J. KSIAM Vol.9, No.2, 55-64, 2005

GENERALIZED DIFFERENCE METHODS FOR ONE–DIMENSIONAL VISCOELASTIC PROBLEMS

HUANRONG LI

ABSTRACT. In this paper, generalized difference methods(GDM) for one–dimensional viscoelastic problems are proposed and analyzed. The new initial values are given in the generalized difference scheme, so we obtain optimal error estimates in L^p and $W^{1,p}(2 \le p \le \infty)$ as well as some superconvergence estimates in $W^{1,p}(2 \le p \le \infty)$ between the GDM solution and the generalized Ritz–Volterra projection of the exact solution.

1. INTRODUCTION

Consider the following initial boundary value problem for the one–dimensional equation of viscoelasticity :

$$\begin{array}{lll} (a) & u_{tt} = \dfrac{\partial}{\partial x}\{a(x,t)\dfrac{\partial u_t}{\partial x} + b(x,t)\dfrac{\partial u}{\partial x}\} + f(x,t), & (x,t) \in (a,b) \times (0,T], \\ (b) & u(a,t) = 0, \quad u(b,t) = 0, & t \in [0,T], \\ (c) & u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in I = [a,b]. \end{array} \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$. $a(x,t)$, $b(x,t)$, $f(x,t)$, $u_0(x)$ and $u_1(x)$ are smooth enough to ensure the analysis validity and $a(x,t)$ is bounded from above and below:

$$0 < a_0 \le a(x,t) \le M, \quad (x,t) \in (a,b) \times [0,T]. \tag{1.2}$$

Since we shall show that the approximate solution is uniformly convergent to the exact solution of (1.1), the above assumptions only need to hold in a neighborhood of the exact solution.

The problem(1.1) describes many physical processes such as heat transfer with memory[1,2], gas diffusion[3], propagation of sound in viscous media[4,5] and fluid dynamics.

The finite element methods to problem(1.1) have been studied by several authors. Y.P.Lin and Cannon[6] demonstrated optimal order error estimates in the L^2 norms and L^p norms error estimates in $R^d(d \le 4)$. Optimal maximum norm estimates are given by other author. However, the generalized difference methods haven't been used

---

2000 Mathematics Subject Classification: 65N15, 65N30.

Key words and phrases: Generalized Difference Methods, One–Dimensional Viscoelastic Problems, Optimal Error Estimates, Superconvergence.

© THE KOREAN SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, 2005

to deal with the viscoelastic problem(1.1). In fact, the generalized difference methods have the same convergence orders as the corresponding finite element methods, but they require less computational expenses, and keep the mass conservation[7,8].

The aim of this paper is to provide a theory for the generalized difference methods for the two–dimensional problem(1.1)of viscoelasticity. We derive the optimal error estimates in L^p and $W^{1,p}$ for $2 \leq p \leq \infty$. Moreover, some superconvergence is also obtained.

The paper is organized in the following way. In section 2, the new initial values are given and the semi-discrete generalized difference schemes are formulated in piecewise linear finite element spaces. Some important lemmas are introduced in section 3, which are essential in our analysis. Main results of this paper are given in section 4.

2. Semi-Discrete Generalized Difference Schemes

In this paper we will follow the notations and symbols in [7]. For examples, $T_h = \{I_i; I_i = [x_{i-1}, x_i], 1 \leq i \leq n\}$, and $T_h^* = \{I_i^*; I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], 1 \leq i \leq n-1,\ I_0^* = [x_0, x_{\frac{1}{2}}],\ I_n^* = [x_{n-\frac{1}{2}}, x_n]\}$denote the primal partition and its dual partition, respectively. Let $h_i = x_i - x_{i-1}$, $h = max\{h_i; 1 \leq i \leq n\}$. The partitions are assumed to be regular, that is, there exists a constant $\mu > 0$ such that $h_i \geq \mu h$, $i = 1, 2, \cdots n$. The trial function space $U_h \subset H_0^1(I) \equiv \{u \in H^1(I); u(a) = u(b) = 0\}$ is defined as a piecewise linear function space over T_h and $U_h = span\{\varphi_i(x), 1 \leq i \leq n-1\}$. The test function space $V_h = span\{\psi_i(x), 1 \leq i \leq n-1\} \subset L^2(I)$ is defined as a piecewise constant function space over T_h^*.

