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NUMERICAL METHODS FOR SOME NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study the numerical solutions of the stochastic differential equations of the form

$$du(x,t) \;=\; f(x,t,u)dt + g(x,t,u)dW(t) + \sum_{|q|\leq 2m} A_q(x,t)D^q u(x,t)dt$$

where $0 \leq t \leq T, x \in R^\nu, (R^\nu$ is the $\nu-$dimensional Euclidean space). Here $u \in R^n$, W(t) is an n-dimensional Brownian motion,

$$f : R^{n+\nu+1} \rightarrow R^n, g : R^{n+\nu+1} \rightarrow R^{n\times n},$$

and

$$A_q : R^\nu \times [0,T] \rightarrow R^{n\times n},$$

where $(A_q, \mid q \mid\leq 2m)$ is a family of square matrices whose elements are sufficiently smooth functions on $R^\nu \times [0,T]$ and $D^q = D_1^{q_1}.....D_\nu^{q_\nu}, D_i = \frac{\partial}{\partial x_i}$.
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1. INTRODUCTION

Desmond J. Higam, X-Uerong Mao, and Andrew M. Stuart studied the numerical solution of the stochastic differential equations of the form

$dy(t) = f(y(t))dt + g(y(t))dW(t), where\ 0 \leq t \leq T, y(0) = y_0, y(t) \in R^n$ for each t and $W(t)$ is an n-dimensional Brownian motion (see [9])

In this paper we study the numerical solutions of the stochastic differential equations of the form

$$du(x,t) = f(x,t,u)dt + g(x,t,u)dW(t) + \sum_{|q|\leq 2m} A_q(x,t)D^q u(x,t)dt \qquad (1.1)$$

where $0 \leq t \leq T, x \in R^\nu$.

Here $u \in R^n$ and W(t) is an n-dimensional Brownian motion. It is supposed that $f : R^{n+\nu+1} \to R^n$, $g : R^{n+\nu+1} \to R^{n\times n}$, where

$$(A_q(x,t), \mid q \mid \leq 2m)$$

is a family of square matrices whose elements are sufficiently smooth functions on $R^\nu \times [0,T]$

$$D^q = D_1^{q_1}.....D_\nu^{q_\nu}, D_i = \frac{\partial}{\partial x_i}$$

$q = (q_1, ...q_\nu)$is ν-dimensional multi-index, (see [5]). Equation (1.1)is called parabolic in the region $G = \{(x,t); x \in R^\nu, t \geq 0\}$ if for any point $(x,t) \in G$ the real parts of the λ-roots of the equation Det $[(-1)^m \sum_{|q|=2m} A_q(x,t)\sigma^q - \lambda I] = 0$ satisfy the inequality $Re\lambda(x,t,\sigma) \leq -\delta \mid \sigma \mid^m$, where δ is a positive constant, $\sigma \in R^\nu$,

$$\mid \delta \mid = (\delta_1^2 + ... + \delta_\nu^2)^{\frac{1}{2}}, \sigma^q = \sigma_1^{q_1}...\sigma_\nu^{q_\nu},$$

and I is the unit matrix.

We suppose that the following conditions are satisfied:

1. The coefficients of the operator $\sum_{|q|\leq 2m} A_q(x,t)D^q$ are continuous in $t \in [0,T]$, moreover, the continuity in t of the coefficients $\sum_{|q|\leq 2m} A_q(x,t)D^q$ is uniform with respect to $x \in R^\nu$.
2. The coefficients of $\sum_{|q|\leq 2m} A_q(x,t)D^q$ are bounded on $R^\nu \times [0,T]$ and satisfy the Holder condition with respect to x.

