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ON THE DISSIPATIVE HELMHOLTZ EQUATION IN A CRACKED DOMAIN WITH THE DIRICHLET-NEUMANN BOUNDARY CONDITION

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ABSTRACT. The Dirichlet-Neumann problem for the dissipative Helmholtz equation in a connected plane region bounded by closed curves and containing cuts is studied. The Neumann condition is given on the closed curves, while the Dirichlet condition is specified on the cuts. The existence of a classical solution is proved by potential theory. The integral representation of the unique classical solution is obtained. The problem is reduced to the Fredholm equation of the second kind and index zero, which is uniquely solvable. Our results hold for both interior and exterior domains.

1. INTRODUCTION

We consider a boundary value problem for the dissipative Helmholtz equation in a 2-D multiply connected domain bounded by closed curves and open arcs (cracks or cuts). The domain may be exterior or interior. The Neumann boundary condition is given on the closed curves, while the Dirichlet condition is posed on the cuts. Problems in domains with cuts are used in applied sciences to model physical processes in cracked domains. Cuts model cracks, wings, screens, spits, wavebreackers in mechanics and engineering. Problems with mixed boundary conditions were not studied in cracked domains for the Helmholtz equation by rigorous mathematical methods before, though these problems are important for applications. Dirichlet and Neumann problems for the dissipative Helmholtz equation in cracked domains were studied in [5,6] by the boundary integral equation method. In the present paper we extend the approach developed in [5,6] to the problem with the mixed Dirichlet-Neumann boundary condition. We obtain an integral representation for a solution of the problem in the form of potentials and reduce the problem to a uniquely solvable Fredholm integral equation of the second kind in the appropriate Banach space. Our approach is constructive since our integral equation can be computed by a standard code as well as a solution of the problem [7], so the results of the paper can be used for a numerical simulation in cracked domains.

The Dirichlet problem for the Helmholtz equation outside cuts in a plane has been studied in [3,10,11].

2. Formulation of the problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [4].

Let γ be a set of curves, which may be closed and open. We say that $\gamma \in C^{2,\lambda}$ (or $\gamma \in C^{2,0}$) if curves γ are of class $C^{2,\lambda}$ (or $C^{2,0}$), where the Hölder exponent $\lambda \in (0,1]$.

In the plane $x = (x_1, x_2) \in R^2$ we consider the multiply connected domain bounded by simple open curves $\Gamma_1^1, ..., \Gamma_{N_1}^1 \in C^{2,\lambda}, \ \ \lambda \in (0,1]$ and simple closed curves

$$\Gamma_1^2, ..., \Gamma_{N_2}^2 \in C^{2,0},$$

so that the curves do not have common points, in particular, end-points. We shall consider both the case of an exterior domain and the case of an interior domain, when the curve Γ_1^2 encloses all other. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \quad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \quad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The connected domain bounded by Γ^2 and containing Γ^1 will be called $\mathcal{D}$, so that $\partial\mathcal{D} = \Gamma^2$; $\Gamma^1 \subset \mathcal{D}$. We assume that each curve Γ_n^k is parametrized by the arc length s: $\Gamma_n^k = \{x : \ \ x = x(s) = (x_1(s), x_2(s)), \ \ s \in [a_n^k, b_n^k]\}, \ \ n = 1, ..., N_k, \ \ k = 1, 2$, so that $a_1^1 < b_1^1 < ... < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < ... < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is to the right when the parameter s increases on Γ_n^2. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except $a_n^2, \ \ b_n^2$, which correspond to the same point x for $n = 1, ..., N_2$. Below the sets of the intervals on the Os axis

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \quad \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \quad \bigcup_{k=1}^{2} \bigcup_{n=1}^{N_k} [a_n^k, b_n^k]$$

will be denoted by $\Gamma^1, \ \ \Gamma^2$ and Γ also. We put

$$C^0(\Gamma_n^2) = \{\mathcal{F}(s) : \ \ \mathcal{F}(s) \in C^0[a_n^2, b_n^2], \quad \mathcal{F}(a_n^2) = \mathcal{F}(b_n^2)\ \}$$

and $C^0(\Gamma^2) = \bigcap_{n=1}^{N_2} C^0(\Gamma_n^2)$.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \ \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s), \ \ \sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), \ -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$. So, $\mathbf{n}_x$ is an inward normal to $\mathcal{D}$ on Γ^2.

We consider Γ^1 as a set of cuts. The side of Γ^1 which is on the left, when the parameter s increases will be denoted by $(\Gamma^1)^+$ and the opposite side will be denoted by $(\Gamma^1)^-$.

