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NEW TAYLOR-LIKE EXPANSIONS FOR FUNCTIONS OF TWO VARIABLES AND ESTIMATES OF THEIR REMAINDERS

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ABSTRACT. In this article, a generalisation of Sard's inequality for Appell polynomials is obtained. Estimates for the remainder are also provided.

1. INTRODUCTION

Let $x \in [a,b]$ and $y \in [c,d]$. If $f(x,y)$ is a function of two variables we shall adopt the following notation for partial derivatives of $f(x,y)$:

$$
\begin{aligned}
f^{(i,j)}(x,y) &\triangleq \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, \\
f^{(0,0)}(x,y) &\triangleq f(x,y), \\
f^{(i,j)}(\alpha,\beta) &\triangleq f^{(i,j)}(x,y)|_{(x,y)=(\alpha,\beta)}
\end{aligned}
\tag{1}
$$

for $0 \le i,j \in \mathbb{N}$ and $(\alpha,\beta) \in [a,b] \times [c,d]$.

A. H. Stroud has pointed out in [6] that one of the most important tools in the numerical integration of double integrals is the following Taylor's formula [6, p. 138 and p. 157] due to A. Sard [5]:

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Theorem A. *If $f(x,y)$ satisfies the condition that all the derivatives $f^{(i,j)}(x,y)$ for $i+j \le m$ are defined and continous on $[a,b]\times[c,d]$, then $f(x,y)$ has the expansion*

$$
\begin{aligned}
f(x,y) = & \sum_{i+j\le m} \frac{(x-a)^i}{i!}\frac{(y-c)^j}{j!} f^{(i,j)}(a,c) \\
& + \sum_{j<q} \frac{(y-c)^j}{j!} \int_a^x \frac{(x-u)^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(u,c)\,du \\
& + \sum_{i<p} \frac{(x-a)^i}{i!} \int_c^y \frac{(y-v)^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a,v)\,dv \\
& + \int_a^x \int_c^y \frac{(x-u)^{p-1}}{(p-1)!}\frac{(y-v)^{q-1}}{(q-1)!} f^{(p,q)}(u,v)\,dv\,du,
\end{aligned} \tag{2}
$$

where i,j are nonnegative integers; p,q are positive integers; and $m \triangleq p+q \ge 2$.

Essentially, the representation (2) is used for obtaining the fundamental Kernel Theorems and Error Estimates in numerical integration of double integrals [6, p. 142, p. 145 and p. 158] and has both theoretical and practical importance in the domain as a whole.

Definition 1. A sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ is called *harmonic* [4] if it satisfies the recursive formula

$$P_i'(x) = P_{i-1}(x) \tag{3}$$

for $i \in \mathbb{N}$ and $P_0(x) = 1$.

A slightly different concept that specifies the connection between the variables is the following one.

Definition 2. We say that a sequence of polynomials $\{P_i(t,x)\}_{i=0}^{\infty}$ satisfies the *Appell condition* [2] if

$$\frac{\partial P_i(t,x)}{\partial t} = P_{i-1}(t,x) \tag{4}$$

and $P_0(t,x) = 1$ for all defined (t,x) and $n \in \mathbb{N}$.

It is wellknown that the Bernoulli polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x| < 2\pi, \quad t \in \mathbb{R}. \tag{5}$$

It can be shown that the polynomials $B_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the two formulae

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t) = 1; \tag{6}$$

$$\text{and} \quad B_i(t+1) - B_i(t) = it^{i-1}. \tag{7}$$

The Euler polynomials can be defined by the expansion

$$\frac{2e^{tx}}{e^x+1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x| < \pi, \quad t \in \mathbb{R}. \tag{8}$$

It can also be shown that the polynomials $E_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the two properties

$$E_i'(t) = iE_{i-1}(t), \quad E_0(t) = 1; \tag{9}$$

$$\text{and} \quad E_i(t+1) + E_i(t) = 2t^i. \tag{10}$$

For further details about Bernoulli polynomials and Euler polynomials, please refer to [1, 23.1.5 and 23.1.6].

There are many examples of Appell polynomials. For instance, for i a nonegative integer, $\theta \in \mathbb{R}$ and $\lambda \in [0,1]$,

$$P_{i,\lambda}(t) \triangleq P_{i,\lambda}(t;x;\theta) = \frac{[t-(\lambda\theta+(1-\lambda)x)]^i}{i!}, \tag{11}$$

$$P_{i,B}(t) \triangleq P_{i,B}(t;x;\theta) = \frac{(x-\theta)^i}{i!} B_i\left(\frac{t-\theta}{x-\theta}\right) \text{ ([4])}, \tag{12}$$

$$P_{i,E}(t) \triangleq P_{i,E}(t;x;\theta) = \frac{(x-\theta)^i}{i!} E_i\left(\frac{t-\theta}{x-\theta}\right) \text{ ([4])}. \tag{13}$$

In [4], the following generalized Taylor's formula was established.

