delta t)/\Delta t, F'(t_0)) = 0.$$

Here $F'(t)$ is called the Hukuhara derivative of F at t_0.

Definition 2.6. *A mapping $F : I \to E^n$ is called differentiable at a $t_0 \in I$ if, for any $\alpha \in [0,1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) : \alpha \in [0,1]\}$ define a fuzzy number $F(t_0) \in E^n$.*

If $F : I \to E^n$ is differentiable at $t_0 \in I$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.1. Let $F : I \to E^n$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.

Theorem 2.2. Let $F : I \to E^n$ be differentiable and assume that the derivative F' is integrable over I. Then, for each $s \in I$, we have

$$F(s) = F(a) + \int_a^s F'(t)dt.$$

Definition 2.7. *A mapping $f : I \times E^n \to E^n$ is called levelwise continuous at a point $(t_0, x_0) \in I \times E^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that*

$$d([f(t,x)]^\alpha, [f(t_0,x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in I, x \in E^n$.

Corollary 2.1 [2] Suppose that $F : I \to E^n$ is continuous. Then the function

$$G(t) = \int_a^t F(s)ds, \quad t \in I$$

is differentiable and $G'(t) = F(t)$.

Now, if F is continuously differentiable on I, then we have the following mean value theorem

$$D(F(b), F(a)) \leq (b - a) \cdot \sup\{D(F'(t), \hat{0}), t \in I\}.$$

As a consequence, we have that

$$D(G(b), G(a)) \leq (b - a) \cdot \sup\{D(F(t), \hat{0}), t \in I\}.$$

Theorem 2.3. Let X be a compact metric space and Y any metric space. A subset Ω of the space $C(X,Y)$ of continuous mappings of X into Y is totally bounded in the metric of uniform convergence if and only if Ω is equicontinuous on X, and $\Omega(x) = \{\phi(x) : \phi \in \Omega\}$ is a totally bounded subset of Y for each $x \in X$.

3. Main Results

Definition 3.1. *A mapping $x : J \to E^n$ is a solution to the problem (1)-(2) if and only if it is levelwise continuous and satisfies the integral equation*

$$\begin{aligned} x(t) &= x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) \\ &\quad + \int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds \qquad (3) \end{aligned}$$

for all $t \in J$.

Let $M + Na = b$, a positive number, where

$$M = \max D\left(f\left(t, x(\sigma_1(t)), \int_0^t h\left(t, s, x(\sigma_2(s)), \int_0^s k(s,\tau,x(\sigma_3(\tau)))d\tau\right) ds\right), \hat{0}\right) \quad \textit{and}$$

$N = D(g(t_1, t_2, \cdots, t_p, x(\cdot)), \hat{0})$, $\hat{0} \in E^n$.

Let $Y = \{\xi \in E^n : H(\xi, x_0) \le b\}$ be the space of continuous functions with $H(\xi, \psi) = \sup_{0\le t\le a} D(\xi(t), \psi(t))$.

Theorem 3.1. Assume that:

(i) The mapping $f : J \times Y \to E^n$ is levelwise continuous in t on J and there exists a constant G_0 such that

$$D(f(t, x_1, x_2), f(t, y_1, y_2)) \le G_0[D(x_1, y_1) + D(x_2, y_2)]$$

(ii) The mapping $h : J \times J \times Y \to E^n$ is levelwise continuous and there exists a constant G_1 such that

$$D(h(t, s, x_1, x_2), h(t, s, y_1, y_2)) \le G_1[D(x_1, y_1) + D(x_2, y_2)]$$

(iii) The mapping $k : J \times J \times Y \to E^n$ is levelwise continuous and there exists a constant G_2 such that

$$D(k(t, s, x), k(t, s, y)) \le G_2 D(x, y)$$

(iv) There exists a constant G_3 such that for all $x, y \in Y$ and $\sigma_i : J \to J$, $i = 1, 2, 3$

$$D(x(\sigma_i(t)), y(\sigma_i(t))) \le G_3 D(x(t), y(t))$$

(v) $g : J^p \times Y \to E^n$ is a function and there exists a constant $G_4 > 0$ such that

$$D(g(t_1, t_2, \cdots, t_p, x(\cdot)), g(t_1, t_2, \cdots, t_p, y(\cdot))) \le G_4 D(x, y).$$

Then there exists a unique solution $x(t)$ of (1)-(2) defined on the interval $[0, a]$.