We say, that the function $u(x)$ belongs to the smoothness class **K** if

1) $u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$, and $u(x)$ is continuous at the end-points of the cuts Γ^1,

2) $\nabla u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}\backslash\Gamma^2\backslash X)$, where X is a point-set, consisting of the end-points of Γ^1 :

$$X = \bigcup_{n=1}^{N_1} \left(x(a_n^1) \cup x(b_n^1)\right),$$

3) in the neighbourhood of any point $x(d) \in X$ for some constants $C > 0$, $\epsilon > -1$ the inequality

$$|\nabla u| \leq C\,|x - x(d)|^{\epsilon}, \tag{1}$$

holds, where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$,

4) there exists a uniform for all $x(s) \in \Gamma^2$ limit of $(\mathbf{n}_x, \nabla_{\hat{x}} u(\hat{x}))$ as $\hat{x} \in \mathcal{D}\backslash\Gamma^1$ tends to $x \in \Gamma^2$ along the normal $\mathbf{n}_x$.

Remark. In the definition of the class K we consider Γ^1 as a set of cuts in the domain $\mathcal{D}$. According to this definition, $u(x)$ and $\nabla u(x)$ are continuously extensible on cuts $\Gamma^1\backslash X$ from the left and from the right, but their values on $\Gamma^1\backslash X$ from the left and from the right may be different, so that $u(x)$ and $\nabla u(x)$ may have a jump across $\Gamma^1\backslash X$.

Let us formulate the Dirichlet-Neumann problem for the dissipative Helmholtz equation in the domain $\mathcal{D}\backslash\Gamma^1$ which may be interior or exterior.

Problem **U**. Find a function $u(x)$ of the class **K** which satisfies the Helmholtz equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) + \beta^2 u(x) = 0, \quad x \in \mathcal{D}\backslash\Gamma^1, \quad \beta = \text{const}, \quad \text{Im}\beta > 0, \tag{2a}$$

and the boundary conditions

$$u(x(s))|_{(\Gamma^1)^+} = F^+(s), \quad u(x(s))|_{(\Gamma^1)^-} = F^-(s), \quad \left.\frac{\partial u(x(s))}{\partial \mathbf{n}_x}\right|_{\Gamma^2} = F(s). \tag{2b}$$

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity

$$u = o(|x|^{-1/2}), \quad |\nabla u(x)| = o(|x|^{-1/2}), \quad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{2c}$$

All conditions of the problem **U** must be satisfied in the classical sense. By $\partial u/\partial \mathbf{n}_x$ on Γ^2 we mean the limit ensured in the point 4) of the definition of the smoothness class **K**.

On the basis of the energy equalities [1,8] we can easily prove the following assertion.

Theorem 1. *If* $\Gamma^1 \in C^{2,\lambda}$, $\lambda \in (0,1]$, $\Gamma^2 \in C^{2,0}$, *then the problem* **U** *has at most one solution.*

The theorem holds for both interior and exterior domain $\mathcal{D}$.

3. Integral equations at the boundary

Below we assume that

$$F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1), \quad F(s) \in C^0(\Gamma^2), \quad \lambda \in (0,1], \tag{3a}$$

$$F^+(a_n^1) = F^-(a_n^1), \quad F^+(b_n^1) = F^-(b_n^1), \quad n = 1, ..., N_1. \tag{3b}$$

Note that the Hölder exponent λ in the smoothness conditions for the arcs Γ^1 and the Hölder exponent λ in (3a) for the functions $F^+(s)$, $F^-(s)$ are assumed to be the same. If these exponents are different, then as λ we can take the smallest one.

If $\mathcal{B}_1(\Gamma^1)$, $\mathcal{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then for functions given on Γ we introduce the Banach space $\mathcal{B}_1(\Gamma^1) \cap \mathcal{B}_2(\Gamma^2)$ with the norm $\|\cdot\|_{\mathcal{B}_1(\Gamma^1)\cap\mathcal{B}_2(\Gamma^2)} = \|\cdot\|_{\mathcal{B}_1(\Gamma^1)} + \|\cdot\|_{\mathcal{B}_2(\Gamma^2)}$.