For numerical analysis, we need to introduce the interpolation operators Π_h from $H_0^1(I) \bigcap C(I)$ to U_h defined by

$$\Pi_h w = \sum_{i=1}^{n-1} w(x_i)\varphi_i(x), w \in H_0^1(I), \tag{2.1}$$

and $\Pi_h^* : H_0^1(I) \bigcap C(I) \longmapsto V_h$, defined by

$$\Pi_h^* w = \sum_{i=1}^{n-1} w(x_i)\psi_i(x), w \in H_0^1(I). \tag{2.2}$$

Using the interpolation theory, we have

$$\begin{array}{lll} (a) & |w - \Pi_h w|_{m,p} \leq C h^{k-m} |w|_{k,p}\ , & m = 0, 1,\ k = 1, 2,\ 1 \leq p \leq \infty, \\ (b) & \|w - \Pi_h^* w\|_{0,p} \leq C h |w|_{1,p}\ , & 1 \leq p \leq \infty. \end{array} \tag{2.3}$$

where $|\cdot|_{m,p}$ and $\|\cdot\|_{m,p}$ stand for the semi-norm and norm of the Sobolev space $W^{m,p}(I)$ respectively, $|\cdot|_m$ and $\|\cdot\|_m$stand for the semi-norm and norm of the Sobolev space $H^m(I) = W^{m,2}(I)$ respectively, and C is a positive constant independent of h.

Let's define, for any $u,\ v \in H_0^1(I)$, $u_h \in U_h$ and $v_h \in V_h$, some bilinear forms as follows:

$$\begin{aligned} a(u,\ v) &= \int_a^b a(x,t)u'v'dx; \\ b(u,\ v) &= \int_a^b b(x,t)u'v'dx; \\ c(u,\ v) &= b(u,\ v) - a_t(u,\ v); \\ a^*(u_h,\ v_h) &= \sum_{j=1}^{n-1} v_j a^*(u_h, \psi_j); \\ b^*(u_h,\ v_h) &= \sum_{j=1}^{n-1} v_j b^*(u_h, \psi_j); \\ c^*(u_h,\ v_h) &= b^*(u_h,\ v_h) - a_t^*(u_h,\ v_h), \end{aligned} \tag{2.4}$$

where $a^*(u_h, \psi_j)$ and $b^*(u_h, \psi_j)$ defined by

$$a^*(u_h, \psi_j) = a_{j-\frac{1}{2}} u_h'(x_{j-\frac{1}{2}}) - a_{j+\frac{1}{2}} u_h'(x_{j+\frac{1}{2}}),$$
$$b^*(u_h, \psi_j) = b_{j-\frac{1}{2}} u_h'(x_{j-\frac{1}{2}}) - b_{j+\frac{1}{2}} u_h'(x_{j+\frac{1}{2}})$$

with $u_h'(x_{j-\frac{1}{2}}) = \frac{u_j - u_{j-1}}{h_j}, u' = \frac{\partial u}{\partial x}, v' = \frac{\partial v}{\partial x}, u_j = u_h(x_j), v_j = v_h(x_j), u_0 = 0, u_n = 0, x_{j-\frac{1}{2}} = \frac{1}{2}(x_{j-1} + x_j), a_{j-\frac{1}{2}} = a(x_{j-\frac{1}{2}}, t), b_{j-\frac{1}{2}} = b(x_{j-\frac{1}{2}}, t)$ and the coefficients of $a_t(\cdot;\cdot,\cdot)$, $a_t^*(\cdot;\cdot,\cdot)$ and $b_t(\cdot;\cdot,\cdot)$, $b_t^*(\cdot;\cdot,\cdot)$ which appear in the following are obtained from differentiating the corresponding coefficients of $a(\cdot;\cdot,\cdot)$, $a^*(\cdot;\cdot,\cdot)$, $b(\cdot;\cdot,\cdot)$ and $b^*(\cdot;\cdot,\cdot)$ with respect to t, respectively.

The generalized weak form of (1.1) is to find a map $u(t) : [0,T] \longmapsto H_0^1(I)$, such that

$$\begin{array}{lll} (a) & (u_{tt}, v) + a(u_t, v) + b(u,v) = (f,v), & \forall\ v \in H_0^1(I), \\ (b) & u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in I. \end{array} \tag{2.5}$$

