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# A SURFACE RECONSTRUCTION METHOD FOR SCATTERED POINTS ON PARALLEL CROSS SECTIONS

PHILSU KIM

ABSTRACT. We consider a surface reconstruction problem from geometrical points (i.e., points given without any order) distributed on a series of smooth parallel cross sections in $\mathbb{R}^3$. To solve the problem, we utilize the natural points ordering method in $\mathbb{R}^2$, described in [18], which is a method of reconstructing a curve from a set of sample points and is based on the concept of diffusion motions of a small object from one point to the other point. With only the information of the positions of these geometrical points, we construct an acceptable surface consisting of triangular facets using a heuristic algorithm to link a pair of parallel cross-sections constructed via the natural points ordering method. We show numerical simulations for the proposed algorithm with some sets of sample points.

## 1. INTRODUCTION

The problem that we treat is to reconstruct a 3D surface from a set of sample points on a series of parallel planar cross-sections (or called slices) corresponding to different levels. It occurs from various fields, for instance, medical imaging, digitization of objects, and GIS systems and so on, and a lot of algorithms with the work of Keppel [15] as its starting point have been proposed. The problem has been explored in four major directions: (i) Delaunay triangulation method [2, 3, 8], (ii) partial differential equation(PDE) method [4, 5, 19] , (iii) optimal method [1, 12, 14, 15], and (iv) heuristic method [6, 13, 16]. The most general method of them is the Delaunay triangulation one, which solves the problem in a $n$-dimensional space for reconstructing a surface from a set of sample points arbitrarily distributed in space. This does not require any particular structures such as that the sample points lie on a series of parallel planar cross-sections, however it is somewhat expensive compared with other methods. The optimal method and heuristic method are more simple and efficient algorithms than the Delaunay triangulation one. According to the number of contours in each cross section and the position of them, these two methodologies have a problem requiring some meticulous care, so called branching problem. The PDE method is very effective one to solve the branching problem.

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A common assumption in the methodologies (ii)-(iv) is that the set of sample points on each cross section are well-ordered, that is, a parametrization on each cross section is possible. However, it is a far cry from realistic situations, and hence the problem of reconstructing a curve from a set of scattered sample points on a planar curve is important to reconstruction methods of a surface. A number of related numerical results can be found in [7, 9, 10, 11, 17]. Most of these methods are based on the use of Delaunay triangulation, and give a reasonable curve under a given criterion such as local feature size.

At most recent, the author [18] developed an efficient method, so called a natural points ordering method, for solving the problem of reconstructing a curve from a set of sample points arbitrarily distributed in $\mathbb{R}^2$. It consists of defining the order of the sample points and piecewise edges one by one using only a local decision criterion, so called a natural distance, which is based on a property of the stochastic process (Brownian motion) which models diffusing motions of a small object from one point to the other point. The purpose of this paper is to utilize this algorithm to develop a heuristic method giving an acceptable surface from a set of sample points on a series of parallel planar cross-sections.

The rest of this paper is organized as follows. In the first part of section 2, we precisely describe the problem of interest. We then review the concept of the natural distance [18] and utilize it to reconstruct each cross-section. The second part of section 2 is devoted to solve the tiling problem between two consecutive cross sections. We develop a heuristic method which creates triangular patches sequentially by satisfying a local criterion, so called a smallest inner angle criterion, and then "stitching" all triangulations together.

In section 3, we execute three numerical simulations for the algorithm and shows the potentiality of the present algorithm. Finally, we finish the paper with some conclusions and comments

## 2. A HEURISTIC METHOD FOR SOLVING TILING PROBLEM

We begin with this section precisely setting up the problem of interest and then briefly recall the concept of the natural distance and the natural points ordering method to solve the described problem.

**2.1. Setting of the problem.** Suppose that there are arbitrarily scattered sample points $p_{ij} = (x_{ij}, y_{ij}, z_i)$ on a smooth surface $\Sigma$ in $\mathbb{R}^3$, where $i = 1, 2 \cdots, M$, $j = 1, 2, \cdots, N_i$. Denote the set of all sample points by $\mathcal{P}$ and all sample points with level $z = z_i$ as $\mathcal{P}_i$, $i = 1, 2, \cdots, M$, where we assume $z_1 < z_2 < \cdots < z_M$. We call each set $\mathcal{P}_i$ as a cross section or a contour. The problem to solve can then be summarized as the following questions. For given the set $\mathcal{P} = \bigcup_{i=1}^{M} \mathcal{P}_i$ of sample points,

- how can we reconstruct a piecewise interpolating curve from the set $\mathcal{P}_i$ of sample points with only knowledge of the coordinates $p_{ij} = (x_{ij}, y_{ij}, z_j)$?;

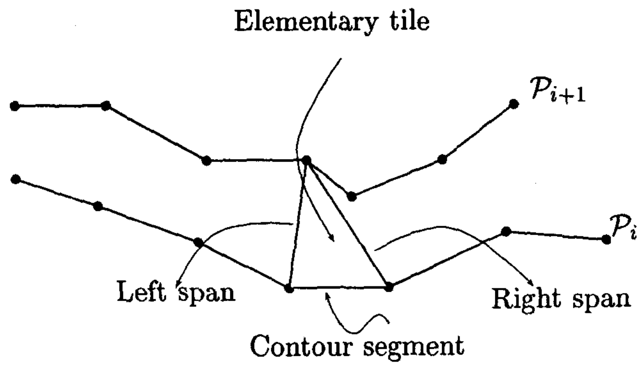



FIGURE 1. Contour segment and elementary tile

- how can we connect the $p_{ij}$'s and $p_{i+1k}$'s with straight lines in such a way as to form a triangular facet(or called an elementary tile) surface spanning two nested cross sections $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$? This last question is called a *tiling problem.*