By $\int_{\Gamma^k} ... d\sigma$ we mean

$$\sum_{n=1}^{N_k} \int_{a_n^k}^{b_n^k} ... d\sigma.$$

We consider the angular potential from [3] for the equation (2a) on Γ^1

$$v_1[\nu](x) = \frac{i}{4} \int_{\Gamma^1} \nu(\sigma) V(x, \sigma) d\sigma. \tag{4}$$

The kernel $V(x,\sigma)$ is defined on each curve Γ_n^1 $(n = 1, ..., N_1)$ by the formula

$$V(x, \sigma) = \int_{a_n^1}^{\sigma} \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)} (\beta |x - y(\xi)|) \, d\xi, \quad \sigma \in [a_n^1, b_n^1],$$

where $\mathcal{H}_0^{(1)}(z)$ is the Hankel function of the first kind [9]:

$$\mathcal{H}_0^{(1)}(z) = \frac{\sqrt{2}\exp(iz - i\pi/4)}{\pi\sqrt{z}} \int_0^\infty \exp(-t) t^{-1/2} \left(1 + \frac{it}{2z}\right)^{-1/2} dt,$$

$$y = y(\xi) = (y_1(\xi), y_2(\xi)), \quad |x - y(\xi)| = \sqrt{(x_1 - y_1(\xi))^2 + (x_2 - y_2(\xi))^2}.$$

Below we suppose that $\nu(\sigma)$ belongs to $C^{0,\lambda}(\Gamma^1)$ and satisfies the following additional conditions

$$\int_{a_n^1}^{b_n^1} \nu(\sigma) \, d\sigma = 0, \quad n = 1, ..., N_1. \tag{5}$$

As shown in [3], for such $\nu(\sigma)$ the angular potential $v_1[\nu](x)$ belongs to the class **K**. In particular, the condition (1) is satisfied for any $\epsilon \in (-1, 0)$. Moreover, integrating

$v_1[\nu](x)$ by parts and using (5) we express the angular potential in terms of a double layer potential

$$v_1[\nu](x) = -\frac{i}{4}\int\limits_{\Gamma^1} \rho(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\mathcal{H}_0^{(1)}\left(\beta\left|x-y(\sigma)\right|\right)d\sigma, \tag{6}$$

with the density

$$\rho(\sigma) = \int\limits_{a_n^1}^{\sigma} \nu(\xi)\,d\xi, \quad \sigma\in[a_n^1, b_n^1], \quad n=1,...,N_1. \tag{7}$$

Consequently, $v_1[\nu](x)$ satisfies both equation (2a) outside Γ^1 and the condition at infinity (2c).

Let us construct a solution of the problem **U** . This solution can be obtained with the help of potential theory for the Helmholtz equation (2a). We seek a solution of the problem in the following form

$$u[\nu,\mu](x) = v_1[\nu](x) + w[\mu](x)\ , \tag{8}$$

where $v_1[\nu](x)$ is given by (4), (6) and

$$w[\mu](x) = w_1[\mu](x) + w_2[\mu](x)\ , \tag{9}$$

$$w_1[\mu](x) = \frac{i}{4}\int\limits_{\Gamma^1} \mu(\sigma)\mathcal{H}_0^{(1)}\left(\beta\left|x-y(\sigma)\right|\right)d\sigma,$$

$$w_2[\mu](x) = \frac{i}{4}\int\limits_{\Gamma^2} \mu(\sigma)\mathcal{H}_0^{(1)}\left(\beta\left|x-y(\sigma)\right|\right)d\sigma.$$

As noted above, we shall look for the density $\nu(\sigma)$ satisfying the conditions (5) and belonging to $C^{0,\lambda}\left(\Gamma^1\right)$.

We shall seek $\mu(s)$ from the Banach space $C_q^{\omega}(\Gamma^1)\cap C^0(\Gamma^2)$, $\omega\in(0,1]$, $q\in[0,1)$ with the norm $\|\cdot\|_{C_q^{\omega}(\Gamma^1)\cap C^0(\Gamma^2)} = \|\cdot\|_{C_q^{\omega}(\Gamma^1)} + \|\cdot\|_{C^0(\Gamma^2)}$. We say, that $\mu(s)\in C_q^{\omega}(\Gamma^1)$ if

$$\mu(s)\prod_{n=1}^{N_1}\left|s-a_n^1\right|^q\left|s-b_n^1\right|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Hölder space with the exponent ω and

$$\|\mu(s)\|_{C_q^{\omega}(\Gamma^1)} = \left\|\mu(s)\prod_{n=1}^{N_1}\left|s-a_n^1\right|^q\left|s-b_n^1\right|^q\right\|_{C^{0,\omega}(\Gamma^1)}.$$

It can be checked directly with the help of [3], that for such $\mu(s)$ the function $w_1[\mu](x)$ satisfies equation (2a) outside Γ^1 and belongs to the class **K**. In particular, the inequality (1) holds with $\epsilon=-q$ if $q\in(0,1)$. The potential $w_2[\mu](x)$ satisfies equation (2a) outside Γ^2 and belongs to the class $\mathbf{K}$ [3,8]. In the case of the exterior domain $\mathcal{D}$ the function (8) satisfies the condition at infinity (2c). Therefore, the function (8) satisfies all the conditions of the problem except the boundary conditions (2b).