For error estimates, we next introduce the *Ritz* projection operator $R_h = R_h(t) : H_0^1(I) \mapsto U_h,\ 0 \le t \le T$, defined by

$$a(u - R_h u,\ v_h) = 0, \quad \forall\ v_h \in U_h, \tag{2.6}$$

the generalized *Ritz* projection operator $R_h^* = R_h^*(t) : H_0^1(I) \longmapsto U_h,\ 0 \le t \le T$, defined by

$$a^*(u - R_h^* u,\ v_h) = 0, \quad \forall\ v_h \in V_h, \tag{2.7}$$

and the generalized *Ritz-Volterra* projection operator $V_h^* = V_h^*(t) : H_0^1(I) \mapsto U_h, 0 \le t \le T$, defined by

$$a^*(u - V_h^* u,\ v_h) + \int_0^t c^*(u - V_h^* u,\ v_h) d\tau = 0, \quad \forall\ v_h \in V_h, \tag{2.8}$$

Differentiating (2.8) with respect to t, we can obtain the equivalence of (2.8):

$$\begin{cases} a^*((u - V_h^* u)_t,\ v_h) + b^*(u - V_h^* u,\ v_h) = 0, & \forall\ v_h \in V_h, \\ a^*(u(0) - V_h^* u(0),\ v_h) = 0, & \forall\ v_h \in V_h. \end{cases} \tag{2.8'}$$

Obviously, $V_h^*(0) = R_h^*(0)$. Then, the semi-discrete generalized difference schemes of (1.1) is to find a map $u_h(t) : [0,T] \longmapsto U_h$, such that

$$\begin{array}{lll} (a) & (u_{h,tt},\ v_h) + a^*(u_{h,t},\ v_h) + b^*(u_h,\ v_h) = (f,\ v_h), & \forall\ v_h \in V_h, \\ (b) & u_h(0) = u_{0h},\ \ u_{h,t}(0) = u_{1h}, & x \in I. \end{array} \tag{2.9}$$

where $u_{0h} = V_h^* u_0 = R_h^* u_0,\ \ u_{1h} \in U_h$ satisfies

$$a^*(u_{1h},\ v_h) = (f(0),\ v_h) - b^*(u_{0h},\ v_h) - ((V_h^*u)_{tt}(0),\ v_h),\ \forall\ v_h \in V_h,$$

here, $(V_h^*u)_{tt}(0)$ satisfies: $\forall\ v_h \in V_h$,

$$a_t^*((u - V_h^*u)_t(0),\ v_h) + a^*((u - V_h^*u)_{tt}(0),\ v_h) + b^*((u - V_h^*u)_t(0),\ v_h) \\ +b_t^*((u - V_h^*u)(0),\ v_h) = 0,$$

and $u_{tt}(0) = \dfrac{\partial}{\partial x}\{a(x,0)\dfrac{\partial u_1}{\partial x} + b(x,0)\dfrac{\partial u_0}{\partial x}\} + f(x,0)$, $(V_h^*u)_t(0)$ is uniquely determined by

$$a^*((u - V_h^*u)_t(0),\ v_h) + b^*((u - V_h^*u)(0),\ v_h) = 0,\ \ \forall\ v_h \in V_h.$$

3. Some Lemmas

Noting that for any $u_h \in U_h$, we have, by (2.2)

$$|u_h|_{1,p} = (\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |u_h'|^p dx)^{\frac{1}{p}} = \{\sum_{i=1}^{n} h_i (\frac{u_i - u_{i-1}}{h_i})^p\}^{\frac{1}{p}}.$$

Define some discrete norms in U_h:

$$\|u_h\|_{0,h} = \{\sum_{i=1}^{n} h_i (u_i^2 + u_{i-1}^2)\}^{\frac{1}{2}},$$
$$|u_h|_{1,h} = \{\sum_{i=1}^{n} (\frac{(u_i - u_{i-1})^2}{h_i})\}^{\frac{1}{2}},$$
$$\|u_h\|_{1,h} = (\|u_h\|_{0,h}^2 + |u_h|_{1,h}^2)^{\frac{1}{2}}.$$

Then we can easily prove the following lemmas.

Lemma 3.1(See[7,8])There exist two positive constants C_1 and C_2, independent of h, such that for any $u_h \in U_h$,

$$\begin{array}{ll} (a) & |u_h|_{1,h} = |u_h|_1; \\ (b) & C_1\|u_h\|_{0,h} \le \|u_h\|_0 \le C_2\|u_h\|_{0,h}; \\ (c) & C_1\|u_h\|_{1,h} \le \|u_h\|_1 \le C_2\|u_h\|_{1,h}. \end{array} \tag{3.1}$$

Lemma 3.2(See[7,8]) For $\forall\ u_h,\ w_h \in U_h$,

(1) $(u_h, \Pi_h^* w_h) = (w_h, \Pi_h^* u_h)$

(2) Let $|||u_h|||_0^2 = (u_h, \Pi_h^* u_h)$,, then $|||\cdot|||$ is equivalent to $\|\cdot\|_0$ in U_h