Under these conditions there exists for the system

$$\frac{\partial V}{\partial t} = \sum_{|q|\leq 2m} A_q(x,t)D^q V \qquad (1.2)$$

a fundamental matrix of solution $K(x,y,t,\theta)$ which satisfy the following conditions

(1)

$$\frac{\partial K}{\partial t}, D^q K \in C(G_1), \mid q \mid \leq 2m$$

where $G_1 = R^{2\nu} \times (0,T) \times (0,T)$

(2) $\mid D^q K(x,y,t,\theta) \mid \leq \frac{b_1}{(t-\theta)^\beta} e^z \; ; t > \theta, z = \frac{-b_2 |x-y|^{2m}}{t-\theta}$,

$$\beta = 1/2(\nu + \mid q \mid), \mid q \mid \leq 2m$$

where $\mid x \mid$ is the norm $(x_1^2 + x_2^2 + ... + x_\nu^2)^{\frac{1}{2}}$, $\mid K \mid$ a suitable norm of the square matrix K , b_1 and b_2 are positive constant.

(3) The function V defined by

$$V(x,t) = \int_{R^\nu} K(x,y,t,0) V_0(y) dy$$

represents the unique solution of the parabolic system

$$\frac{\partial V}{\partial t} = \sum_{|q| \leq 2m} A_q(x,t) D^q V \tag{1.3}$$

with the initial conditions

$$V(x,0) = V_0(x) \tag{1.4}$$

$$(\frac{\partial V}{\partial t}, D^q V \in C(G_2), \mid q \mid \leq 2m),$$

$$G_2 = R^\nu \times (0,T), (C(G) \text{is the set of all continuous functions on G})$$

The existence of such functions depends on the parabolicity of the system (1.3) and on the smoothness of the coefficients of such systems (see [1], [4]).

We shall use the notations

$$\sup_x \mid V(x,t) \mid = \parallel V(.,t) \parallel,$$

$$\sup_x \mid e(x,t) \mid = \parallel e(.,t) \parallel,$$

where

$$e(x,t) = V(x,t) - u(x,t).$$

We assume that f and g satisfy the Lipschitz condition.

$$\parallel f(.,t,u) - f(.,t,v) \parallel + \parallel g(.,t,u) - g(.,t,v) \parallel \leq \gamma \mid u - v \mid, \tag{1.6}$$

we assume also that f and g satisfy the growth condition.

$$\parallel f(.,t,u) \parallel^2 + \parallel g(.,t,u) \parallel^2 \leq \gamma^2 (1 + \mid u \mid^2), \tag{1.7}$$

where $\gamma > 0$
The convergence theory for numerical methods were studied in many papers (see [2], [3], [6], [8], [15], [10], [12], [13], [14]). In this paper we focus on the mean square convergence, which implies convergence in probability. In section 2 we prove that the Euler-Maruyama method converges strongly if the exact and numerical solution have bounded pth moment for some $p > 2$.
In section 3 we prove that u (x, t) has bounded moments. In section 4 we show that the Euler-Maruyama converges strongly at the optimal rate

2. Some numerical methods

Given a step size $\Delta t > 0$, the Euler-Maruyama (EM) method applied to (1.1) computes approximations

$$\begin{aligned} V_k(x) &= u(x, t_k), \qquad \text{where} \qquad t_k = k\Delta t, u_0(x) = V_0(x), \\ V_{k+1}(x) &= V_k(x) + \Delta t f(x, t_k, V_k) + g(x, t_k, V_k)\Delta W_k \\ &\quad + \Delta t \sum_{|q| \leq 2m} a_q D^q V_k(x), \ (2.1) \end{aligned}$$

and $\Delta W_k = W(t_{k+1}) - W(t_k)$.
In our analysis we find it convenient to work with continuous time approximations and hence we define
$V(x,t)$ by $V(x,t) = V_k(x); t \in [t_k, t_{k+1})$ and set

$$\overline{V}(x,t) = \int_{R^\nu} K(x,y,t,0)V_o(y)dy + \int_0^T \int_{R^\nu} K(x,y,t,s)f(y,s,V(y,s))dyds$$

$$+ \int_0^t \int_{R^\nu} K(x,y,t,s)g(y,s,V(y,s))dydW(s). \qquad (2.2)$$

Note that $\overline{V}(x, t_k) = V(x, t_k) = V_k(x)$; that is $\overline{V}(x,t)$ and $V(x,t)$ coincide with the discrete solution at the grid points.
We refer to V(x, t) and $\overline{V}(x,t)$ as continuous time extensions of the discrete approximations $\{V_k(x)\}$. We will study the error in $\overline{V}(x,t)$ in the supremum norm, this will, of course, give an immediate bound for the error in the discrete approximation.
In this paper $| \, . \, |$ is used to denote the Euclidean vector norm.