To satisfy the boundary conditions we put (8) in (2b) and arrive at the system of the integral equations for the densities $\mu(s)$, $\nu(s)$

$$\pm\frac{1}{2}\rho(s)+\frac{i}{4}\int\limits_{\Gamma^1}\nu(\sigma)V\left(x(s),\sigma\right)d\sigma+ \tag{10a}$$
$$+\frac{i}{4}\int\limits_{\Gamma^1}\mu(\sigma)\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right)d\sigma+$$
$$+\frac{i}{4}\int\limits_{\Gamma^2}\mu(\sigma)\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right)d\sigma=F^{\pm}(s),\quad s\in\Gamma^1,$$

$$\frac{i}{4}\int\limits_{\Gamma^1}\nu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}V\left(x(s),\sigma\right)d\sigma+\frac{i}{4}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right)d\sigma- \tag{10b}$$
$$-\frac{1}{2}\mu(s)+\frac{i}{4}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right)d\sigma=F(s),\quad s\in\Gamma^2,$$

where $\rho(s)$ is defined in terms of $\nu(s)$ in (7).

To derive limit formulae for the angular potential we used its expression in the form of a double layer potential (6).

Equation (10a) is obtained as $x\to x(s)\in(\Gamma^1)^{\pm}$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$, the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations written above we have the conditions (5).

Subtracting the integral equations (10a) and using (7) we find

$$\rho(s)=\left(F^+(s)-F^-(s)\right)\in C^{1,\lambda}(\Gamma^1),$$

$$\nu(s)=\left(F'^+(s)-F'^-(s)\right)\in C^{0,\lambda}(\Gamma^1),\quad F'^{\pm}(s)=\frac{d}{ds}F^{\pm}(s). \tag{11}$$

We note that $\nu(s)$ is found completely and satisfies all required conditions, in particular, (5). Hence, the angular potential (4), (6) is found completely as well.

We introduce the functions $f_1(s)$ and $f_2(s)$ by the formulae

$$f_1(s)=\frac{1}{2}\left(F^+(s)+F^-(s)\right)-\frac{i}{4}\int\limits_{\Gamma^1}\left(F'^+(\sigma)-F'^-(\sigma)\right)V\left(x(s),\sigma\right)d\sigma,\quad s\in\Gamma^1, \tag{12a}$$

(12b)
$$f_2(s) = F(s) - \frac{i}{4}\int\limits_{\Gamma^1} \nu(\sigma)\frac{\partial}{\partial \mathbf{n}_x} V\left(x(s),\sigma\right) d\sigma =$$
$$= F(s) + \frac{i}{4}\int\limits_{\Gamma^1} \rho(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\frac{\partial}{\partial \mathbf{n}_y}\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right) d\sigma, \quad s\in\Gamma^2.$$

As shown in [3], if $s \in \Gamma^1$, then $f_1(s) \in C^{1,p_0}(\Gamma^1)$ where $p_0 = \lambda$ if $0 < \lambda < 1$ and $p_0 = 1 - \epsilon_0$ for any $\epsilon_0 \in (0,1)$ if $\lambda = 1$. Clearly, $f_2(s) \in C^0(\Gamma^2)$.

Adding the integral equations (10a) and taking into account (10b) we obtain the integral equations for $\mu(s)$ on Γ^1 and Γ^2

(13a)
$$w[\mu](x(s))|_{\Gamma^1} = \frac{i}{4}\int\limits_{\Gamma^1} \mu(\sigma)\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right) d\sigma +$$
$$+\frac{i}{4}\int\limits_{\Gamma^2} \mu(\sigma)\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right) d\sigma = f_1(s), \quad s\in\Gamma^1,$$

(13b)
$$-\frac{1}{2}\mu(s) + \frac{i}{4}\int\limits_{\Gamma} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right) d\sigma = f_2(s), \quad s\in\Gamma^2,$$

where $f_1(s)$ and $f_2(s)$ are given in (12), and the limit values of the function (9) as $x \to x(s) \in \Gamma^1$, $x \in \mathcal{D}$ are denoted by $w[\mu](x(s))|_{\Gamma^1}$.

Thus, if $\mu(s)$ is a solution of equations (13) from the space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, then the potential (8) with $\nu(s)$ from (11) satisfies all conditions of the problem **U**.