According to the technique given in [7,8], it is easy to derive the following conclusions:

Lemma 3.3 There exist four positive constants $\bar{\alpha}$, $\bar{M}_1$, $\bar{M}_2$ and $\bar{M}$, independent of h, such that for $\forall\, u,\ v \in U$,

$$\begin{array}{ll} (a) & a(u,\ u) \geq \bar{\alpha}\|u\|_1^2; \\ (b) & |a(u,\ v)| \leq \bar{M}_1\|u\|_1\|v\|_1; \\ (c) & |b(u,\ v)| \leq \bar{M}_2\|u\|_1\|v\|_1; \\ (d) & |c(u,\ v)| \leq \bar{M}\|u\|_1\|v\|_1. \end{array} \tag{3.2}$$

Lemma 3.4 There exist four positive constants α, M_1, M_2 and M, independent of h, such that

$$\begin{array}{ll} (a) & a^*(u_h,\ \Pi_h^* u) \geq \alpha\|u_h\|_1^2, \quad \forall\, u_h \in U_h; \\ (b) & |a^*(u_h,\ \Pi_h^* v_h)| \leq M_1\|u_h\|_1\|v_h\|_1, \quad \forall\, u_h,\ v_h \in U_h; \\ (c) & |b^*(u_h,\ \Pi_h^* v_h)| \leq M_2\|u_h\|_1\|v_h\|_1, \quad \forall\, u_h,\ v_h \in U_h; \\ (d) & |c^*(u_h,\ \Pi_h^* v_h)| \leq M\|u_h\|_1\|v_h\|_1, \quad \forall\, u_h,\ v_h \in U_h. \end{array} \tag{3.3}$$

For simplicity, we set

$$\begin{aligned} d_1(u-u_h,\ w_h) &= a(u-u_h,\ w_h) - a^*(u-u_h,\ \Pi_h^* w_h), \\ d_2(u-u_h,\ w_h) &= b(u-u_h,\ w_h) - b^*(u-u_h,\ \Pi_h^* w_h). \end{aligned}$$

We now present a very useful lemma:

Lemma 3.5 If $u \in W^{3,p}(I)$, for any $u_h,\ v_h,\ w_h \in U_h$, we have

$$\begin{array}{ll} (a) & |d_1(u-u_h,\ w_h)| \leq Ch^2(h^{-1}|u-u_h|_{1,p} + |u|_{3,p})\|w_h\|_{1,p'}; \\ (b) & |d_1(v_h,\ w_h)| \leq Ch\|v_h\|_{1,p}\|w_h\|_{1,p'}; \\ (c) & |d_2(u-u_h,\ w_h)| \leq Ch^2(h^{-1}|u-u_h|_{1,p} + |u|_{3,p})\|w_h\|_{1,p'}; \\ (d) & |d_2(v_h,\ w_h)| \leq Ch\|v_h\|_{1,p}\|w_h\|_{1,p'}. \end{array} \tag{3.4}$$

where $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemma 3.6[9] For any $u_h,\ v_h,\ w_h \in U_h$, we can get

$$\begin{array}{ll} (a) & |a^*(u_h,\ \Pi_h^* w_h) - a^*(w_h,\ \Pi_h^* u_h)| \leq Ch\|u_h\|_1\|w_h\|_1; \\ (b) & |b^*(u_h,\ \Pi_h^* w_h) - b^*(w_h,\ \Pi_h^* u_h)| \leq Ch\|u_h\|_1\|w_h\|_1. \end{array} \tag{3.5}$$

Remark: If the coefficients of the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $a^*(\cdot,\cdot)$ and $b^*(\cdot,\cdot)$ are replaced by other functions, the lemmas 3.3–3.6 are still valid.

Let X be a Banach space with norm $\|\cdot\|_X$ and $\phi : [0,T] \longmapsto X$. Define

$$\|\phi\|_{L^2(X)}^2 = \int_0^T \|\phi(t)\|_X^2 dt \quad \textit{and } \|\phi\|_{L^\infty(X)} = ess \sup_{0\leq t\leq T} \|\phi(t)\|_X.$$

Let the space $H^k(W^{s,p})$ be defined by

$$H^k(0,T;W^{s,p}) = \{u \in W^{s,p};\ \frac{\partial^j u}{\partial t^j} \in L^2(0,T;W^{s,p}),\ j=0,1,\cdots,k\}$$

and for any $u \in H^k(W^{s,p})$, we set

$$\|u(t)\|_{k,s,p} = \sum_{j=1}^{k}\{\|\frac{\partial^j u}{\partial t^j}\|_{s,p} + \int_0^t \|\frac{\partial^j u}{\partial t^j}\|_{s,p} d\tau\},\ t\in[0,T].$$

Similar to the proof given in [7,9,10], we can deduce the properties of the generalized *Ritz — —Volterra* projection.