Theorem B. *Let $\{P_i(x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that $f^{(n)}(x)$ is absolutely continuous for some $n \in \mathbb{N}$, then, for any $x \in I$, we have*

$$f(x) = f(a) + \sum_{k=1}^{m} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right] + R_n(f;a,x), \tag{14}$$

where

$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt. \tag{15}$$

The fundamental aim of this article is to obtain a generalisation of the Taylor-like formula (2) for Appell polynomials and to study its impact on the numerical integration of double integrals.

2. Two New Taylor-like Expansions

Following a similar argument to the proof of Theorem 2 in [4], we obtain the following result.

Theorem 1. *If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$, then we have the generalised integration by parts formula for $x \in [a,b]$*

$$\int_a^b g(t)\,dt = \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(b,x)g^{(k-1)}(b) - P_k(a,x)g^{(k-1)}(a)\right] + (-1)^n \int_a^b P_n(t,x)g^{(n)}(t)\,dt. \tag{16}$$

Proof. By integration by parts we obtain, on using the Appell condition (4),

$$\begin{aligned} &(-1)^n \int_a^b P_n(t,x)g^{(n)}(t)\,dt \\ &= (-1)^n P_n(t,x)g^{(n-1)}(t)\Big|_a^b + (-1)^{n-1}\int_a^b P_{n-1}(t,x)g^{(n-1)}(t)\,dt \qquad (17)\\ &= (-1)^n\left[P_n(b,x)g^{(n-1)}(b) - P_n(a,x)g^{(n-1)}(a) - \int_a^b P_{n-1}(t,x)g^{(n-1)}(t)\,dt\right]. \end{aligned}$$

Clearly, the same procedure can be used for the term $\int_a^b P_{n-1}(t,x)g^{(n-1)}(t)\,dt$. Therefore, formula (16) follows from successive integration by parts. ∎

Theorem 2. *Let D be a domain in $\mathbb{R}^2$ and the point $(a,c) \in D$. Also, let $\{P_i(t,x)\}_{i=0}^{\infty}$ and $\{Q_j(s,y)\}_{j=0}^{\infty}$ be two Appell polynomials. If $f : D \to \mathbb{R}$ is such that $f^{(i,j)}(x,y)$ are continuous on D for all $0 \le i \le m$ and $0 \le j \le n$, then*

$$f(x,y) = f(a,c) + C(f,P_m,Q_n) + D(f,P_m,Q_n) + S(f,P_m,Q_n) + T(f,P_m,Q_n), \tag{18}$$

where

$$\begin{aligned} C(f,P_m,Q_n) &= \sum_{i=1}^{m}(-1)^{i+1}\left[P_i(x,x)f^{(i,0)}(x,c) - P_i(a,x)f^{(i,0)}(a,c)\right] \\ &\quad + \sum_{j=1}^{n}(-1)^{j+1}\left[Q_j(y,y)f^{(0,j)}(a,y) - Q_j(c,y)f^{(0,j)}(a,c)\right], \end{aligned} \tag{19}$$

$$\begin{aligned} &D(f,P_m,Q_n) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(x,x)\left[Q_j(y,y)f^{(i,j)}(x,y) - Q_j(c,y)f^{(i,j)}(x,c)\right] \\ &\quad - \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(a,x)\left[Q_j(y,y)f^{(i,j)}(a,y) - Q_j(c,y)f^{(i,j)}(a,c)\right], \end{aligned} \tag{20}$$

(21) $S(f,P_m,Q_n)$

$$= (-1)^m \int_a^x P_m(t,x) f^{(m+1,0)}(t,c)\,dt + (-1)^n \int_c^y Q_n(s,y) f^{(0,n+1)}(a,s)\,ds$$
$$+ \sum_{i=1}^m (-1)^{n+i+1} \int_c^y Q_n(s,y)\big[P_i(x,x) f^{(i,n+1)}(x,s) - P_i(a,x) f^{(i,n+1)}(a,s)\big]\,ds$$
$$+ \sum_{j=1}^n (-1)^{m+j+1} \int_a^x P_m(t,x)\big[Q_j(y,y) f^{(m+1,j)}(t,y) - Q_j(c,y) f^{(m+1,j)}(t,c)\big]\,dt$$

and

$$T(f,P_m,Q_n) = (-1)^{m+n} \int_a^x \int_c^y P_m(t,x) Q_n(s,y) f^{(m+1,n+1)}(t,s)\,ds\,dt. \tag{22}$$