The following theorem holds.

Theorem 2. *Let $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$ and the conditions (3) hold. If the equations (13) have a solution $\mu(s)$ from the Banach space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0,1]$, $q \in [0,1)$, then a solution of the problem* **U** *is given by (8), where $\nu(s)$ is defined in (11).*

If $s \in \Gamma^2$, then (13b) is an equation of the second kind. The kernel in the integral term in (13b) is a continuous function for $s \in \Gamma^2$, $\sigma \in \Gamma$, because $\Gamma^2 \in C^{2,0}$ (see [3] for details). If $s \in \Gamma^1$, then (13a) is an equation of the first kind and its kernel has the logarithmic singularity when $s = \sigma \in \Gamma^1$, because

(14)
$$\mathcal{H}_0^{(1)}(z) = \frac{2i}{\pi}\ln\frac{z}{\beta} + h(z),$$

where $h(z) \in C^1[0,+\infty)$. Moreover, as $z \to 0+0$
$$h(z) = const + O(z^2 \ln z), \quad h'(z) = O(z\ln z), \quad h''(z) = O(\ln z).$$

Our further treatment will be aimed to the proof of the solvability of (13) in the Banach space $C_q^{\omega}(\Gamma^1) \cap C^0(\Gamma^2)$. Moreover, we reduce (13) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

By differentiating (13a) on Γ^1 in accordance with [4, sect.14], we reduce (13a) to the following singular integral equation on Γ^1

$$\frac{\partial}{\partial s}w[\mu](x(s))=\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma+ \tag{15a}$$

$$+\frac{i}{4}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\partial}{\partial s}h\left(\beta\,|x(s)-y(\sigma)|\right)d\sigma+$$

$$+\frac{i}{4}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial s}\mathcal{H}_0^{(1)}\left(\beta\,|x(s)-y(\sigma)|\right)d\sigma=f_1'(s),\quad s\in\Gamma^1,$$

where the function $h(z)$ is defined by (14), and $\varphi_0(x,y)$ is the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$ and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$ if $x\neq y$.

Equation (13b) on Γ^2 we rewrite in the form

$$\mu(s)+\int\limits_{\Gamma}\mu(\sigma)A_2(s,\sigma)d\sigma=-2f_2(s),\quad s\in\Gamma^2, \tag{15b}$$

where

$$A_2(s,\sigma)=-\frac{i}{2}\frac{\partial}{\partial\mathbf{n}_x}\mathcal{H}_0^{(1)}\left(\beta\,|x(s)-y(\sigma)|\right)\in C^0(\Gamma^2\times\Gamma).$$

Remark. Evidently, $f_2(a_n^2)=f_2(b_n^2)$ and $A_2(a_n^2,\sigma)=A_2(b_n^2,\sigma)$ for any $\sigma\in\Gamma$ $(n=1,...,N_2)$. Hence, if $\mu(s)$ is a solution of equation (15b) from $C^0\left(\bigcup\limits_{n=1}^{N_2}[a_n^2,b_n^2]\right)$, then, according to the equality (15b), $\mu(s)$ automatically satisfies matching conditions $\mu(a_n^2)=\mu(b_n^2)$ for $n=1,...,N_2$ and therefore belongs to $C^0(\Gamma^2)$. This observation is true for equation (13b) also and can be helpful for finding numerical solutions, since we may abandon matching conditions $\mu(a_n^2)=\mu(b_n^2)$ $(n=1,...,N_2)$, which are fulfilled automatically.

We note that equation (15a) is equivalent to (13a) on Γ^1 if (15a) is accompanied by the following additional conditions

$$w[\mu](x(a_n^1))=f_1(a_n^1),\quad n=1,...,N_1\ . \tag{16}$$

The system (15), (16) is equivalent to the equations (13).

It follows from [3, lemma 3] that

$$\left[\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}-\frac{1}{\sigma-s}\right]\in C^{0,\lambda}(\Gamma^1\times\Gamma^1).$$

Therefore we can rewrite (15a) in the form

$$2\frac{\partial}{\partial s}w[\mu](x(s)) = \frac{1}{\pi}\int_{\Gamma^1}\mu(\sigma)\frac{d\sigma}{\sigma - s} + \int_{\Gamma}\mu(\sigma)Y(s,\sigma)d\sigma = \tag{17}$$

$$= 2f_1'(s), \quad s \in \Gamma^1,$$

where

$$Y(s,\sigma) = \Bigg\{(1-\delta(\sigma))\left[\frac{1}{\pi}\left(\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|} - \frac{1}{\sigma - s}\right) + \right.$$

$$\left. + \frac{i}{2}\frac{\partial}{\partial s}h\left(\beta\,|x(s)-y(\sigma)|\right)\right] +$$

$$+ \frac{i}{2}\delta(\sigma)\frac{\partial}{\partial s}\mathcal{H}_0^{(1)}\left(\beta\,|x(s)-y(\sigma)|\right)\Bigg\} \in C^{0,p_0}(\Gamma^1\times\Gamma),$$