More concisely, we settle the problem to solve following [13]. A *contour segment* is a linear approximation of the curve connecting consecutive points in a single cross section. An *elementary tile(or triangular patches)* is a triangular face composed of a single contour segment and two spans connecting the endpoints of a contour segment with a common point on the adjacent contour. The spans will be designated as "left" and "right" for obvious reasons (see Figure 1). Then the problem of reconstructing a 3D surface from a set of sample points on a series of parallel planar cross-sections is to find a set of elementary tiles which defines a surface satisfying two constraints:

(C1) Each contour segment will appear in exactly one elementary tile.
(C2) If a span appears as the left(right) span of some tile in the set, it will appear as the right (left) span exactly one other tile in the set.

A set of tiles which satisfies these two conditions is called an *acceptable surface.*

As described in the introduction, the main focus of this paper is to utilize the natural points ordering method reviewed in subsequent subsection and introduce a new heuristic method for solving the described problem above. Therefore, we consider only the simple case that the set $\mathcal{P}$ of sample points satisfies the following restrictions:

- Sample points on each set $\mathcal{P}_i$ belong to a simple smooth closed curve, which satisfies a local convexity described in Definition 2.1
- Each cross section $\mathcal{P}_i$ consists of only one contour. That is, there are no branching problem.
- For each index $i$, two consecutive cross section $P_i$ and $\mathcal{P}_{i+1}$ are so close and similar-shaped that the situations such as Figure 2 do not occur.

**2.2. The natural points ordering method.** For given two points $p$ and $q$ in $\mathbb{R}^2$ which are already ordered in the direction $\overrightarrow{pq}$, the *natural distance* is an answer to

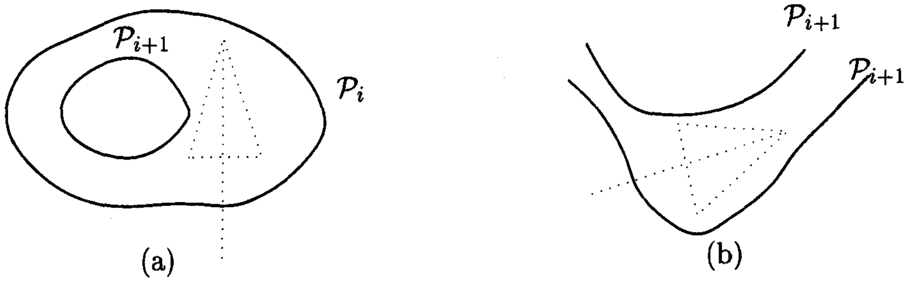



FIGURE 2. (a) Far away situation (b) Dissimilar situation

the question: for each $r$ which is lying on the opposite side of $p$, centered around the line perpendicular to $\overrightarrow{pq}$ containing $q$ (see Figure 3), how can we construct a distance between two points $q$ and $r$ which reflects the 'smoothness' of the piecewise linear arc $\widehat{pqr}$ and the 'closeness' between $q$ and $r$ simultaneously? In order to give a reasonable answer, we considered a small object moving from $q$ in the direction of $\overrightarrow{pq}$ with constant velocity 1 and simultaneously diffusing randomly on the line which is perpendicular to $\overrightarrow{pq}$. A trace of such object corresponds to a graph of sample path of Brownian motion starting from $q$ when we think of $\overrightarrow{pq}$ as a directional vector of time axis. The authors[18] then defined the *natural distance* $f(r)$ from $q$ to $r$ for directional vector $\overrightarrow{pq}$ as

$$f(r) = t(r) + K\frac{s(r)}{\sqrt{t(r)}}, \tag{1}$$

where

$$t(r) = |\overrightarrow{qr}|\cos\theta, \quad s(r) = |\overrightarrow{qr}|\sin\theta,$$

where $\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ is the signed angle between $\overrightarrow{pq}$ and $\overrightarrow{qr}$ defined by $\cos\theta = \frac{\overrightarrow{pq}\cdot\overrightarrow{qr}}{|\overrightarrow{pq}||\overrightarrow{qr}|}$ (see Figure 3).

Then the first factor $t(r)$ was construed as the diffusion time of the small object from $q$ to $r$ for directional vector $\overrightarrow{pq}$ and call it the *time distance*, while the second factor $\frac{s(r)}{\sqrt{t(r)}}$ as the transition density measuring how smoothly moves the small object from $q$ to $r$, and the probability that the small object moves from $q$ to $r$ with reaching time $t(r)$, and call it the *standardized probability distance*. For a detailed illustration, one refers to the paper [18]. We call $K$ the *subjective weight*, since it represents how much weight is given to the second factor. That is, the larger the subjective weight $K$ is, the more sensitive than the natural distance it is to the magnitude of the change of the probability distance $\frac{s(\cdot)}{\sqrt{t(\cdot)}}$ than that of the time distance $t(\cdot)$.