Proof. Define an operator $\Phi : Y \to Y$ by

$$\begin{aligned} \Phi x(t) &= x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) \\ &\quad + \int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds. \end{aligned} \tag{4}$$

First, we show that $\Phi : Y \to Y$ is continuous whenever $\xi \in Y$ and that $H(\Phi\xi, x_0) \le b$.

$$\begin{aligned} &D(\Phi\xi(t+h), \Phi\xi(t)) \\ &= D\Big(x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot)) \\ &\quad + \int_0^{t+h} f\left(s, \xi(\sigma_1(s)), \int_0^s h\left(s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta\right) d\tau\right) ds, \\ &\quad x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot)) \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^t f\left(s, \xi(\sigma_1(s)), \int_0^s h\left(s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta\right) d\tau\right) ds\Big) \\
&\leq D\Big(\int_0^{t+h} f\left(s, \xi(\sigma_1(s)), \int_0^s h\left(s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta\right) d\tau\right) ds, \\
&\qquad \int_0^t f\left(s, \xi(\sigma_1(s)), \int_0^s h\left(s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta\right) d\tau\right) ds\Big) \\
&\leq \int_t^{t+h} D\Big(f\Big(s, \xi(\sigma_1(s)), \int_0^s h\Big(s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta\Big) d\tau\Big) ds, \hat{0}\Big) ds \\
&\leq hM \to 0 \text{ as } h \to 0.
\end{aligned}
$$

That is, the map Φ is continuous. Now

$$
\begin{aligned}
&D(\Phi\xi(t), x_0) \\
&\quad = D\Big(x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot)) \\
&\qquad + \int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds,\ x_0\Big) \\
&\quad \leq D(g(t_1, t_2, \cdots, t_p, \xi(\cdot)), \hat{0}) \\
&\qquad + \int_0^t D\left(f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right), \hat{0}\right) ds \\
&\quad \leq N + Mt
\end{aligned}
$$

and so

$$
H(\Phi\xi, x_0) = \sup_{0 \leq t \leq a} D(\Phi\xi(t), x_0) \leq N + Ma \leq b.
$$

Thus Φ is a mapping from Y into Y. Since $C([0,a], E^n)$ is a complete metric space with the metric H, we only show that Y is a closed subset of $C([0,a], E^n)$. Let $\{\psi_n\}$ be a sequence in Y such that $\psi_n \to \psi \in C([0,a], E^n)$ as $n \to \infty$. Then

$$
D(\psi(t), x_0) \leq D(\psi(t), \psi_n(t)) + D(\psi_n(t), x_0),
$$

that is,

$$
H(\psi, x_0) \quad = \quad \sup_{0 \leq t \leq a} D(\psi(t), x_0) \leq H(\psi, \psi_n) + H(\psi_n, x_0) \leq \epsilon + b
$$

for sufficiently large n and arbitrary $\epsilon > 0$. So $\psi \in Y$. This implies that Y is closed subset of $C([0,a], E^n)$. Therefore Y is a complete metric space.

By using Proposition 2.1 and assumptions (i)-(v), we will show that Φ is a contraction mapping. For $\xi, \psi \in Y$,