Proof. Let $P_m(t,x)$ be an Appell polynomial. Applying formula (14) to the function $f(x,y)$ with respect to variable x yields

$$\begin{aligned} f(x,y) = f(a,y) &+ \sum_{i=1}^m (-1)^{i+1}\big[P_i(x,x) f^{(i,0)}(x,y) - P_i(a,x) f^{(i,0)}(a,y)\big] \\ &+ (-1)^m \int_a^x P_m(t,x) f^{(m+1,0)}(t,y)\,dt. \end{aligned} \tag{23}$$

Similarly, for the functions $f^{(i,0)}(x,y)$, $f^{(i,0)}(a,y)$, $f^{(m+1,0)}(t,y)$ and $f(a,y)$, we have

$$\begin{aligned} f^{(i,0)}(x,y) = f^{(i,0)}(x,c) &+ (-1)^n \int_c^y Q_n(s,y) f^{(i,n+1)}(x,s)\,ds \\ &+ \sum_{j=1}^n (-1)^{j+1}\big[Q_j(y,y) f^{(i,j)}(x,y) - Q_j(c,y) f^{(i,j)}(x,c)\big], \end{aligned} \tag{24}$$

$$\begin{aligned} f^{(i,0)}(a,y) = f^{(i,0)}(a,c) &+ (-1)^n \int_c^y Q_n(s,y) f^{(i,n+1)}(a,s)\,ds \\ &+ \sum_{j=1}^n (-1)^{j+1}\big[Q_j(y,y) f^{(i,j)}(a,y) - Q_j(c,y) f^{(i,j)}(a,c)\big], \end{aligned} \tag{25}$$

$$\begin{aligned} f^{(m+1,0)}(t,y) = f^{(m+1,0)}(t,c) &+ (-1)^n \int_c^y Q_n(s,y) f^{(m+1,n+1)}(a,s)\,ds \\ &+ \sum_{j=1}^n (-1)^{j+1}\big[Q_j(y,y) f^{(m+1,j)}(t,y) - Q_j(c,y) f^{(m+1,j)}(t,c)\big], \end{aligned} \tag{26}$$

$$\begin{aligned} f(a,y) = f(a,c) &+ (-1)^n \int_c^y Q_n(s,y) f^{(0,n+1)}(a,s)\,ds \\ &+ \sum_{j=1}^n (-1)^{j+1}\big[Q_j(y,y) f^{(0,j)}(a,y) - Q_j(c,y) f^{(0,j)}(a,c)\big]. \end{aligned} \tag{27}$$

Substituting formulae (24)–(27) into (23) produces

$$\begin{aligned}
f(x,y) &= f(a,c) + \sum_{i=1}^{m}(-1)^{i+1}\big[P_i(x,x)f^{(i,0)}(x,c) - P_i(a,x)f^{(i,0)}(a,c)\big] \\
&+ \sum_{j=1}^{n}(-1)^{j+1}\big[Q_j(y,y)f^{(0,j)}(a,y) - Q_j(c,y)f^{(0,j)}(a,c)\big] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(x,x)\big[Q_j(y,y)f^{(i,j)}(x,y) - Q_j(c,y)f^{(i,j)}(x,c)\big] \\
&- \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(a,x)\big[Q_j(y,y)f^{(i,j)}(a,y) - Q_j(c,y)f^{(i,j)}(a,c)\big] \\
&+ (-1)^m\int_a^x P_m(t,x)f^{(m+1,0)}(t,c)\,dt + (-1)^n\int_c^y Q_n(s,y)f^{(0,n+1)}(a,s)\,ds \qquad (28)\\
&+ \sum_{i=1}^{m}(-1)^{n+i+1}\int_c^y Q_n(s,y)\big[P_i(x,x)f^{(i,n+1)}(x,s) - P_i(a,x)f^{(i,n+1)}(a,s)\big]\,ds \\
&+ \sum_{j=1}^{n}(-1)^{m+j+1}\int_a^x P_m(t,x)\big[Q_j(y,y)f^{(m+1,j)}(t,y) - Q_j(c,y)f^{(m+1,j)}(t,c)\big]\,dt \\
&+ (-1)^{m+n}\int_a^x\int_c^y P_m(t,x)Q_n(s,y)f^{(m+1,n+1)}(t,s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