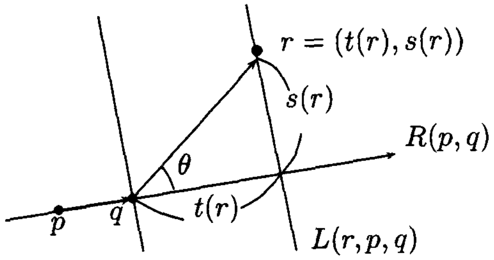



FIGURE 3. New coordinates for a scattered point $r$ with $\theta > 0$ based on two initial points $p$ and $q$

TABLE 1. An algorithm for the natural points ordering method for given sample points

Step 0. Assumption: A set of sample points $\mathcal{S}$ and the first and second ordered points $p_1$ and $p_2$ are already given.
Step 1. Let $i = 2$ and let $C_{i+1}$ be the set of candidate points of $p_{i+1}$ consisting of the points located on the opposite side of $p_{i-1}$ centering around the line perpendicular to $\overrightarrow{p_{i-1}p_i}$ containing $p_i$.
Step 2. Set $\widetilde{\mathcal{S}}$ to be an ordered set with $p_1$, $p_2$.
Step 3. Initialize the starting direction vector $\overrightarrow{p_{i-1}p_i}$ and calculate it and $|\overrightarrow{p_{i-1}p_i}|$.
Step 4. For each candidate point $r \in C_{i+1}$, calculate $\overrightarrow{qr}$, $t(r)$ and $s(r)$.
Step 5. Find the next point $p_{i+1}$ such that $p_{i+1} = \text{argmin}_{r \in C_{i+1}} f(r)$.
Step 6. Append the point $p_{i+1}$ to $\widetilde{\mathcal{S}}$.
Step 7. Set $i$ to be $i+1$ and find the set $C_{i+1}$ of candidate points.
Step 8. If $C_{i+1}$ is nonempty then go to Step 3, otherwise stop.

Using the natural distance described above, the authors [18] developed a method to solve the curve reconstruction problem in $\mathbb{R}^2$ so called the *natural points ordering method.* It provides an algorithm to us for consecutive myopic choices of sample points in a natural way. In fact, it is designed to choose the points where the natural distance from the starting point(chosen one step before) is minimized. The algorithm can be summarized as in Table 1.

In Table 1, for a unique set $\widetilde{S}$ of ordered points, it is sufficient to fulfill the assumption that in each steps, the new coordinate representations $(t(r), s(r))$ for each candidate points in $C_\bullet$ are different one another.

**2.3. Tiling method.** To describe the heuristic method we use, suppose we are given sample points distributed on two consecutive cross sections $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ as in section 1

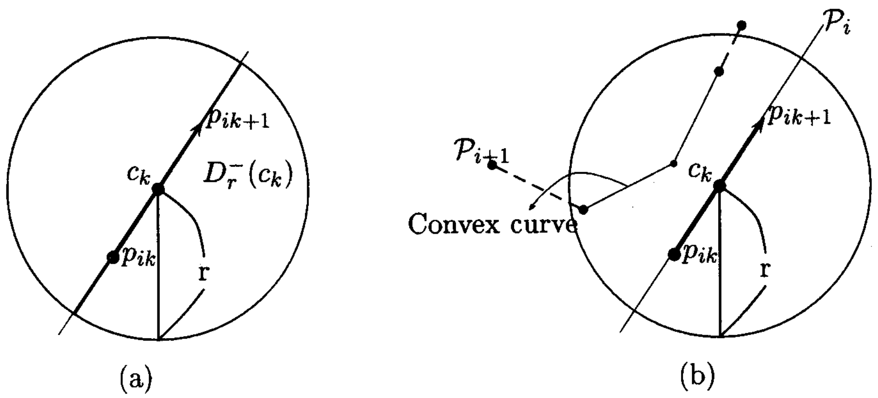



FIGURE 4. Local convexity

and these set are well-ordered by the algorithm given in Table 1. Since both cross-sections are closed, we may think of the points $p_{ij}$ as being extended periodically; i.e. $p_{iN_i+j} = p_{ij}$, $j = 1, 2 \cdots, N_1$. Hereafter, we assume that two consecutive cross sections $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ are on a same plane without any special mention.

**Definition 2.1.** *For given two consecutive cross sections $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$, we say that a discrete set $\mathcal{P}_{i+1}$ has a local convexity with respect to a discrete set $\mathcal{P}_i$ provided the following conditions are satisfied: For each contour segment $L_k$ connecting consecutive points $p_{ik}$ and $p_{ik+1}$ in $\mathcal{P}_i$ with direction $\overrightarrow{p_{ik}p_{ik+1}}$, let $c_k$ be the mid point of $p_{ik}$ and $p_{ik+1}$, and $D_r(c_k)$ the closed disk with center $c_k$ and radius $r$. Further we denote $D_r^+(c_k)$ is as the left hand side of $L_k$ and $D_r^-(c_k)$ is as the remaining portion of $D_r(c_k)$ (see Figure 4 (a)). Then there exists a positive number $\delta$ such that for any index $k$,*

(i) *either $D_\delta^+(c_k) \cap \mathcal{P}_{i+1} \neq \emptyset$ or $D_\delta^-(c_k) \cap \mathcal{P}_2 \neq \emptyset$; and*
(ii) *furthermore, the piecewise linear approximation of the nonempty set is convex (see Figure 4 (b)).*

*Here, we say the positive number $\delta$ as a local convexity radius.*

Hereafter, for each $i$, we assume that the cross-section $\mathcal{P}_{i+1}$ has a local convexity with respect to $\mathcal{P}_i$ with sufficiently small local convexity radius $\delta$.