Let us suppose for some $p>2$ there is a constant B such that

$$E[\sup_{0\leq t\leq T} \| \overline{V}(.,t) \|] \leq B,\ E[\sup_{0\leq t\leq T} \| u(.,t) \|^p] \leq B \ (2.3)$$

Theorem 2.1. Under the conditions (1.6) and (2.3), for any $T>0$ the Euler-Maruyama solution (2.1) with continuous time extension (2,2) satisfies

$$\lim_{\Delta t\to 0} E[\sup_{0\leq t\leq T} \| \overline{V}(0,t)-u(0,t) \|^2]=0. \tag{2.4}$$

Proof.
Set

$\tau_R := inf\{t\geq 0 : \| \overline{V}(.,t) \| \geq R\}, \rho_R := inf\{t\geq 0 : \| u(.,t) \| \geq R\}, \theta_R := \tau_R \wedge \rho_R$

($\tau_R \wedge \rho_R$ the minimum of τ_R *and* ρ_R)
Recall the Young inequality; $ab \leq \frac{\delta}{r}a^r + \frac{1}{q^{\delta q/r}}b^q, \forall a,b,\delta>0$ where $\frac{1}{r}+\frac{1}{q}=1$.
We thus have for any $\delta>0$

$$\begin{aligned} E[\sup_{0\leq t\leq T} \| e(.,t) \|^2] &= E[\sup_{0\leq t\leq T} \| e(.,t) \|^2_{1\{\tau_R>T,\rho_R>T\}}] \\ &+ E[\sup_{0\leq t\leq T} \| e(.,t) \|^2_{1\{\tau_R\leq T or \rho_R\leq T\}}] \\ &\leq E[\sup_{0\leq t\leq T} \| e(.,t\wedge\theta_R) \|^2_1 \{\theta_R>T\}] \\ &+ \frac{2\delta}{p}E[\sup_{0\leq t\leq T} \| e(.,t) \|^p] \end{aligned}$$

$$+\frac{1-\frac{2}{p}}{\delta^{2/(p-2)}}P(\tau_R\leq T \ or \ \rho_R\leq T). \tag{2.5}$$

Now, by using (2.3)we get,

$$\begin{aligned} P(\tau_R\leq T) &= E[1_{\tau_R\leq T}\frac{\| \overline{V}(.,\tau_R) \|^p}{R^p}] \\ &\leq \frac{1}{R_p}E[\sup_{0\leq t\leq T} \| \overline{V}(.,t) \|^p] \leq \frac{B}{R^p}, \end{aligned}$$

A similar results can be derived for ρ_R, so that

$$P(\tau_R\leq T \ or \ \rho_R\leq T) \leq \frac{2B}{R^p}.$$

Using these bounds along with

$$E[\sup_{0\leq t\leq T} \| e(.,t) \|^p] \leq 2^{p-1}[\sup_{0\leq x<T}(\| \overline{V}(.,t)) \|^p + \| u(.,t) \|^p] \leq 2^p B$$

and by substituting in (2.5), we get

$$E[\sup_{0\leq t\leq T} \| e(.,t) \|^2] \leq E[\sup_{0\leq t\leq T} [\| (\overline{V}(.,t\wedge\theta_R)) - u(.,t\wedge\theta_R)] \|^2] + \frac{2^{p+1}\delta B}{p} + \frac{2(p-2)B}{p\delta^2/(p-2)R^p} \tag{2.6}$$

By estimating the first term on the right of (2.6), we get

$$u(x,t\wedge\theta_R) = \int_{R^\nu} K(x,y,t\wedge\theta_R,0)u_o(y)dy + \int_0^{t\wedge\theta_R}\int_{R_\nu} K(x,y,t\wedge\theta_R,s)g(y,s,u(y,s))dydW(s) \tag{2.7}$$