The proof of Theorem 2 is complete. ∎

REMARK 1. If we take

$$P_i(t,x) = P_{m,\lambda}(t,x;a), \quad Q_j(s,y) = Q_{j,\mu}(s,y;c) \qquad (29)$$

for $0 \le i \le m$, $0 \le j \le n$ and $\lambda,\mu \in [0,1]$ in Theorem 2, then the expressions simplify to give, on using (11),

$$\begin{aligned}
C(f,P_m,Q_n) &= \sum_{i=1}^{m}\frac{(x-a)^i}{i!}\big[(1-\lambda)^i f^{(i,0)}(a,c) + \lambda^i f^{(i,0)}(x,c)\big] \\
&\quad + \sum_{j=1}^{n}\frac{(y-c)^j}{j!}\big[(1-\mu)^j f^{(0,j)}(a,c) + \mu^j f^{(0,j)}(a,y)\big],
\end{aligned} \qquad (30)$$

$$\begin{aligned}
D(f,P_m,Q_n) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\lambda^i(x-a)^i(y-c)^j}{i!\cdot j!}\big[\mu^j f^{(i,j)}(x,y) + (1-\mu)^j f^{(i,j)}(x,c)\big] \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(1-\lambda)^i(x-a)^i(y-c)^j}{i!\cdot j!}\big[\mu^j f^{(i,j)}(a,y) + (1-\mu)^j f^{(i,j)}(a,c)\big],
\end{aligned} \qquad (31)$$

$$
\begin{aligned}
S(f, P_m, Q_n) &= (-1)^m \int_a^x \frac{[t-(\lambda a+(1-\lambda)x)]^m}{m!} f^{(m+1,0)}(t,c)\,\mathrm{d}t \\
&\quad + (-1)^n \int_c^y \frac{[s-(\mu c+(1-\mu)y)]^n}{n!} f^{(0,n+1)}(a,s)\,\mathrm{d}s
\end{aligned} \tag{32}
$$

$$
+ \sum_{i=1}^{m} \int_c^y \frac{[\mu c+(1-\mu)y-s]^n (x-a)^i}{n!\cdot i!} \left[(\lambda-1)^i f^{(i,n+1)}(a,s) - \lambda^i f^{(i,n+1)}(x,s)\right] ds
$$

$$
+ \sum_{j=1}^{n} \int_a^x \frac{[\lambda a+(1-\lambda)x-t]^m (y-c)^j}{m!\cdot j!} \left[(\mu-1)^j f^{(m+1,j)}(t,c) - \mu^j f^{(m+1,j)}(t,y)\right] dt,
$$

and

$$
T(f, P_m, Q_n) = \int_a^x \int_c^y \frac{[(\lambda a+(1-\lambda)x)-t]^m[(\mu c+(1-\mu)y)-s]^n}{m!\cdot n!} f^{(m+1,n+1)}(t,s)\,ds\,dt. \tag{33}
$$

Notice that, taking $\lambda = 0$ and $\mu = 0$ in (29), then we can deduce Theorem A from Theorem 2.

Other choices of Appell type polynomials will provide generalizations of Theorem A.

The following approximation of double integrals in terms of Appell polynomials holds.

Theorem 3. *Let $\{P_i(t,x)\}_{i=0}^{\infty}$ and $\{Q_j(s,y)\}_{j=0}^{\infty}$ be two Appell polynomials and $f : [a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ such that $f^{(i,j)}(x,y)$ are continuous on $[a,b]\times[c,d]$ for all $0 \le i \le m$ and $0 \le j \le n$. We then have*

$$
\int_a^b \int_c^d f(t,s)\,ds\,dt = A(f,P_m,Q_n) + B(f,P_m,Q_n) + R(f,P_m,Q_n), \tag{34}
$$

where

$$
\begin{aligned}
&A(f, P_m, Q_n) = \\
&\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j} P_i(a,b)\left[Q_j(d,d) f^{(i-1,j-1)}(a,d) - Q_j(c,d) f^{(i-1,j-1)}(a,c)\right] \\
&- \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j} P_i(b,b)\left[Q_j(d,d) f^{(i-1,j-1)}(b,d) - Q_j(c,d) f^{(i-1,j-1)}(b,c)\right],
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
B(f,P_m,Q_n) &= \sum_{j=1}^{n}(-1)^j Q_j(c,d)\int_a^b f^{(0,j-1)}(t,c)\,dt \\
&\quad - \sum_{j=1}^{n}(-1)^j Q_j(d,d)\int_a^b f^{(0,j-1)}(t,d)\,dt \\
&\quad + \sum_{i=1}^{m}(-1)^i P_i(a,b)\int_c^d f^{(i-1,0)}(a,s)\,ds \\
&\quad - \sum_{i=1}^{m}(-1)^i P_i(b,b)\int_c^d f^{(i-1,0)}(b,s)\,ds
\end{aligned} \tag{36}
$$

and

$$
R(f,P_m,Q_n) = (-1)^{m+n}\int_a^b\int_c^d P_m(t,b)Q_n(s,d)f^{(m,n)}(t,s)\,ds\,dt. \tag{37}
$$