For simplicity of descriptions, let $i = 1$ hereafter, and $k$ be a fixed index. For each $p_{2j} \in \mathcal{P}_2$, let $p'_{2j}$ be the end point of the projection vector of $\overrightarrow{p_{1k}p_{2j}}$ on the ray containing the contour segment $L_k$ (See Figure 5). Let $E_k$ be a subset of $\mathcal{P}_2$ defined by

$$E_k = \underset{p_{2j} \in D_\delta(c_k) \cap \mathcal{P}_2}{\operatorname{argmax}} \frac{\overrightarrow{r_k p_{2j}} \cdot \overrightarrow{p_{2j}p'_{2j}}}{|\overrightarrow{r_k p_{2j}}||\overrightarrow{p_{2j}p'_{2j}}|}, \quad k = 1, 2, \cdots, N_i.$$

Then the set $E_k$ has the following property;

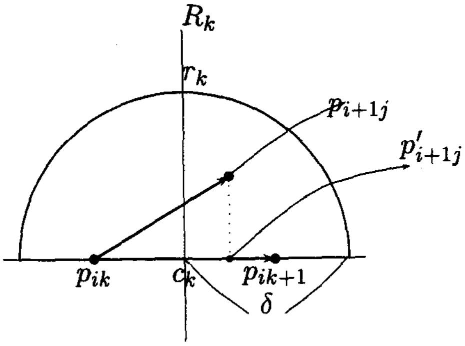



FIGURE 5. Projection of $p_{i+1j}$

**Theorem 2.2.** *Assume that $\delta$ is the smallest radius such that $D_\delta^+(c_k) \cap \mathcal{P}_2$ is nonempty and the linear approximation of its elements if exists is convex. Then the set $E_k$ is one of the following cases;*

(i) *$E_k$ is a set with single element;*
(ii) *if not the case (i), either the linear approximation of the set $E_k$ is a part of a straight line passing through the point $r_k$ or there are only two points, say $q_{j_1}$ and $q_{j_2}$, such that $q_{j_1}$ and $q_{j_2}$ are symmetric with respect to the line $R_k$.*

*Proof.* If the elements of $E_k$ are one or two, then the assertion holds clearly. So we assume that the number of element of $E_k$ is larger than three and the linear approximation of the set $E_i$ is not a part of a straight line. Then there exist three points $q_1, q_2, q_3 \in \mathcal{P}_2$ such that $q_1, q_2 \in E_k$ and $q_3 \in D_\delta^+ \cap \mathcal{P}_2 \setminus E_k$. If $|\overrightarrow{c_k q_3}| > |\overrightarrow{c_k q_j}|$, $j = 1, 2$, or $|\overrightarrow{c_k q_3}|$ is larger than one of $|\overrightarrow{c_k q_j}|$, $j = 1, 2$, then either there exist a positive number $\gamma$ such that $D_\gamma^+ \cap \mathcal{P}_2$ is nonempty and the linear approximation of its elements is convex, which is a contradiction to the assumption, or the linear approximation of $q_1, q_2$, and $q_3$ is either concave or a wedge, which is also a contradiction to the assumption. If $|\overrightarrow{c_k q_3}| < |\overrightarrow{c_k q_j}|$, $j = 1, 2$, then the angle between two vectors $\overrightarrow{r_k c_k}$ and $\overrightarrow{r_k q_3}$ is less than that between two vectors $\overrightarrow{r_k c_k}$ and $\overrightarrow{r_k q_1}$. So

$$\frac{\overrightarrow{r_k q_1} \cdot \overrightarrow{q_1 q_1'}}{|\overrightarrow{r_k q_1}||\overrightarrow{q_1 q_1'}|} < \frac{\overrightarrow{r_k q_3} \cdot \overrightarrow{q_3 q_3'}}{|\overrightarrow{r_k q_3}||\overrightarrow{q_3 q_3'}|},$$

which is a contradiction to the fact $q_1 \in E_k$. □

Using this theorem, we now introduce our heuristic method for solving the tiling problem. For the simplicity of the notations, we let $\mathcal{T}_{ij}$ be the $(i+j)$th triangular facet. Then if the triangle $\mathcal{T}_{ij}$ has two vertices in $\mathcal{P}_1$, say $p_{1l}$, $p_{1l+1}$, consecutively, and one vertex in $\mathcal{P}_2$, say $p_{2m}$, (it calls a triangle of Type I), then $i$ and $j$ mean $i = l - 1$ and $j = m$, while if $\mathcal{T}_{ij}$ has two vertices in $\mathcal{P}_2$, say $p_{2l}$, $p_{2l+1}$, consecutively, and one vertex in $\mathcal{P}_1$, say $p_{1m}$, (it calls a triangle of Type II), then $i = m - 1$ and $j = l$. Further, let $T_{kl}$ be the $l$th vertex of the triangle $\mathcal{T}_{ij}$ with $k = i + j$. As described in the following, we will directly construct the triangles of Type II from nested two triangles of Type I without any computation. We are now ready to introduce our algorithm.

*Initial step.* To construct first triangular facet $\mathcal{T}_{01}$, start with $T_{11} = p_{11}$ and $T_{12} = p_{12}$. In order to choose the third vertex of $\mathcal{T}_{01}$, consider the set $E_1$. If $E_1$ is a set with single element, we choose the third vertex $T_{13}$ of the first triangle as the point in $E_1$. If the number of element of $E_1$ is more than one, the above theorem shows that either the linear approximation of the set $E_1$ is a part of a straight line passing through the point $r_1$ or there are only two points, say $p_{2j_1}$ and $p_{2j_2}$, which are symmetric with respect to the line $R_1$.