Lemma 3.7 $V_h^* u$ is defined by (2.8) or (2.8)', then

$$\begin{array}{llll} (a) & \|D_t^l(u-V_h^*u)\|_{1,p} \le Ch\|u\|_{l,2,p}, & l=0,1,2,3, & 2\le p\le\infty; \\ (b) & \|D_t^l(u-V_h^*u)\|_{0,p} \le Ch^2\|u\|_{l,3,p}, & l=0,1,2,3, & 2\le p\le\infty. \end{array} \quad (3.6)$$

For convenience, we write $u_h - u = (u_h - V_h^*u) + (V_h^*u - u) = \xi + \eta$ in this paper.

Lemma 3.8 If u_0, u_1 and u_{0h}, u_{1h} are the initial values of (1.1) and (2.9), respectively, then

$$\begin{array}{ll} (a) & \xi(0) = \xi_{tt}(0) = 0; \\ (b) & \|\xi_t(0)\|_1 \le Ch^2(\|u_0\|_3 + \|u_1\|_3 + \|u_{tt}(0)\|_3) \end{array}. \quad (3.7)$$

Proof. Obviously, $\xi(0) = u_{0h} - V_h^*u_0 = 0$. By noting that

$((V_h^*u)_{tt}(0),\ v_h) = (f(0),\ v_h) - a^*(u_{1h},\ v_h) - b^*(u_{0h},\ v_h),\ v_h) = (u_{h,tt}(0),\ v_h),\ \forall\, v_h \in V_h,$

we can know $(V_h^*u)_{tt}(0) = u_{h,tt}(0)$,, i.e. $\xi_{tt}(0) = 0$, the conclusion of (3.7a) is proved.

To show (3.7b), apply (2.5), (2.9) and (2.8$'$) to get the error equation:

$$(\xi_{tt},\ v_h) + a^*(\xi_t,\ v_h) + b^*(\xi,\ v_h) = -(\eta_{tt},\ v_h),\ \forall\ v_h \in V_h, \quad (3.8)$$

Integrating (3.8) with respect to t and noting $\xi(0) = 0$, we can obtain the equivalence of (3.8), by (2.4b)

$$(\xi_t,\ v_h) + a^*(\xi,\ v_h) + \int_0^t c^*(\xi,\ v_h)d\tau = -(\eta_t - \xi_t(0) - \eta_t(0),\ v_h),\ \forall\ v_h \in V_h, \quad (3.8')$$

Setting $t = 0$ and $v_h = \Pi_h^*\xi_t(0)$ in (3.8), we have, by (3.7a)

$$a^*(\xi_t(0),\ \Pi_h^*\xi_t(0)) = -(\eta_{tt}(0),\ \Pi_h^*\xi_t(0)),$$

Also from (3.3a) and (3.5b),

$$\alpha\|\xi_t(0)\|_1^2 \le \|\eta_{tt}(0)\|\|\xi_t(0)\| \le Ch^2(\|u_0\|_3 + \|u_1\|_3 + \|u_{tt}(0)\|_3)\|\xi_t(0)\|_1.$$

Hence, this completes the proof of (3.7b). □

Lemma 3.9 If u and u_h are the solution of (1.1) and (2.9), respectively, then

$$\|\xi_t\| + \|\xi\|_1 \le Ch^2\{\|u_0\|_3 + \|u_1\|_3 + \|u_{tt}(0)\|_3 + \int_0^t (\|u\|_3 + \|u_t\|_3 + \|u_{tt}\|_3)d\tau\}. \quad (3.9)$$

Proof. Taking $v_h = \Pi_h^* \xi_t$ in (3.8), we have

$$(\xi_{tt},\ \Pi_h^*\xi_t) + a^*(\xi_t,\ \Pi_h^*\xi_t) + b^*(\xi,\ \Pi_h^*\xi_t) = -(\eta_{tt},\ \Pi_h^*\xi_t),$$