Proof. Using the generalized integration by parts formula consecutively yields

$$
\begin{aligned}
&\int_a^b\int_c^d P_m(t,b)Q_n(s,d)f^{(m,n)}(t,s)\,ds\,dt \\
&= \int_a^b P_m(t,b)\left[\int_c^d Q_n(s,d)f^{(m,n)}(t,s)\,ds\right]dt \\
&= (-1)^m\int_a^b P_m(t,b)\left\{\int_c^d f^{(m,0)}(t,s)\,ds\right. \\
&\quad \left. + \sum_{j=1}^{n}(-1)^j\left[Q_j(d,d)f^{(m,j-1)}(t,d) - Q_j(c,d)f^{(m,j-1)}(t,c)\right]\right\}dt \\
&= (-1)^m\int_a^b\int_c^d P_m(t,b)f^{(m,0)}(t,s)\,ds\,dt \\
&\quad + \sum_{j=1}^{n}(-1)^{m+j}Q_j(d,d)\int_a^b P_m(t,b)f^{(m,j-1)}(t,d)\,dt \\
&\quad - \sum_{j=1}^{n}(-1)^{m+j}Q_j(c,d)\int_a^b P_m(t,b)f^{(m,j-1)}(t,c)\,dt \\
&= (-1)^m\int_c^d(-1)^n\left\{\int_a^b f(t,s)\,dt\right. \\
&\quad \left. + \sum_{i=1}^{m}(-1)^i\left[P_i(b,b)f^{(i-1,0)}(b,s) - P_i(a,b)f^{(i-1,0)}(a,s)\right]\right\}ds \\
&\quad + \sum_{j=1}^{n}(-1)^{n+j}Q_j(d,d)\left\{(-1)^m\left[\int_a^b f^{(0,j-1)}(t,d)\,dt\right.\right.
\end{aligned}
$$

$$+\sum_{i=1}^{m}(-1)^{j}\Big(P_i(b,b)f^{(i-1,j-1)}(b,d)-P_i(a,b)f^{(i-1,j-1)}(a,d)\Big)\Big]\Big\}$$

$$-\sum_{j=1}^{n}(-1)^{n+j}Q_j(c,d)\Big\{(-1)^{m}\Big[\int_a^b f^{(0,j-1)}(t,c)\,dt$$

$$+\sum_{i=1}^{m}(-1)^{i}\Big(P_i(b,b)f^{(i-1,j-1)}(b,c)-P_i(a,b)f^{(i-1,j-1)}(a,c)\Big)\Big]\Big\}$$

$$=(-1)^{m+n}\int_a^b\int_c^d f(t,s)\,ds\,dt$$

$$+\sum_{i=1}^{m}(-1)^{m+n+i}\int_c^d\big[P_i(b,b)f^{(i-1,0)}(b,s)-P_i(a,b)f^{(i-1,0)}(a,s)\big]\,ds$$

$$+\sum_{j=1}^{n}(-1)^{m+n+j}Q_j(d,d)\int_a^b f^{(0,j-1)}(t,d)\,dt$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{m+n+i+j}P_i(b,b)Q_j(d,d)f^{(i-1,j-1)}(b,d)$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{m+n+i+j}P_i(a,b)Q_j(d,d)f^{(i-1,j-1)}(a,d)$$

$$-\sum_{j=1}^{n}(-1)^{m+n+j}Q_j(c,d)\int_a^b f^{(0,j-1)}(t,c)\,dt$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{m+n+i+j}P_i(a,b)Q_j(c,d)f^{(i-1,j-1)}(a,c)$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{m+n+i+j}P_i(b,b)Q_j(c,d)f^{(i-1,j-1)}(b,c)$$

$$=(-1)^{m+n}\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(b,b)\big[Q_j(d,d)f^{(i-1,j-1)}(b,d)$$
$$-Q_j(c,d)f^{(i-1,j-1)}(b,c)\big]$$

$$+(-1)^{m+n}\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_i(a,b)\big[Q_j(c,d)f^{(i-1,j-1)}(a,c)$$
$$-Q_j(d,d)f^{(i-1,j-1)}(a,d)\big]$$

$$+(-1)^{m+n}\sum_{i=1}^{m}(-1)^{i}P_i(b,b)\int_c^d f^{(i-1,0)}(b,s)\,ds$$

$$
\begin{aligned}
&-(-1)^{m+n}\sum_{i=1}^{m}(-1)^{j}P_i(a,b)\int_c^d f^{(i-1,0)}(a,s)\,ds\\
&+(-1)^{m+n}\sum_{j=1}^{n}(-1)^{j}Q_j(d,d)\int_a^b f^{(0,j-1)}(t,d)\,dt\\
&-(-1)^{m+n}\sum_{j=1}^{n}(-1)^{i}Q_j(c,d)\int_a^b f^{(0,j-1)}(t,c)\,dt\\
&+(-1)^{m+n}\int_a^b\int_c^d f(t,s)\,ds\,dt.
\end{aligned}
$$

The proof of Theorem 3 is complete. ∎

REMARK 2. As usual, let B_i, $i \in \mathbb{N}$, denote Bernoulli numbers.