$$
\begin{aligned}
&D(\Phi\xi(t), \Phi\psi(t)) \\
&\quad = D\Big(x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot))
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t f\left(s,\xi(\sigma_1(s)),\int_0^s h\left(s,\tau,\xi(\sigma_2(\tau)),\int_0^\tau k(\tau,\theta,\xi(\sigma_3(\theta)))d\theta\right)d\tau\right)ds,\\
&\qquad x_0+g(t_1,t_2,\cdots,t_p,\psi(\cdot))\\
&+\int_0^t f\left(s,\psi(\sigma_1(s)),\int_0^s h\left(s,\tau,\psi(\sigma_2(\tau)),\int_0^\tau k(\tau,\theta,\psi(\sigma_3(\theta)))d\theta\right)d\tau\right)ds\Bigg)\\
\le\ & D(g(t_1,t_2,\cdots,t_p,\xi(\cdot)),g(t_1,t_2,\cdots,t_p,\psi(\cdot)))\\
&+\int_0^t D\Bigg(f\left(s,\xi(\sigma_1(s)),\int_0^s h\left(s,\tau,\xi(\sigma_2(\tau)),\int_0^\tau k(\tau,\theta,\xi(\sigma_3(\theta)))d\theta\right)d\tau\right)ds,\\
&f\left(s,\psi(\sigma_1(s)),\int_0^s h\left(s,\tau,\psi(\sigma_2(\tau)),\int_0^\tau k(\tau,\theta,\psi(\sigma_3(\theta)))d\theta\right)d\tau\right)ds\Bigg)\\
\le\ & G_4D(\xi(\cdot),\psi(\cdot))+G_0\int_0^t D(\xi(\sigma_1(s)),\psi(\sigma_1(s)))ds\\
&+G_0\int_0^t D\Bigg(\int_0^s h\left(s,\tau,\xi(\sigma_2(\tau)),\int_0^\tau k(\tau,\theta,\xi(\sigma_3(\theta)))d\theta\right)d\tau,\\
&\int_0^s h\left(s,\tau,\psi(\sigma_2(\tau)),\int_0^\tau k(\tau,\theta,\psi(\sigma_3(\theta)))d\theta\right)d\tau\Bigg)ds\\
\le\ & G_4D(\xi(\cdot),\psi(\cdot))+G_0G_3\int_0^t D(\xi(s),\psi(s))ds+G_0G_1G_3\int_0^t\int_0^s D(\xi(\tau),\psi(\tau))d\tau ds\\
&+G_0G_1G_2G_3\int_0^t\int_0^s\int_0^\tau D(\xi(\theta),\psi(\theta))d\theta d\tau ds.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
H(\Phi\xi,\Phi\psi)\le \sup_{0\le t\le a}&\Bigg\{G_4D(\xi(\cdot),\psi(\cdot))+G_0G_3\int_0^t D(\xi(s),\psi(s))ds\\
&+G_0G_1G_3\int_0^t\int_0^s D(\xi(\tau),\psi(\tau))d\tau ds\\
&\quad+G_0G_1G_2G_3\int_0^t\int_0^s\int_0^\tau D(\xi(\theta),\psi(\theta))d\theta d\tau ds\Bigg\}\\
\le\ & G_4D(\xi(\cdot),\psi(\cdot))+aG_0G_3D(\xi(t),\psi(t))\\
&+a^2G_0G_1G_3D(\xi(t),\psi(t))+a^3G_0G_1G_2G_3D(\xi(t),\psi(t))\\
\le\ & pH(\xi,\psi),
\end{aligned}
$$

where the constant $p=G_4+G_0G_3a+G_0G_1G_3a^2+G_0G_1G_2G_3a^3$. Taking sufficiently small a such that $p<1$, we obtain Φ to be a contraction mapping. Therefore Φ has a unique fixed point $x\in C([0,a],E^n)$ such that $\Phi x=x$, that is,

$$x(t)=x_0+g(t_1,t_2,\cdots,t_p,x(\cdot))$$

$$+\int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s,\tau,x(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,x(\sigma_3(\theta)))d\theta\right)d\tau\right)ds. \quad \square$$

Theorem 3.2. Let f, h, k, σ and g be as in Theorem 3.1. Denote by $x(t, x_0), y(t, y_0)$ the solutions of equation (1) corresponding to x_0, y_0, respectively. Then there exists constant $q > 0$ such that

$$H(x(\cdot, x_0), y(\cdot, y_0)) \leq qD(x_0, y_0)$$

for any $x_0, y_0 \in E^n$ and $q = 1/(1-p)$.

Proof. Let $x(t, x_0), y(t, y_0)$ be solutions of equations (1) corresponding to x_0, y_0, respectively. Then

$$\begin{aligned}
&D(x(t,x_0), y(t,y_0)) \\
&= D\Big(x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) \\
&\quad + \int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s,\tau,x(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,x(\sigma_3(\theta)))d\theta\right)d\tau\right)ds, \\
&\quad y_0 + g(t_1, t_2, \cdots, t_p, y(\cdot)) \\
&\quad + \int_0^t f\left(s, y(\sigma_1(s)), \int_0^s h\left(s,\tau,y(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,y(\sigma_3(\theta)))d\theta\right)d\tau\right)ds\Big) \\
&\leq D(x_0, y_0) + D(g(t_1, t_2, \cdots, t_p, x(\cdot)), g(t_1, t_2, \cdots, t_p, y(\cdot))) \\
&\quad + \int_0^t D\Big(f\Big(s, x(\sigma_1(s)), \int_0^s h\Big(s,\tau,x(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,x(\sigma_3(\theta)))d\theta\Big)d\tau\Big), \\
&\qquad f\left(s, y(\sigma_1(s)), \int_0^s h\left(s,\tau,y(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,y(\sigma_3(\theta)))d\theta\right)d\tau\right)\Big)ds \\
&\leq D(x_0, y_0) + G_4 D(x(\cdot), y(\cdot)) + G_0 G_3 \int_0^t D(x(s), y(s))ds \\
&\quad + G_0 G_1 G_3 \int_0^t \int_0^s D(x(\tau), y(\tau))d\tau ds + G_0 G_1 G_2 G_3 \int_0^t \int_0^s \int_0^\tau D(x(\theta), y(\theta))d\theta d\tau ds.
\end{aligned}$$

Thus $H(x(\cdot, x_0), y(\cdot, y_0)) \leq D(x_0, y_0) + pH(x(\cdot, x_0), y(\cdot, y_0))$. that is,

$$H(x(\cdot, x_0), y(\cdot, y_0)) \leq 1/(1-p)D(x_0, y_0).$$

This completes the proof of the theorem. $\square$

Next we generalize the above theorem for the fuzzy delay integrodifferential equation (1)-(2) with nonlocal condition.