From (2.2) , (2.7) and by using Cauchy Schwartz inequality we get

$$\begin{aligned} &\| \overline{V}(.,t\wedge\theta_R) - u(.,t\wedge\theta_R) \|^2 \\ \leq\ & 2C\{\int_0^{t\wedge\theta_R} [\| f(.,s,V(.,s) - f(.,s,u(.,s)) \|^2]ds \\ +\ & \int_0^{t\wedge\theta_R} [\| g(.,s,V(.,s)) - g(.,s,u(.,s)) \|^2] \| dW(s) \| . \end{aligned}$$

So, from (1.6) and Doob's martingale inequality [12], we have for any $\tau \leq T$

$$\begin{aligned} E[\sup_{0\leq t\leq T} &\| \overline{V}(.,t\wedge\theta_R) - u(.,t\wedge\theta_R) \|^2] \\ &\leq 2CE[\int_0^{\tau\wedge\theta_R} \| V(.,s) - u(.,s) \|^2 ds] \\ &\leq 4CE[\int_0^{\tau\wedge\theta_R} [\| V(.,s) - \overline{V}(.,s) \|^2 + \| \overline{V}(.,s) - u(.,s) \|^2]ds] \\ &\leq 4CE[\int_0^{\tau\wedge\theta_R} \| V(.,s) - \overline{V}(.,s) \|^2 ds \\ &\qquad + \int_0^{\tau} \| \overline{V}(.,s\wedge\theta_R) - u(.,s\wedge\theta_R) \|^2 ds] \\ &\qquad \leq 4C(E[\int_0^{\tau\wedge\theta_R} \| V(.,s) - \overline{V}(.,s) \|^2 ds] \\ &\quad + \int_0^{\tau} E\sup_{0\leq r\leq s} \| \overline{V}(.,r\wedge\theta_R - V(.,r\wedge\theta_R \|^2 ds). \end{aligned} \tag{2.8}$$

Let k_c be the integer for which $c \in [t_{k_c}, t_{k_{c+1}})$.
Then :

$$V(x,c) - \overline{V}(x,c) = V_{k_c}(x) - \int_{R^\nu} K(x,y,c,0)V_{k_c}(y)dy$$

$$\begin{aligned}
&- \int_{t_{k_c}}^{c}\int_{R^\nu} K(x,y,c,s)f(y,s,V(y,s))dyds \\
&- \int_{t_{k_c}}^{c}\int_{R^\nu} K(x,y,c,s)g(y,s,V(y,s))dydW(s) \\
&= V_{k_c}(x) - \int_{R^\nu} K(x,y,c,0)V_{kc}(y)dy \\
&- \int_{R^\nu} K(x,y,c,0)f(y,c,V_{k_c}(y))(C-(t_{k_c}))dy \\
&- \int_{R^\nu} K(x,y,c,0)g(y,c,V_{k_c}(y))(W(c)-W(t_{k_c}))dy.
\end{aligned}$$

Hence

$$\begin{aligned}
&\| \quad V(.,c) - \overline{V}(.,c) \|^2 \leq 4 \| V_{k_c}(.) \|^2 [2 + \| f(.,c,V_{k_c}(.) \|^2 (\Delta t)^2 \\
&+ \quad \| g(.,c,V_{k_c}(.) \|^2 \| W(c) - W(t_{k_c}) \|^2].
\end{aligned}$$

From (1.7), (2.3) and Lyapunov inequality [11], yields

$$\begin{aligned}
E\int_0^{\tau\wedge\theta_R} &\| \quad V(.,s) - \overline{V}(.,s) \|^2 \, ds \\
&\leq \quad 4\overline{C}E[\int_0^{\tau\wedge\theta_R} \| V_{k_s}(.) \|^2][2 + (1 + \| V_{k_s}(.) \|^2)((\Delta t)^2 + m\Delta t)ds] \\
&\leq \quad 4\overline{C}\int_0^T E[3 \| V_{k_s}(.) \|^2 + \| V_{k_s}(.) \|^4]((\Delta t)^2 + m\Delta t)ds \\
&\leq \quad 4\overline{C}T(B^{2/P} + B^{4/p})\Delta t(\Delta t + m) \\
&\leq \quad 4\tilde{C}TB^{4/P}\Delta t(\Delta t + m).
\end{aligned}$$