¿From properties (6) and (7), (9) and (10) of the Bernoulli and Euler polynomials respectively, we can easily obtain that, for $i \geq 1$,

$$B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2}, \tag{38}$$

and, for $j \in \mathbb{N}$,

$$E_j(0) = -E_j(1) = -\frac{2}{j+1}(2^{j+1}-1)B_{j+1}. \tag{39}$$

It is also a well known fact that $B_{2i+1} = 0$ for all $i \in \mathbb{N}$.

As an example, taking $P_i(t,x) = P_{i,B}(t,x;a)$ and $Q_j(s,y) = P_{j,E}(s,y;c)$ from (12) and (13) for $0 \leq i \leq m$ and $0 \leq j \leq n$ in Theorem 3 and using (38) and (39) yields

$$
\begin{aligned}
A(f,P_m,Q_n) &= \sum_{i=1}^{m}\sum_{j=2}^{n}\frac{(a-b)^i(c-d)^j}{i!\cdot j!}\cdot\frac{2(2^{j+1}-1)}{j+1}B_iB_{j+1}\\
&\times\left[f^{(i-1,j-1)}(a,d)+f^{(i-1,j-1)}(a,c)-f^{(i-1,j-1)}(b,d)-f^{(i-1,j-1)}(b,c)\right]\\
&+(b-a)\sum_{i=1}^{m}\frac{(2^{i+1}-1)(c-d)^j}{(i+1)!}B_{j+1}\\
&\times\left[f^{(i-1,0)}(a,d)+f^{(i-1,0)}(a,c)+f^{(i-1,0)}(b,d)+f^{(i-1,0)}(b,c)\right],
\end{aligned}
\tag{40}
$$

$$B(f,P_m,Q_n) = \tag{41}$$
$$2\sum_{j=1}^{n}\frac{(1-2^{j+1})(c-d)^j}{(j+1)!}B_{j+1}\int_a^b\left[f^{(0,j-1)}(t,c)+f^{(0,j-1)}(t,d)\right]dt$$
$$+\sum_{j=2}^{n}\frac{(a-b)^j}{j!}B_j\int_c^d\left[f^{(i-1,0)}(a,s)-f^{(i-1,0)}(b,s)\right]ds$$
$$+\frac{b-a}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds,$$

and

$$R(f,P_m,Q_n) = \frac{(a-b)^m(c-d)^n}{m!\cdot n!}\int_a^b\int_c^d B_m\Big(\frac{t-a}{b-a}\Big)E_n\Big(\frac{s-c}{d-c}\Big)f^{(m,n)}(t,s)\,ds\,dt. \tag{42}$$

3. Estimates of the Remainders

In this section, we will give some estimates for the remainders of expansions in Theorem 2 and Theorem 3.

We firstly need to introduce some notation.

For a function $\ell:[a,b]\times[c,d]\to\mathbb{R}$, then for any $x,y\in[a,b]$, $u,\ v\in[c,d]$ we define

$$\|\ell\|_{[x,y]\times[u,v],\infty} := ess\sup\{|\ell(t,s)|\},$$
$$t\in[x,y] \text{ or } [y,x] \text{ and } s\in[u,v] \text{ or } [v,u]$$

and

$$\|\ell\|_{[x,y]\times[u,v],p} := \left|\int_x^y\int_u^v |h(t,s)|^p\,dsdt\right|^{\frac{1}{p}},\quad p\geq 1.$$

The following result establishing bounds for the remainder in the Taylor-like formula (18) holds.

Theorem 4. *Assume that* $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ *and* f *satisfy the assumptions of Theorem 2. Then we have the representation (18) and the remainder satisfies the estimate*

$$|T(f,P_m,Q_n)| \leq \begin{cases} \|P_m(\cdot,x)\|_{[a,x],\infty}\|Q_n(\cdot,y)\|_{[c,y],\infty}\left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],1}, \\ \|P_m(\cdot,x)\|_{[a,x],p}\|Q_n(\cdot,y)\|_{[c,y],p}\left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],p}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|P_m(\cdot,x)\|_{[a,x],1}\|Q_n(\cdot,y)\|_{[c,y],1}\left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],\infty}. \end{cases} \tag{43}$$

The proof follows in a straightforward fashion on using Hölder's inequality applied for the integral representation of the remainder $T(f, P_m, Q_n)$ provided by equation (22). We omit the details.

The integral remainder in the cubature formula (34) may be estimated in the following manner.