$\delta(\sigma) = 0$ if $\sigma \in \Gamma^1$ and $\delta(\sigma) = 1$ if $\sigma \in \Gamma^2$, $p_0 = \lambda$ if $0 < \lambda < 1$ and $p_0 = 1 - \epsilon_0$ for any $\epsilon_0 \in (0,1)$ if $\lambda = 1$ (see [3, theorem 6]).

4. The Fredholm integral equation and the solution of the problem

Inverting the singular integral operator in (17) we arrive at the following integral equation of the second kind [3], [4]:

$$\mu(s) + \frac{1}{Q_1(s)}\int_{\Gamma}\mu(\sigma)A_1(s,\sigma)d\sigma + \frac{1}{Q_1(s)}\sum_{n=0}^{N_1-1}G_n s^n = \tag{18}$$

$$= \frac{1}{Q_1(s)}\Phi_1(s), \quad s \in \Gamma^1,$$

where

$$A_1(s,\sigma) = -\frac{1}{\pi}\int_{\Gamma^1}\frac{Y(\xi,\sigma)}{\xi - s}Q_1(\xi)d\xi, \qquad \Phi_1(s) = -\frac{1}{\pi}\int_{\Gamma^1}\frac{2Q_1(\sigma)f_1'(\sigma)}{\sigma - s}d\sigma,$$

$$Q_1(s) = \prod_{n=1}^{N_1}\left|\sqrt{s-a_n^1}\sqrt{b_n^1 - s}\right|\operatorname{sign}(s - a_n^1),$$

and $G_0, \ldots, G_{N_1-1}$ are arbitrary constants.

It can be shown using the properties of singular integrals [2], [4], that $\Phi_1(s)$, $A_1(s,\sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Consequently, any integrable on Γ^1 and continuous on Γ^2 solution of equation (18) belongs to $C^{\omega}_{1/2}(\Gamma^1)$ with some $\omega \in (0,1]$, and below we look for $\mu(s)$ on Γ^1 in this space.

We put

$$Q(s) = (1-\delta(s))\,Q_1(s) + \delta(s), \quad s \in \Gamma.$$

Instead of $\mu(s) \in C^{\omega}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ we introduce the new unknown function

$$\mu_*(s) = \mu(s)Q(s) \in C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$$

and rewrite (18), (15b) in the form of one equation

$$\mu_*(s) + \int_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)d\sigma + (1-\delta(s)) \sum_{n=0}^{N_1-1} G_n s^n = \Phi(s), \quad s \in \Gamma, \tag{19}$$

where

$$A(s,\sigma) = (1-\delta(s))\, A_1(s,\sigma) + \delta(s)A_2(s,\sigma),$$
$$\Phi(s) = (1-\delta(s))\, \Phi_1(s) - 2\delta(s)f_2(s).$$

To derive equations for $G_0, ..., G_{N_1-1}$ we substitute $\mu(s)$ from (18), (15b) in the conditions (16), then in terms of $\mu_*(s)$ we obtain

$$\int_{\Gamma} Q^{-1}(\xi)\mu_*(\xi)l_n(\xi)d\xi + \sum_{m=0}^{N_1-1} B_{nm}G_m = H_n, \quad n = 1, ..., N_1 \ , \tag{20}$$

where

$$l_n(\xi) = -w\left[Q^{-1}(\cdot)A(\cdot,\xi)\right](x(a_n^1)), \quad H_n = -w\left[Q^{-1}(\cdot)\Phi(\cdot)\right](x(a_n^1)) + f_1(a_n^1),$$

$$B_{nm} = -w\left[Q^{-1}(\cdot)\,(1-\delta(\cdot))\,(\cdot)^m\right](x(a_n^1)). \tag{21}$$

By $\cdot$ we denote the variable of integration in the potential (9).

Thus, the system of equations (15), (16) for $\mu(s)$ has been reduced to the system (19), (20) for the function $\mu_*(s)$ and constants $G_0, ..., G_{N_1-1}$. It is clear from our consideration that any solution of system (19), (20) gives a solution of the system (15), (16).