If the first case occurs, we let $T_{13} = \underset{q \in E_1}{\text{argmin}} |\overrightarrow{qc_1}|$, while if the second case occurs, we let $T_{13} = \underset{q \in E_1}{\text{argmax}} \cos^{-1}\left(\frac{\overrightarrow{T_{11}q} \cdot \overrightarrow{T_{11}T_{12}}}{|\overrightarrow{T_{11}q}||\overrightarrow{T_{11}T_{12}}|}\right)$. Once the vertex $T_{13}$ is chosen, then we reorder the ordered set $\mathcal{P}_2$ starting with $T_{13}$.

*Middle steps.* Assume that $\mathcal{T}_{01}, \mathcal{T}_{11}, \cdots, \mathcal{T}_{1m_1}, \cdots, \mathcal{T}_{ln}, \cdots, \mathcal{T}_{ln_l}, \cdots, \mathcal{T}_{k-1j}$, $k \geq 2$, $j \geq 1$ have already been constructed, where $\mathcal{T}_{k-1j}$ is a triangle of Type I. Then three vertices of $\mathcal{T}_{k-1j}$ are $p_{1k}, p_{1k+1}, p_{2j}$. In order to find a next triangle facet, we let $A = p_{1k+1}$, $B = p_{1k+2}$ and consider the set $E_{k+1}$. To find an acceptable surface, we reset $E_{k+1}$ by eliminating the points $p_{2c}$ with $c < j$ if it exist. Now let $C = p_{2e}$ be the points in $E_{k+1}$ such that:

- If $E_{k+1}$ is a set with single element, we let $C$ be the point of $E_{k+1}$;
- If the number of element of $E_{k+1}$ is more than one, then either the linear approximation of the set $E_{k+1}$ is a part of a straight line passing through the point $r_{k+1}$ or there are only two points, which are symmetric with respect to the line $R_{k+1}$. If the first case occurs, we let $C$ be the point such that $C = \underset{q \in E_{k+1}}{\text{argmin}} |\overrightarrow{qc_{k+1}}|$, while if the second case occurs, we let $C$ be the point such that
$$C = \underset{q \in E_{k+1}}{\text{argmax}} \cos^{-1}\left(\frac{\overrightarrow{Aq} \cdot \overrightarrow{AB}}{|\overrightarrow{Aq}||\overrightarrow{AB}|}\right).$$

If $e = j$, we construct the next triangle as $\mathcal{T}_{ke}$ with vertices $T_{k+e1} = A$, $T_{k+e2} = B$ and $T_{k+e3} = C$. When $e > j$, first construct triangles of Type II, $\mathcal{T}_{kj}, \mathcal{T}_{kj+1}, \cdots, \mathcal{T}_{ke-2}$, $\mathcal{T}_{ke-1}$ and then construct a triangle of Type I, $\mathcal{T}_{ke}$. Repeat these procedures until the point $C = p_{2N_2+1} = p_{21}$ is chosen. We let the last chosen triangle is $\mathcal{T}_{fN_2+1}$. (See Figure 6 (a)).

*Final step.* In the last of the middle steps, if the index $f + 1$ equals to $N_1$, stop the algorithm. Otherwise, construct triangles of Type I, $\mathcal{T}_{f+1N_2+1}, \cdots, \mathcal{T}_{N_1N_2+1}$. (See Figure 6 (b)).

This algorithm can be summarized as in Table 2.

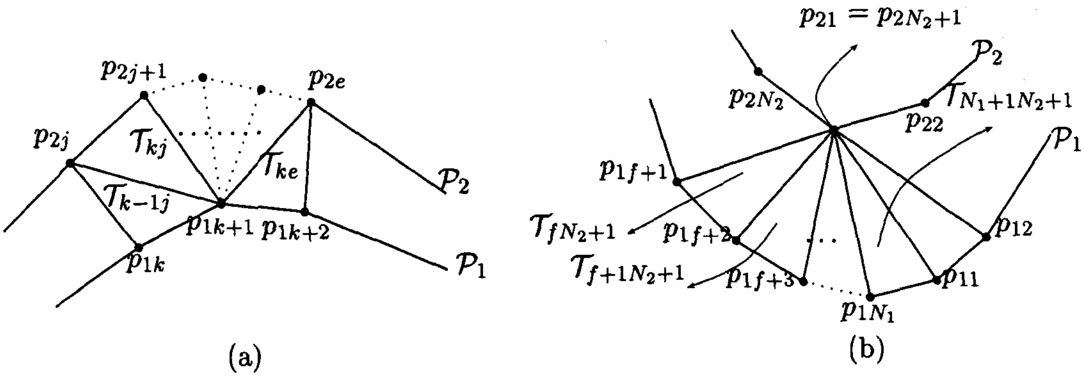



FIGURE 6. (a) Triangles of Type II between nested two triangle of Type I. (b) Triangles of Type I in the final step.