By substituting in (2.8) we have for a new constant $C > 0$

$$\begin{aligned}
&E[\sup_{0\leq t\leq T} \quad \| \quad \overline{V}(.,t\wedge\theta_R) - u(.,t\wedge\theta_R) \|^2] \\
&\qquad \leq \quad CB^{4/p}\Delta t(\Delta t + m) + \overline{C}\int_0^T E\sup_{0\leq r\leq s}[\| \overline{V}(.,r\wedge\theta_R) - u(.,r\wedge\theta_R \|^2]ds
\end{aligned}$$

Apply the Gronwal inequality we obtain

$$E \sup_{0\le t\le T} \| \quad \overline{V}(.,t\wedge\theta_R) - u(.,t\wedge\theta_R) \|^2] \le C\Delta t e^{4\overline{C}T}$$

where C is a universal constant independent of $\Delta t, R$ *and* δ. Substituting in (2.6) we obtain

$$E[\sup_{0\le t\le T} \| e(.,t) \|^2] \le C\Delta t e^{4\overline{C}T} + \frac{2^{p+1}\delta B}{p} + \frac{(p-2)2B}{p^{\delta^2}/(p-2)R^b}$$

Given $\epsilon > 0$, we can choose $\delta > 0$ such that $(2^{p+1}\delta B)/p < \frac{\epsilon}{3}$. Then choose R so that $\dfrac{(1-2/p)2B}{\delta^{2(p-2)}R^p} < \dfrac{\epsilon}{3}$, and then choose Δt sufficiently small such that $C\Delta t e^{e^{4\overline{C}T}} < \dfrac{\epsilon}{3}$ we get,

$$E[\sup_{0\le t\le T} \| e(.,t) \|^2] \le \epsilon,$$

as required.

3. Bounded moments

we show that the stochastic differential equation has bounded pth moment for each $p > 2$.

Lemma 3.1. Under the conditions (1.6) and (1.7), for each $p > 2$, there is $C = C(p,T) > 0$ such that

$$E[\sup_{0\le t\le T} \| u(.,t) \|^p] \le C(1 + E \| u_0(.) \|^p)$$

Proof.we have

$$\begin{aligned} u(x,t) &= \int_{R^\nu} K(x,y,t,0)u_0(y)dy \\ &+ \int_0^t \int_{R^\nu} K(x,y,t,s)f(y,s,u(y,s))dyds \\ &+ \int_0^t \int_{R^\nu} K(x,y,t,s)g(y,s,u(y,s))dydW(s) \end{aligned}$$

By using (1.7) we have for some
$\gamma = \gamma(p)$and $t_1 \in [0, T]$

$$\sup_{0 \leq t \leq t_1} \| u(.,t) \|^p \leq C\{\| u_0(.) \|^p +$$

$$\int_0^{t_1} [\| f(.,s,u(.,s)) \|^p + \| g(.,s,u(.,s)) \|^p] ds,$$

we take the expectation to give

$$E[\sup_{0 \leq t \leq t_1} \| u(.,t) \|^p] \leq \overline{C}\{E \| u_0(.) \|^p + \int_0^{t_1} E \| u(.,s) \|^p \, ds\} \tag{3.1}$$

since, we have

$$E[\| u(.,t) \|^p] \leq C\{E[\| u_0(.) \|^p] + \int_0^t E[\| u(.,s) \|^p] ds\}.$$

By using Gronwall lemma, we get :

$$E[\| u(.,t) \|^p] \leq E \| u_0(.) \|^p \, e^{Ct}, t \in [0, t_1]$$

From the last inequality and (3.1), we get the required result.