As noted above, $\Phi_1(s)$ and $A_1(s,\sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. More precisely (see [3], [4]), $\Phi_1(s) \in C^{0,p}(\Gamma^1)$, $p = \min\{1/2, \lambda\}$ and $A_1(s,\sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma \in \Gamma$. We arrive at the following assertion.

Lemma 1. *Let* $\Gamma^1 \in C^{2,\lambda}$, $\lambda \in (0,1]$, $\Gamma^2 \in C^{2,0}$, $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. *If* $\mu_*(s)$ *from* $C^0(\Gamma)$ *satisfies the equation (19), then*

$$\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2).$$

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ holds if conditions (3) hold.

Hence below we shall seek $\mu_*(s)$ from $C^0(\Gamma)$.

Since $A(s,\sigma) \in C^0(\Gamma \times \Gamma)$, the integral operator from (19):

$$\mathbf{A}\mu_* = \int_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself.

We rewrite (19) in the operator form

$$(I + \mathbf{A})\mu_* + PG = \Phi \ , \tag{22}$$

where P is the operator of multiplication of the row $P = (1 - \delta(s))\,(s^0, ..., s^{N_1-1})$ by the column $G = (G_0, ..., G_{N_1-1})^T$. The operator P is finite-dimensional from E_{N_1} into $C^0(\Gamma)$ and therefore compact.

Now we rewrite equations (20) in the form

$$I_{N_1}G + L\mu_* + (B - I_{N_1})G = H \ , \tag{23}$$

where $H = (H_1, ..., H_{N_1})^T$ is a column of N_1 elements, I_{N_1} is the identity operator in E_{N_1}, B is a $N_1 \times N_1$ matrix consisting of the elements B_{nm} from (21). The operator L acts from $C^0(\Gamma)$ into E_{N_1}, so that $L\mu_* = (L_1\mu_*, ..., L_{N_1}\mu_*)^T$, where

$$L_n\mu_* = \int_\Gamma Q^{-1}(\xi)\mu_*(\xi)l_n(\xi)d\xi \ , \qquad n = 1, \ldots, N_1.$$

The operators $(B - I_{N_1})$, L are finite-dimensional and therefore compact.

We consider the columns $\tilde{\mu} = \begin{pmatrix} \mu_* \\ G \end{pmatrix}$, $\tilde{\Phi} = \begin{pmatrix} \Phi \\ H \end{pmatrix}$ in the Banach space $C^0(\Gamma) \times E_{N_1}$ with the norm $\|\tilde{\mu}\|_{C^0(\Gamma)\times E_{N_1}} = \|\mu_*\|_{C^0(\Gamma)} + \|G\|_{E_{N_1}}$.

We write the system (22), (23) in the form of one equation

$$(\mathbf{I} + \mathbf{R})\tilde{\mu} = \tilde{\Phi} \ , \quad \mathbf{R} = \begin{pmatrix} \mathbf{A} & P \\ L & B - I_{N_1} \end{pmatrix} , \tag{24}$$

where $\mathbf{I}$ is an identity operator in the space $C^0(\Gamma) \times E_{N_1}$. It is clear, that $\mathbf{R}$ is a compact operator mapping $C^0(\Gamma) \times E_{N_1}$ into itself. Therefore, (24) is a Fredholm equation of the second kind and index zero in this space.

Let us show that homogeneous equation (24) has only a trivial solution. Then, according to Fredholm's theorems, the inhomogeneous equation (24) has a unique solution for any right-hand side. We shall prove this by a contradiction.

Let $\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$ be a non-trivial solution of the homogeneous equation (24). According to the lemma 1, $\tilde{\mu}^0 = \begin{pmatrix} \mu_*^0 \\ G^0 \end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}$, $p = \min\{\lambda, 1/2\}$. Therefore the function $\mu^0(s) = \mu_*^0(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ and the column G^0 convert the homogeneous equations (18), (15b), (20) into identities. The homogeneous identity (15b) is equivalent to the homogeneous identity (13b). Using the homogeneous identities (18), (15b) we check, that the homogeneous identities (20) are equivalent to

$$w\left[\mu^0\right](x(a_n^1)) = 0 \ , \quad n = 1, ..., N_1. \tag{25a}$$

Besides, acting on the homogeneous identity (18) with a singular operator with the kernel $(s-t)^{-1}$ we find that $\mu^0(s)$ satisfies the homogeneous equation (17) :

$$\frac{\partial}{\partial s} w\left[\mu^0\right](x(s))\Big|_{\Gamma^1} = 0 \ . \tag{25b}$$