**Remark 2.3.** *For the fixed $k$, if we let*

$$A(p_{2j}) = \cos^{-1}\left(\frac{\overrightarrow{r_kc_k}\cdot\overrightarrow{r_kp_{2j}}}{|\overrightarrow{r_kc_k}||\overrightarrow{r_kp_{2j}}|}\right), \quad p_{2j}\in V_\theta(r_k)\cap\mathcal{P}_{2\cdot},$$

*then the function $A(p_{2j})$ is the angle between two vectors $\overrightarrow{r_kc_k}$ and $\overrightarrow{r_kp_{2j}}$. Further, we can see that*

$$E_k = \underset{p_{2\cdot}\in V_\theta(r_k)\cap\mathcal{P}_2}{\operatorname{argmin}} |A(p_{2\cdot})|.$$

*In this sense, we call the above algorithm as a smallest inner angle criterion.*

Note that the calculation of the set $E_k$ is based on the problem of finding the location of the point $r_k$. We thus close this section after some discussion for it.

**Theorem 2.4.** *For each $k$, if we let $r_k$ be the intersection point with the boundary of $D_\delta^+(c_k)$ and the ray $R_k$ such as in see Figure 5, then it can be expressed as*

$$r_k = c_k - \delta\frac{\overrightarrow{d_j}}{|\overrightarrow{d_j}|}, \quad \overrightarrow{d_j} = \frac{\overrightarrow{p_{1k}p_{2j}}\cdot\overrightarrow{p_{1k}p_{1k+1}}}{|\overrightarrow{p_{1k}p_{1k+1}}|^2}\overrightarrow{p_{1k}p_{1k+1}} + \overrightarrow{p_{2j}p_{1k}},$$

*where $p_{2j}$ is any point in the set $D_\delta^+(c_k)\cap\mathcal{P}_2$.*

*Proof.* Let $p_{2j}$ be a point in $D_\delta^+(c_k)\cap\mathcal{P}_2$ and $p'_{2j}$ be its projection onto the line containing the vector $\overrightarrow{p_{1k}p_{1k+1}}$. Then the vector $\overrightarrow{p_{1k}p'_{2j}}$ can be expressed as

$$\overrightarrow{p_{1k}p'_{2j}} = \frac{\overrightarrow{p_{1k}p_{2j}}\cdot\overrightarrow{p_{1k}p_{1k+1}}}{|\overrightarrow{p_{1k}p_{1k+1}}|^2}\overrightarrow{p_{1k}p_{1k+1}}$$

TABLE 2. An algorithm for the tiling problem with given two well-ordered consecutive cross-sections

| |
|---|
| Step 0. Assumption: A set of sample points $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where the points in $\mathcal{P}_i$ are well-ordered as $p_{i1}, \cdots, p_{iN_i}$, $i = 1, 2$. Let $\delta$ and $\theta$ be given real numbers. |
| Step 1. With the starting points $p_{11}$ and $p_{12}$, calculate $r_1$, $c_1$ and $E_1$ |
| Step 2. Find the third vertex $T_{13}$ in $E_1$ for the first triangle $\mathcal{T}_{01}$ |
| Step 3. Reorder the set $\mathcal{P}_2$ starting at $T_{13}$ and let $j = 1$, $k = 1$. |
| Step 4. Calculate $r_{k+1}$, $c_{k+1}$ and $E_{k+1}$. |
| Step 5. Reset $E_{k+1}$ by eliminating the points $p_{2c} \in E_{k+1}$ such that $c < j$ if exists. |
| Step 6. Choose a point $p_{2e} \in E_{k+1}$ such that: if $E_{k+1}$ is a set with single element, we let $p_{2e}$ be the point of $E_{k+1}$; if the piecewise linear approximation of the set $E_{k+1}$ is a part of a straight line passing through the point $r_{k+1}$, $p_{2e} = \underset{q \in E_{k+1}}{\operatorname{argmin}} \lvert \overrightarrow{qc_{k+1}} \rvert$; otherwise $p_{2e} = \underset{q \in E_{k+1}}{\operatorname{argmax}} \cos^{-1}\left( \frac{\overrightarrow{p_{1k+1}q} \cdot \overrightarrow{p_{1k+1}p_{1k+2}}}{\lvert \overrightarrow{p_{1k+1}q} \rvert \lvert \overrightarrow{p_{1k+1}p_{1k+2}} \rvert} \right)$. |
| Step 7. If $j = e$, construct a triangle $\mathcal{T}_{kj}$ of Type I, while if $e > j$, then first construct triangles $\mathcal{T}_{kj}, \mathcal{T}_{kj+1}, \cdots \mathcal{T}_{ke-1}$ of Type II using the points $p_{2j}, p_{2j+1}, \cdots, p_{2e}, p_{1k+1}$ and then the triangle $\mathcal{T}_{ke}$ of Type I with vertices $p_{1k+1}, p_{1k+2}, p_{2e}$. |
| Step 8. Put $j = e$ and $k = k + 1$. |
| Step 9. If $e < N_2 + 1$, then go to Step 4, otherwise go to next step. |
| Step 10. If $k + 1 < N_1 + 1$, construct triangles $\mathcal{T}_{ke}, \mathcal{T}_{k+1e}, \cdots, \mathcal{T}_{N_1e}$ using the points $p_{1k+1}, p_{1k+2}, \cdots, p_{1N_1+1}, p_{2j}$ and then stop. Otherwise, stop. |

and hence the point $p'_{2j}$ is given by

$$p'_{2j} = \frac{\overrightarrow{p_{1k}p_{2j}} \cdot \overrightarrow{p_{1k}p_{1k+1}}}{\lvert \overrightarrow{p_{1k}p_{1k+1}} \rvert^2} \overrightarrow{p_{1k}p_{1k+1}} + p_{1k}.$$