The above is also written as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 + a^*(\xi_t,\ \Pi_h^*\xi_t) + \frac{1}{2}\frac{d}{dt}b^*(\xi,\ \Pi_h^*\xi) = \ & -(\eta_{tt},\ \Pi_h^*\xi_t) + \frac{1}{2}[b^*(\xi_t,\ \Pi_h^*\xi) \\ & -b^*(\xi,\ \Pi_h^*\xi_t)] + \frac{1}{2}b_t^*(\xi,\ \Pi_h^*\xi),\end{aligned} \tag{3.10}$$

Noting $\xi(0) = 0$, lemmas 3.4 and 3.6 and integrating from 0 to t, we get

$$\begin{aligned}\|\xi_t\|^2 + \|\xi\|_1^2 &\le \|\xi_t(0)\|^2 + C\int_0^t \|\eta_{tt}\|\|\xi_t\|d\tau + C\int_0^t h\|\xi_t\|_1\|\xi\|_1 d\tau + C\int_0^t \|\xi\|_1^2 d\tau \\ &\le \|\xi_t(0)\|^2 + C\int_0^t \|\eta_{tt}\|^2 d\tau + C\int_0^t \|\xi_t\|^2 d\tau + C\int_0^t \|\xi\|_1^2 d\tau,\end{aligned} \tag{3.11}$$

here we have applied the inverse properties of the finite element space and the inequality $\|\xi_t\|_1 \le Ch^{-1}\|\xi_t\|$. Thus, the conclusion follows from *Gronwall*'s Lemma, and lemmas 3.7 and 3.8. □

Finally, in order to conclude maximum norm estimates, we introduce Green function $\partial_z G_z^h \in U_h$ and pre-Green function $\partial_z G_z^* \in H_0^1(I)$:

$$\begin{array}{lll}(a) & a(\partial_z G_z^h,\ v_h) = \partial_z v_h(z), & \forall v_h \in U_h; \\ (b) & a(\partial_z G_z^*,\ v) = \partial_z P_h v(z), & \forall v \in H_0^1(I).\end{array} \tag{3.12}$$

where P_h: $L^2(I) \longmapsto U_h$ is L^2 projection operator, and we have the following (see[11])

$$\|P_h u\|_{s,q} \le C\|u\|_{s,q}, \quad s = 0, 1, \quad 2 \le q \le \infty. \tag{3.13}$$

Lemma 3.10[11] For Green functions defined in (3.12), we know

$$\begin{array}{ll}(a) & \|\partial_z G_z^* - \partial_z G_z^h\|_{1,1} + h\|\partial_z G_z^h\|_1 + \|\partial_z G_z^*\|_{0,1} \le C; \\ (b) & \|\partial_z G_z^h\| \le C.\end{array} \tag{3.14}$$

4. Main Results

We next demonstrate a superconvergence results of $u_h - V_h^* u$.

Theorem 4.1 Under the conditions of lemma 3.9, for h sufficiently small, we can deduce

$$\|\xi\|_{1,p} \le Ch^2\{\|u(0)\|_{2,3,p} + \|u\|_{2,3,p}\}, \quad 2 \le p \le \infty. \tag{4.1}$$

Proof. (i) Let us consider the case of $2 \le p < \infty$.

We now introduce an auxiliary problem . Denote ϕ_x to be the derivative of ϕ and let $\Phi \in H_0^1(I)$ be the solution of

$$a(v,\ \Phi) = -(v, \phi_x), \quad v \in H_0^1(I), \tag{4.2}$$

and there is a priori estimate

$$\|\Phi\|_{1,p'} \le C\|\phi\|_{0,p'}, \quad p' = \frac{p}{p-1}. \tag{4.3}$$

By virtue of Green formula , (2.6), and (3.8′),

$$\begin{aligned}
(\xi_x,\ \phi) &= a(\xi,\ \Phi) \\
&= a(\xi,\ R_h\Phi) \\
&= d_1(\xi,\ R_h\Phi) + a^*(\xi,\ \Pi_h^* R_h\Phi) \\
&= d_1(\xi,\ R_h\Phi) - (\xi_t + \eta_t - \xi_t(0) - \eta_t(0),\ \Pi_h^* R_h\Phi) \\
&\quad - \int_0^t c^*(\xi,\ \Pi_h^* R_h\Phi)d\tau \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{4.4}$$