Theorem 5. *Assume that* $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ *and* f *satisfy the assumptions in Theorem 3. Then one has the cubature formula (34) and, the remainder* $R(f, P_m, Q_n)$ *satisfies the estimate:*

$$|R(f, P_m, Q_n)| \le \begin{cases} \|P_m(\cdot,b)\|_{[a,b],\infty} \|Q_n(\cdot,d)\|_{[c,d],\infty} \left\|f^{(m,n)}\right\|_{[a,b]\times[c,d],1}, \\ \|P_m(\cdot,b)\|_{[a,b],p} \|Q_n(\cdot,d)\|_{[c,d],p} \left\|f^{(m,n)}\right\|_{[a,b]\times[c,d],p}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|P_m(\cdot,b)\|_{[a,b],1} \|Q_n(\cdot,d)\|_{[c,d],1} \left\|f^{(m,n)}\right\|_{[a,b]\times[c,d],\infty}. \end{cases} \tag{44}$$

Remark 1. If we consider the particular instances of Appell polynomials provided by (11), (12) and (13), then a number of particular formulae may be obtained. Their remainder may be estimated by the use of Theorems 4 and 5, providing a 2–dimensional version of the results in [4].

For instance, if we consider from (11),

$$P_{m,\lambda}(t,x;a) = \frac{[t-(\lambda\, a+(1-\lambda)\, x)]^m}{m!} \tag{45}$$

$$Q_{n,\mu}(s,y;c) = \frac{[s-(\mu\, c+(1-\mu)\, y)]^n}{n!} \tag{46}$$

then we obtain the following result.

Theorem 6. *Let* $\{P_{m,\lambda}(t,x;a)\}_{m=0}^{\infty}$, $\{Q_{n,\mu}(s,y)\}_{n=0}^{\infty}$ *and* f *satisfy the assumptions of Theorem 2. Then we have the representation (18) and the remainder satisfies for* $a \le x$, $c \le y$, *the estimate*

$$|T(f, P_{m,\lambda}, Q_{n,\mu})| \le \begin{cases} \frac{(x-a)^m (y-c)^n}{m!n!} \lambda_\infty\ \mu_\infty \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],1}, \\ \frac{1}{m!n!}\left[\frac{(x-a)^{mq+1}(y-c)^{nq+1}}{(mq+1)(nq+1)}\right]^{\frac{1}{q}} \lambda_p\ \mu_p \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],q}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(x-a)^{m+1}(y-c)^{n+1}}{(m+1)!(n+1)!} \lambda_1 \mu_1 \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],\infty}. \end{cases} \tag{47}$$

where

$$\lambda_1 = \left[\lambda^{m+1} + (1-\lambda)^{m+1}\right],$$
$$\lambda_p = \left[\lambda^{mq+1} + (1-\lambda)^{mq+1}\right]^{\frac{1}{p}} \quad \text{and}$$
$$\lambda_\infty = \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^m.$$

and similar for μ_1, μ_p *and* μ_∞

Proof. Utilizing equations (45) and (46) and using Hölder's inequality for double integrals and the properties of the modulus on equation (22), then we have that

$$\begin{aligned}
&\left|\int_a^x \int_c^y T(f, P_{m,\lambda}, Q_{n,\mu})\right| \\
&= \left|\int_a^x \int_c^y P_{m,\lambda}(t,x;a)\ Q_{n,\mu}(s,y;c) f^{(m+1,n+1)}\ ds\ dt\right| \\
&\le \int_a^x \int_c^y \left|P_{m,\lambda}(t,x;a)\ Q_{n,\mu}(s,y;c)\right| \left|f^{(m+1,n+1)}\right|\ ds\ dt
\end{aligned}$$

$$(48)\qquad \le \begin{cases} \sup\limits_{(t,s)\in[a,x]\times[c,y]} \left|P_{m,\lambda}(t,x;a) Q_{n,\mu}(s,y;c)\right| \ \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],1}\cdot \\[2ex] \left(\int_a^x \int_c^y \left|P_{m,\lambda}(t,x;a) Q_{n,\mu}(s,y;c)\right|^q dt\, ds\right)^{\frac{1}{q}} \ \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],p}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \int_a^x \int_c^y \left|P_{m,\lambda}(t,x;a) Q_{n,\mu}(s,y;c)\right| dt\, ds \ \left\|f^{(m+1,n+1)}\right\|_{[a,x]\times[c,y],\infty}\cdot \end{cases}$$

Now, the result in equation (48) can be further simplified by the application of equations (45) and (46), given that,

$$\alpha = (1-\lambda)\ x + \lambda\ a \quad \text{and} \quad \beta = (1-\mu)\ y + \mu\ c.$$