Hence if we let $\overrightarrow{d_j} = \overrightarrow{p_{2j}p'_{2j}}$, we get

$$\begin{aligned} \overrightarrow{d_j} = p'_{2j} - p_{2j} &= \frac{\overrightarrow{p_{1k}p_{2j}} \cdot \overrightarrow{p_{1k}p_{1k+1}}}{\lvert \overrightarrow{p_{1k}p_{1k+1}} \rvert^2} \overrightarrow{p_{1k}p_{1k+1}} + p_{1k} - p_{2j} \\ &= \frac{\overrightarrow{p_{1k}p_{2j}} \cdot \overrightarrow{p_{1k}p_{1k+1}}}{\lvert \overrightarrow{p_{1k}p_{1k+1}} \rvert^2} \overrightarrow{p_{1k}p_{1k+1}} + \overrightarrow{p_{2j}p_{1k}}. \end{aligned}$$

Since $\overrightarrow{r_kc_k} = \delta \overrightarrow{d_j} / \lvert \overrightarrow{d_j} \rvert$, we can complete the assertion. □

## 3. Simulation Results

As numerical simulations for the proposed heuristic method giving an acceptable surface, in this section, we consider three examples. First, to illustrate our algorithm,

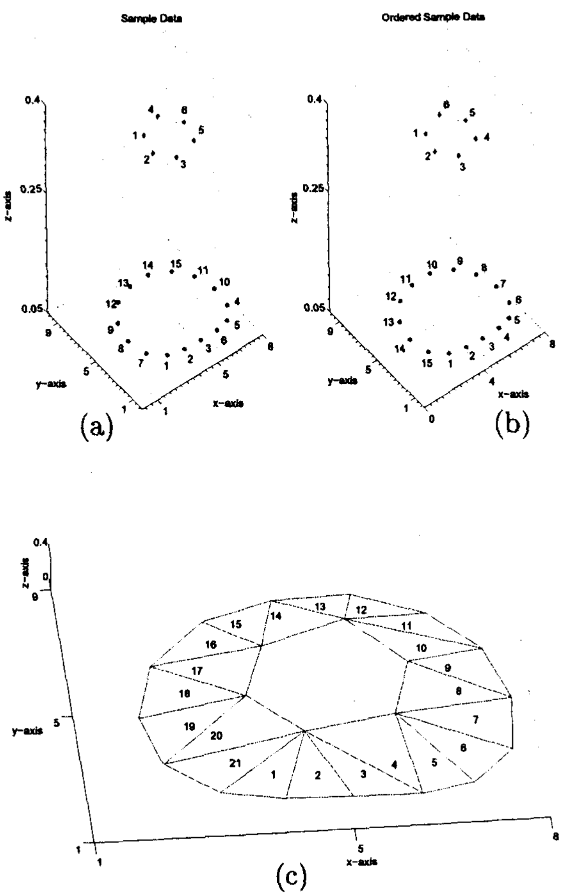
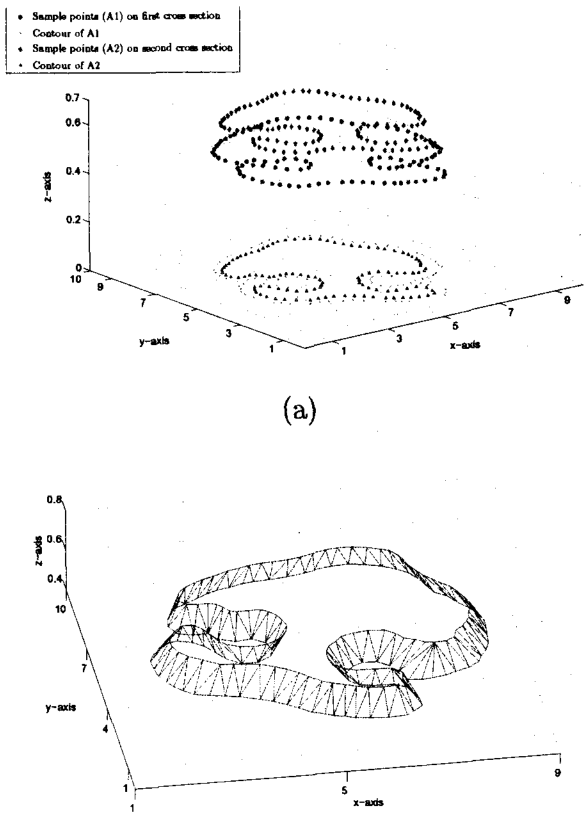



FIGURE 7. (a) A set of sample points scattered in two convex cross-sections; (b) A set of ordered sample points (c) Reconstructed surface from given sample points on two cross-sections

we treat a set of sample points on two simple cross-sections, which are perfectly convex curve in $\mathbb{R}^3$. The second example is a set of sample points on two cross-sections which are not convex, but satisfy the local convexity. Finally, we consider a set of sample points arbitrary distributed on a series of parallel cross sections in the unit half-sphere.

*Example 1.* As shown in Figure 7 (a), the considered set of sample points are scattered on two convex curves. Figure 7 (b) displays the ordered set of sample points on each cross section obtained by the natural points ordering method. Figure 7 (c) shows that the presented algorithm gives an acceptable surface. The numbers in each triangle denote the order of the triangle facets constructed from the algorithm.