4. Convergence rate for Euler Maruyama

In this section we establish a rate of convergence for Euler-Maruyama. So let us assume that the stochastic differential equation solutions and EM solutions satisfy

$$E(\sup_{0 \leq t \leq T} \| u(.,t) \|^p) < \infty, E(\sup_{0 \leq t \leq T} \| V(.,t) \|^p) < \infty$$

$$E(\sup_{0 \leq t \leq T} \| \overline{V}(.,t) \|^p < \infty, for\ all\ p > 1 \tag{4.1}$$

Throughout the following analysis z is a positive integer, whose value may change between occurrences.

Before obtaining a convergence rate for EM, we give the following Lemma.

Lemma 4.1. Under condition (1.6) and (4.1), for any even integer $r \leq 2$, there exists a constant k = k (r) such that

$$\sup_{0 \leq t \leq T} E \mid V(x,t) - \overline{V}(x,t) \mid^r \leq k \Delta t^{r/2}$$

Proof. Let $t \in [k\Delta t, (k+1)\Delta t)$. Then

$$\begin{aligned} \| \; V(.,t) - \overline{V}(.,t) \|^r &\leq C\{\| \int_t^{t_k} f(.,s,V(.,s)ds) \|^r \\ &+ \| \int_{t_k}^t g(.,s,V(.,s)dW(s) \|^r\} \\ &= C\{\| t - t_k \|^r \| f(.,t_k,V_k(.)) \|^r \\ &+ \| g(.,t_k,V_k(.) \|^r \| W(t) - W(t_k) \|^r\} \end{aligned}$$

Hence, for some k = k(r)

$$\begin{aligned} E \;\; \| \; V(.,t) - \overline{V}(.,t) \|^r &\leq k(\Delta t)^r [1 + E sup_{0 \leq t \leq T} \| V(.,t) \|^z] \\ &+ [1 + E sup_{0 \leq t \leq T} \| V(.,t) \|^z](t - t_k)^{r/2}), \end{aligned}$$

for $0 \leq t \leq T$. Since $t - t_k \leq \Delta t$, the result follows by redefinition of k .

Theorem 4.1.

Under the conditions (1.6) and (4.1) the Euler- Maruyama solution (2.1) with continuous time extension (2.2) satisfies

$$E[\sup_{0 \leq t \leq T} | \overline{V}(x,t) - u(x,t)) |^2] = O(\Delta t))$$

Proof. We have (2.3) and

$$\begin{aligned} u(x,t) &= \int_{R^\nu} K(x,y,t,0) u_0(y) dy \\ &+ \int_0^t \int_{R^\nu} K(x,y,t,s) f(y,s,u(y,s)) dy ds \\ &+ \int_0^t \int_{R^\nu} K(x,y,t,s) g(y,s,u(y,s)) dy dW(s) \end{aligned}$$

let $e(x,t) = u(x,t) - \overline{V},(x,t)$,then

$$\begin{aligned} \| \; e(.,t) \|^2 &\leq 2C\{\int_0^t \| f(.,s,u(.,s)) - f(.,s,V(.,s) \|^2 ds \\ &+ \int_0^t \| g(.,s,u(.,s)) - g(.,s,V(.,s) \|^2 ds\} \end{aligned}$$

$$\leq \overline{C}\{\int_0^t \| u(.,s) - V(.,s) \|^2 ds\}$$

$$\leq \overline{C}\{\int_0^t \| u(.,s) - \overline{V}(.,s) \|^2 ds\}$$

$$+ \int_0^t \| V(.,s) - \overline{V}(.,s) \|^2 ds\}$$

By using lemma (4.1) with r = 4 , we get :

$$E[\sup_{0\leq s\leq t} \| e(.,s) \|^2] \leq C \int_0^t E \| e(.,s) \|^2 ds$$

$$+ \int_0^t E(\| V(.,s) - \overline{V}(.,s) \|^4)^{\frac{1}{2}} ds$$

$$\leq C[\int_0^t E \| e(.,s) \|^2 ds] + K\Delta t$$

The result follows from Gronwal lemma .

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