Theorem 3.3. Suppose that $f : J \times E^n \to E^n$, $h : J \times J \times E^n \times E^n \to E^n$ and $k : J \times J \times E^n \to E^n$ are level wise continuous and bounded, $\sigma_i : J \to J$ $(i = 1, 2, 3)$ and $g : J^p \times E^n \to E^n$ are continuous.Then the initial value problem (1)-(2) possesses at least one solution on the interval J.

Proof. Since f, h, k are continuous and bounded and g is a continuous function there exists $r \geq 0$ such that

$$D\left(f\left(t, x(\sigma_1(t)), \int_0^t h\left(t, s, x(\sigma_2(s)), \int_0^s k(s, \tau, x(\sigma_3(\tau)))d\tau\right) ds\right), \hat{0}\right) \leq r, \ t \in J, x \in E^n.$$

Let B be a bounded set in $C(J, E^n)$. The set $\Phi B = \{\Phi x : x \in B\}$ is totally bounded if and only if it is equicontinuous and for every $t \in J$, the set $\Phi B(t) = \{\Phi x(t) : t \in J\}$ is a totally bounded subset of E^n. For $t_0, t_1 \in J$ with $t_0 \leq t_1$ and $x \in B$ we have that

$$\begin{aligned}
&D(\Phi x(t_0), \Phi x(t_1)) \\
&= D\Big(x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) \\
&\quad + \int_0^{t_0} f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds, \\
&\quad x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) \\
&\quad + \int_0^{t_1} f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds\Big) \\
&\leq D\Big(\int_0^{t_0} f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds, \\
&\quad \int_0^{t_1} f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right) ds\Big) \\
&\leq \int_{t_0}^{t_1} D\left(f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right) d\tau\right), \hat{0}\right) ds \\
&\leq |t_1 - t_0| \cdot \sup\left\{D\left(f\left(t, x(\sigma_1(t)), \int_0^t h\left(t, s, x(\sigma_2(s)), \int_0^s k(s, \tau, x(\sigma_3(\tau)))d\tau\right) ds\right), \hat{0}\right)\right\} \\
&\leq |t_1 - t_0| \cdot r.
\end{aligned}$$

This shows that ΦB is equicontinuous. Now, for $t \in J$ fixed, we have

$$D(\Phi x(t), \Phi x(t')) \ \leq \ |t - t'| \cdot r, \quad \text{for every} \ \ t' \in J, \ x \in B.$$

Consequently, the set $\{\Phi x(t) : x \in B\}$ is totally bounded in E^n. By Ascoli's theorem we conclude that ΦB is a relatively compact subset of $C(J, E^n)$. Then Φ is compact, that is, Φ transforms bounded sets into relatively compact sets.

We know that $x \in C(J, E^n)$ is a solution of (1)-(2) if and only if x is a fixed point of the operator Φ defined by (4).

Now, in the metric space $(C(J, E^n), H)$, consider the ball

$$B = \{\xi \in C(J, E^n), H(\xi, \hat{0}) \leq m\}, \ m = a \cdot r.$$

Thus, $\Phi B \subset B$. Indeed, for $x \in C(J, E^n)$,

$$\begin{aligned}
&D(\Phi x(t), \Phi x(0)) \\
&= D\Big(x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot))
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s,\tau,x(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,x(\sigma_3(\theta)))d\theta\right)d\tau\right)ds, \\
&x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot))\Big) \\
\leq\ & \int_0^t D\left(f\left(s, x(\sigma_1(s)), \int_0^s h\left(s,\tau,x(\sigma_2(\tau)), \int_0^\tau k(\tau,\theta,x(\sigma_3(\theta)))d\theta\right)d\tau\right), \hat{0}\right)ds \\
\leq\ & |t| \cdot r \leq a \cdot r.
\end{aligned}
$$

Therefore, defining $\hat{0} : J \to E^n$, $\hat{0}(t) = \hat{0}$, $t \in J$ we have

$$H(\Phi x, \Phi\hat{0}) = \sup\{D(\Phi x(t), \Phi\hat{0}(t)) : t \in J\}.$$

Therefore Φ is compact and, in consequence, it has a fixed point $x \in B$. This fixed point is a solution of the initial value problem (1)-(2). □

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