(a)

(b)

FIGURE 8. (a) Sample points on two cross-sections with elbows (b) Reconstructed surface

*Example 2.* The sample points to be considered in this example are on two non-convex cross sections as shown in Figure 8 (a). As shown in the contour graph of the cross sections, these two cross sections have similar shape with elbows and hence satisfy the local convexity condition. The proposed algorithm well behavior in this case also and gives an acceptable surface as shown in Figure 8 (b).

*Example 3.* We consider 20 parallel planner cross sections in the upper hemisphere with radius 1, where we assume that the cross sections are randomly distributed with starting cross section in the plane $z = 0$. If we let the level of each cross section $\mathcal{P}_i$ as $z_i$, the sample points in $\mathcal{P}_i$ are distributed on the level curve $x^2 + y^2 = \sqrt{1 - z_i^2}$ with step size $\pi/(15(1 - z_i)^2 + 10)$ provided $z_i \neq 1$. If $z_i = 1$, the set $\mathcal{P}_i$ is a single set of point $(0, 0, 1)$. Of course, we assume that the sample points in each cross section $\mathcal{P}_i$ are randomly distributed. In Figure 9 (a), we show the set of sample points with its projection onto the plane $z = -1$. In particular, the graph in the projection of the

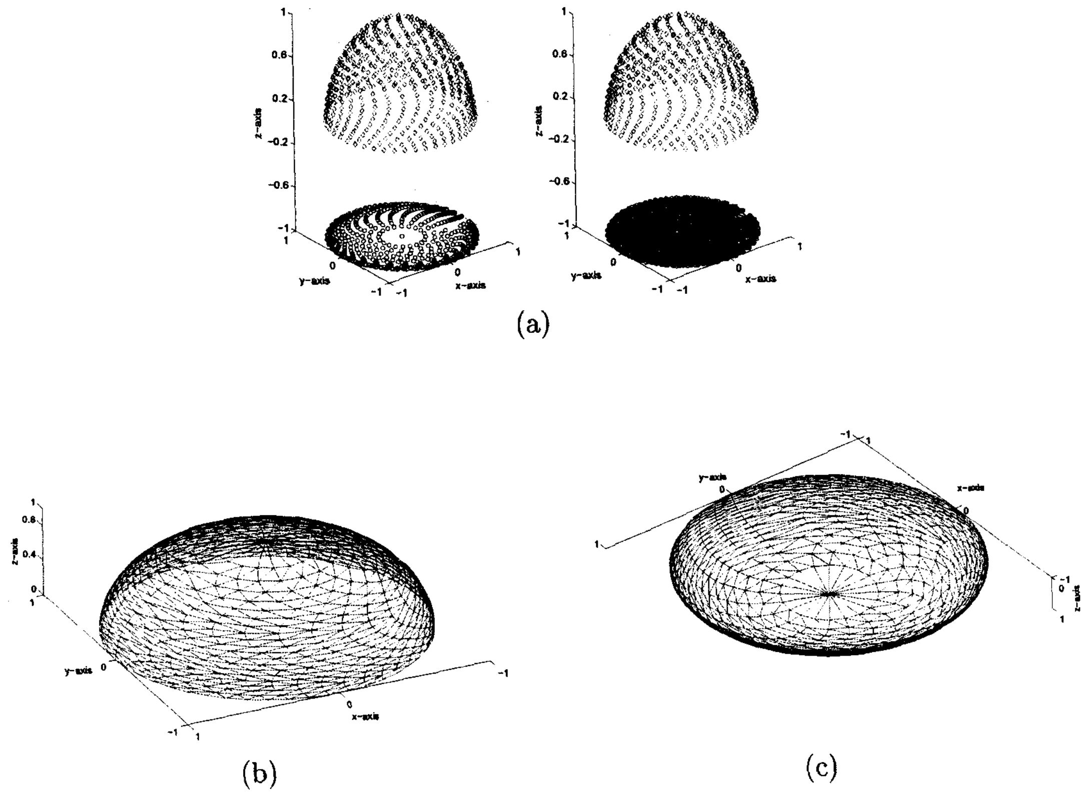


FIGURE 9. Sample points on the hemisphere (a), Reconstructed surface (b) and its view in the bottom of the hemisphere (c)

second figure of Figure 9 (a) is constructed by connecting the sample points with given order, and it shows that the sample points are randomly distributed. The proposed algorithm well behavior in this example also and gives an almost perfect hemisphere surface as shown in Figure 9 (b). Figure 9 (c) is the graph viewed in the bottom direction of the hemisphere.

## 4. Concluding Remark

We develop a heuristic method for reconstructing 3D surface from sample points arbitrary scattered on a series of parallel planner cross sections, where we assume that consecutive cross sections are very similar. The method is established by using the natural points ordering method developed in [18] and a smallest inner angle criterion.

This provides us a way to reconstruct and the order of the triangle facets simultaneously. By stitching all triangle facets together, we can obtain desired or acceptable surfaces. Numerical simulations say that it can be very efficient to reconstruct an acceptable surface from unorganized sample points distributed on a series of parallel planner cross sections.

Although our focus is on reconstructing acceptable surfaces from sample points distributed on parallel cross sections which are very similar, the method may be extendable to other situations, for instance, parallel cross sections with dissimilar portion, parallel cross sections with branching portion and so on. We conjecture that one efficient way to solve these problems is combining the proposed method and a partial differential equation method technique. That is, one adopts the present algorithm for the similar portion of the cross sections, while for the dissimilar portion or the branching portion, one first models a suitable partial differential equation describing these areas and then solve it.

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Philsu Kim:
Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
*E-mail*: kimps@mail.knu.ac.kr