It then follows

$$\begin{aligned}
&\sup_{(t,s)\in[a,x]\times[c,y]} \left|P_{m,\lambda}(t,x;a) Q_{n,\mu}(s,y;c)\right| \\
&= \sup_{t\in[a,c]} \left|P_{m,\lambda}(t,x;a)\right| \sup_{s\in[c,y]} \left|Q_{n,\mu}(s,y;c)\right| \\
&= \max\left\{\frac{(\alpha-a)^m}{m!}, \frac{(x-\alpha)^m}{m!}\right\} \times \max\left\{\frac{(\beta-c)^n}{n!}, \frac{(y-\beta)^n}{n!}\right\} \\
&= \frac{(x-a)^m (y-c)^n}{m!n!} \left[\max\{(1-\lambda),\lambda\}\right]^m \times \left[\max\{(1-\mu),\mu\}\right]^n \\
&= \frac{(x-a)^m (y-c)^n}{m!n!} \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^m \times \left[\frac{1}{2} + \left|\mu - \frac{1}{2}\right|\right]^n
\end{aligned}$$

giving the first inequality in (47) where we have used the fact that

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|.$$

Further, we have

$$\begin{aligned}
&\left(\int_a^x \int_c^y |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|^q \, ds\, dt\right)^{\frac{1}{q}} \\
&\quad = \left(\int_a^x |P_{m,\lambda}(t,x;a)|^q \, dt\right)^{\frac{1}{q}} \left(\int_c^y |Q_{n,\mu}(s,y;c)|^q \, ds\, dt\right)^{\frac{1}{q}} \\
&\quad = \frac{1}{m!n!}\left[\int_a^\alpha (\alpha-t)^{mq}\, dt + \int_\alpha^x (t-\alpha)^{mq}\, dt\right]^{\frac{1}{q}} \\
&\qquad \times \left[\int_c^\beta (\beta-s)^{nq}\, ds + \int_\beta^y (s-\beta)^{nq}\, ds\right]^{\frac{1}{q}} \\
&\quad = \frac{1}{m!n!}\left[\frac{(x-a)^{mq+1}(y-c)^{nq+1}}{(mq+1)(nq+1)}\right]^{\frac{1}{q}} \lambda_p\, \mu_p
\end{aligned}$$

producing the second inequality in (47).
Finally,

$$\begin{aligned}
&\int_a^x \int_c^y |P_{m,\lambda}(t,x;a)Q_{n,\mu}(s,y;c)|\, dt\, ds \\
&= \int_a^x \left|\frac{(t-\alpha)^m}{m!}\right| dt \int_c^y \left|\frac{(s-\beta)^n}{n!}\right| ds \\
&= \left[\int_a^\alpha \frac{(\alpha-t)^m}{m!} dt + \int_\alpha^x \frac{(t-\alpha)^m}{m!} dt\right] \times \left[\int_c^\beta \frac{(\beta-s)^n}{n!} ds + \int_\beta^y \frac{(s-\beta)^n}{n!} ds\right] \\
&= \frac{(x-a)^{m+1}(y-c)^{n+1}}{(m+1)!\,(n+1)!}\left[(1-\lambda)^{m+1} + \lambda^{m+1}\right] \times \left[(1-\mu)^{n+1} + \mu^{n+1}\right]
\end{aligned}$$

gives the last inequality in (47). Thus the theorem is completely proved. ∎

Remark 2. By taking $\lambda = \mu = 0$ or 1, we recapture the result obtained by G. Hanna et al. in [3].

In a similar fashion, we can stat the remainder $R(f, P_{m,\lambda}, Q_{n,\mu})$ estimate in the cubature formula (34) as in the following

Theorem 7. *Let* $\{P_{m,\lambda}(t,x;a)\}_{m=0}^{\infty}$, $\{Q_{n,\mu}(s,y)\}_{n=0}^{\infty}$ *and* f *satisfy the assumptions of Theorem 3, then the remainder* $R(f,P_{m,\lambda},Q_{n,\mu})$ *estimate in the cubature formula (34) satisfies the following*

$$
(49)\qquad |R(f,P_{m,\lambda},Q_{n,\mu})| \le \begin{cases} \frac{(b-a)^m(d-c)^n}{m!n!}\lambda_\infty\ \mu_\infty \left\|f^{(m,n)}\right\|_{[a,b]\times[c,d],1}, \\ \frac{1}{m!n!}\left[\frac{(b-a)^{mq+1}(d-c)^{nq+1}}{(mq+1)(nq+1)}\right]^{\frac{1}{q}}\lambda_p\ \mu_p \left\|f^{(m,n)}\right\|_{[a,b]\times[c,d],q}, \\ \qquad\qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^{m+1}(d-c)^{n+1}}{(m+1)!(n+1)!}\lambda_1\ \mu_1 \left\|f^{(m,n)}\right\|_{[a,b]\times[c,d],\infty}. \end{cases}
$$

The proof is similar to the one in Theorem 6 applied on the interval $[a,b]\times[c,d]$, and we omit the details.

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