Now it suffices to estimate each term in the above.
Noting that lemmas 3.5 and 3.4, and $\|R_h\Phi\|_{1,p'} \le C\|\Phi\|_{1,p'}$, we easily get

$$\begin{aligned}
|I_1| &\le Ch\|\xi\|_{1,p}\|R_h\Phi\|_{1,p'} \\
&\le Ch\|\xi\|_{1,p}\|\Phi\|_{1,p'}
\end{aligned}$$

and

$$|I_3| \le C\int_0^t \|\xi\|_{1,p}d\tau\|\Phi\|_{1,p'}.$$

For I_2, from *Sobolev*'s imbedding inequalities, we have

$$\begin{aligned}
|I_2| &\le (\|\xi_t\| + \|\xi_t(0)\|)\|R_h\Phi\| + (\|\eta_t\|_{0,p} + \|\eta_t(0)\|_{0,p})\|R_h\Phi\|_{0,p'} \\
&\le C(\|\xi_t\| + \|\xi_t(0)\|_1 + \|\eta_t\|_{0,p} + \|\eta_t(0)\|_{0,p})\|\Phi\|_{1,p'}
\end{aligned}$$

Combining the estimates of $I_1 - -I_4$, we obtain also by (4.3) that

$$\begin{aligned}
\|\xi\|_{1,p} &\le C\sup_{\phi\in L^{p'}} \frac{|(\xi_x,\ \phi)|}{\|\phi\|_{0,p'}} \\
&\le Ch\|\xi\|_{1,p} + C\{\|\xi_t\| + \|\xi_t(0)\|_1 + \|\eta_t\|_{0,p} + \|\eta_t(0)\|_{0,p}\} + C\int_0^t \|\xi\|_{1,p}d\tau
\end{aligned}$$

By letting h sufficiently small such that $Ch \le \frac{1}{2}$, the results for $2 \le p < \infty$ now follows by *Gronwall'* Lemma and lemmas 3.7–3.9.

(ii) Let us next consider the case of $p = \infty$.

Applying the definition (3.12a) of Green function, we have

$$\begin{aligned}
\partial_z\xi(z) &= a(\xi, \partial_z G_z^h) \\
&= d_1(\xi, \partial_z G_z^h) - \int_0^t c^*(\xi, \Pi_h^*\partial_z G_z^h)d\tau - (\xi_t + \eta_t - \xi_t(0) - \eta_t(0),\ \Pi_h^*\partial_z G_z^h) \\
&= d_1(\xi, \partial_z G_z^h) - \int_0^t [d_2(\xi, \partial_z G_z^h) - d_{1t}(\xi, \partial_z G_z^h)]d\tau - \int_0^t c(\xi, \partial_z G_z^h - \partial_z G_z^*)d\tau \\
&\quad - \int_0^t c(\xi, \partial_z G_z^*)d\tau - (\xi_t + \eta_t - \xi_t(0) - \eta_t(0), \Pi_h^*\partial_z G_z^h) \\
&= J_1 + \cdots + J_5,
\end{aligned}$$

Now we proceed to estimate these J_i one by one.
From lemmas 3.5 and 3.3, and (3.14a), we get

$$\begin{aligned} |J_1| &\le Ch\|\xi\|_1\|\partial_z G_z^h\|_1 \le C\|\xi\|_1, \\ |J_2| &\le C\int_0^t \|\xi\|_1 d\tau \le C\int_0^t \|\xi\|_{1,\infty} d\tau, \end{aligned}$$

and

$$\begin{aligned} |J_3| &\le C\int_0^t \|\xi\|_{1,\infty} d\tau \|\partial_z G_z^* - \partial_z G_z^h\|_{1,1} \\ &\le C\int_0^t \|\xi\|_{1,\infty} d\tau. \end{aligned}$$

As for J_4, it follows from (3.12b), by integration by parts and (3.14a), that

$$\begin{aligned} |J_4| &= |\int_0^t [\int_a^b b(x,t)\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}(\partial_z G_z^*)dx - \int_a^b a_t(x,t)\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}(\partial_z G_z^*)dx]d\tau| \\ &= |\int_0^t \{\int_a^b [a(x,t)\frac{\partial}{\partial x}(\frac{b(x,t)}{a(x,t)}\xi) - a(x,t)\frac{\partial}{\partial x}(\frac{b(x,t)}{a(x,t)})\xi]\frac{\partial}{\partial x}(\partial_z G_z^*)dx \\ &\quad - \int_a^b [a(x,t)\frac{\partial}{\partial x}(\frac{a_t(x,t)}{a(x,t)}\xi) - a\frac{\partial}{\partial x}(\frac{a_t(x,t)}{a(x,t)})\xi]\frac{\partial}{\partial x}(\partial_z G_z^*)dx\}d\tau| \\ &= |\int_0^t \partial_z P(\frac{b(x,t)}{a(x,t)}\xi)d\tau - \int_0^t\int_a^b \frac{\partial}{\partial x}[a(x,t)\frac{\partial}{\partial x}(\frac{b(x,t)}{a(x,t)})\xi]\partial_z G_z^* dx d\tau \\ &\quad - \int_0^t \partial_z P(\frac{a_t(x,t)}{a(x,t)}\xi)d\tau + \int_0^t\int_a^b \frac{\partial}{\partial x}[a(x,t)\frac{\partial}{\partial x}(\frac{a_t(x,t)}{a(x,t)})\xi]\partial_z G_z^* dx d\tau| \\ &\le C\int_0^t \|\xi\|_{1,\infty}d\tau + C\int_0^t \|\xi\|_{1,\infty}d\tau\|\partial_z G_z^*\|_{0,1} \\ &\le C\int_0^t \|\xi\|_{1,\infty}d\tau \end{aligned}$$

Lastly, it is easy to see , by *Sobolev*'s imbedding inequalities and (3.14b), that

$$\begin{aligned} |J_5| &\le C(\|\xi_t\| + \|\eta_t\| + \|\xi_t(0)\| + \|\eta_t(0))\|)\|\partial_z G_z^h\| \\ &\le C(\|\xi_t\| + \|\eta_t\| + \|\xi_t(0)\|_1 + \|\eta_t(0))\|), \end{aligned}$$

Combining the estimates of $J_1 - -J_5$, we have

$$\|\xi\|_{1,\infty} \le C(\|\xi\|_1 + \|\xi_t\| + \|\eta_t\| + \|\xi_t(0)\|_1 + \|\eta_t(0))\|) + C\int_0^t \|\xi\|_{1,\infty}d\tau.$$

which together with *Gronwall*'s Lemma , and lemmas 3.7 − −3.9 completes the proof of $p = \infty$.

Finally, We can deduce the L^p and $W^{1,p}$ norm error estimates of $u - u_h$, by using $\|\xi\|_{0,p} \le C\|\xi\|_{1,p}$ and lemma 3.7. □

Theorem 4.2 Under the conditions of theorem 4.1, we can conclude that

$$\begin{aligned} (a) \quad & \|u-u_h\|_{0,p} \le Ch^2\{\|u(0)\|_{2,3,p} + \|u\|_{2,3,p}\}, \quad 2 \le p \le \infty, \\ (c) \quad & \|u-u_h\|_{1,p} \le Ch\{\|u(0)\|_{2,3,p} + \|u\|_{2,3,p}\}, \quad 2 \le p \le \infty. \end{aligned} \tag{4.4}$$

References

[1] M. Gurtin, A. Pipkin, A general theory of heat conduction with finite wave speeds, Arch Rational Mech. Anal., 31(1968), 113–126.

[2] R. K. Miller, An integro-difference equation for rigid heat conductions with memory, J. Math. Anal. Appl., 66(1998), 313-332.

[3] M. Raynal, On some nonlinear problems of diffusion in Volterra Equations, S. London and O. Staffans eds., Lecture notes in Math., 737, Springer–Verlag, Berlin, Newyork, 1979, 251–266.

[4] K. Suveika, Mixed problems for an equation describing the propagation of disturbances in viscous media, J.Differential Equations, 19(1982), 337-347.

[5] Y. P. Lin, A mixed boundary problem describing the propagation of disturbances in viscous media solution for quasi-linear equations, J.Math.Anal.Appl, 135(1988), 644-653.

[6] J. R. Cannon and Y. Lin, A Priori L_2 error estimates for finite-element methods for nonlinear diffusion equations with memory, SIAM J. Numer. Anal., 27(1990), No.3, 595–607.

[7] Q. Li, Generalized difference method, Lecture Notes of the Twelfth Mathematical Workshop, Taejon, Korea, 1997.

[8] R.H.Li and Z.Y.Chen, The generalized difference method for differential equations, Jilin University Publishing House, (1994). (In Chinese)

[9] Q. Li and Z. Y. Liu, Finite volume element methods for nonlinear parabolic problems, J. KSIAM, Vol.6, No.2, (2002), 85-97.

[10] H. R. Li and Q. Li, Finite volume element methods for nonlinear parabolic intergrodifferential problems, J. KSIAM, Vol.7, No.2, (2003), 35-49.

[11] Q.D.Zhu and Q.Lin, Superconvergence theory of the finite element method, Hunan Science and Technique press, China, 1989.

Huanrong Li:
Department of Mathematics, Capital Normal University, Beijing, 100037, P.R.China
E-mail: lihuanrong1979@126.com