It follows from (25) that $\mu^0(s)$ satisfies the homogeneous equation (13a). Thus, $\mu^0(s)$ satisfies the homogeneous equations (13). On the basis of theorem 2,

$$u[0,\mu^0](x) \equiv w[\mu^0](x)$$

is a solution of the homogeneous problem **U**. According to theorem 1:

$$w[\mu^0](x) \equiv 0, \quad x \in \mathcal{D}\backslash\Gamma^1.$$

Using the limit formulae for normal derivatives of a single-layer potential on Γ^1, we have

$$\lim_{x\to x(s)\in(\Gamma^1)^+} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) - \lim_{x\to x(s)\in(\Gamma^1)^-} \frac{\partial}{\partial \mathbf{n}_x} w[\mu^0](x) = \mu^0(s) \equiv 0, \quad s \in \Gamma^1.$$

Hence, $w[\mu^0](x) = w_2[\mu^0](x) \equiv 0$, $x \in \mathcal{D}$, and $\mu^0(s)$ satisfies the homogeneous equation (13b), which can be written as

$$-\frac{1}{2}\mu^0(s) + \frac{i}{4}\int\limits_{\Gamma^2} \mu^0(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\mathcal{H}_0^{(1)}\left(\beta\left|x(s)-y(\sigma)\right|\right) d\sigma = 0, \quad s \in \Gamma^2. \tag{26}$$

The Fredholm integral equation (26) is known in classical mathematical physics. We arrive at (26) when solving the Neumann problem for the Helmholtz equation (2a) in the domain $\mathcal{D}$ by the single layer potential. Since $\mu^0(s) \in C^0\left(\Gamma^2\right)$, we obtain from properties of the single layer potential [8] that
(*) $w_2\left[\mu^0\right](x) \in C^0(\overline{R^2\backslash\mathcal{D}}) \cap C^2\left(R^2\backslash\overline{\mathcal{D}}\right)$;
(**) there exists a uniform for all $x \in \Gamma^2$ limit of $(\mathbf{n}_x, \nabla_{\hat{x}} w_2(\hat{x}))$ as $\hat{x} \in R^2\backslash\overline{\mathcal{D}}$ tends to $x \in \Gamma^2$ along the normal $\mathbf{n}_x$.
Moreover [8], the single layer potential $w_2[\mu^0](x)$ belongs to $C^0(R^2)$, therefore

$$w_2|_{\Gamma^2} = 0.$$

Note that $w_2\left[\mu^0\right](x)$ satisfies the following homogeneous Dirichlet problem in the domain $R^2\backslash\overline{\mathcal{D}}$:

$$\Delta w_2(x) + \beta^2 w_2(x) = 0, \quad x \in R^2\backslash\overline{\mathcal{D}}, \quad \operatorname{Im}\beta > 0,$$

$$w_2|_{\Gamma^2} = 0.$$

Besides, $w_2\left[\mu^0\right](x)$ satisfies the conditions (2c) as $|x| \to \infty$ if $R^2\backslash\overline{\mathcal{D}}$ is an exterior domain. It can be shown by the method of energy equalities that there is only the trivial solution of this homogeneous Dirichlet problem if smoothness conditions (*), (**) are imposed. Therefore,

$$w_2\left[\mu^0\right](x) \equiv 0, \quad x \in R^2\backslash\overline{\mathcal{D}},$$

so $w_2\left[\mu^0\right](x) \equiv 0$ in $R^2\backslash\Gamma^2$. Using the jump formula for the normal derivative of the single layer potential on the integration curve, we obtain

$$\lim_{\substack{\hat{x}\to x(s)\in\Gamma^2\\ \hat{x}\in R^2\backslash\overline{\mathcal{D}}}}\left(\mathbf{n}_x, \nabla_{\hat{x}} w_2\left[\mu^0\right](\hat{x})\right) - \lim_{\substack{\hat{x}\to x(s)\in\Gamma^2\\ \hat{x}\in \mathcal{D}}}\left(\mathbf{n}_x, \nabla_{\hat{x}} w_2\left[\mu^0\right](\hat{x})\right) =$$

$$= \mu^0(s) \equiv 0, \quad s \in \Gamma^2.$$

Here the limits are understood along the normal according to the property (**) and the point 4) of the definition of the class **K**. Therefore, the equation (26) has only the trivial solution $\mu^0(s) \equiv 0$ in $C^0(\Gamma^2)$.

Consequently, if $s \in \Gamma$, then $\mu^0(s) \equiv 0$, $\mu^0_*(s) = \mu^0(s)Q^{-1}(s) \equiv 0$ and it follows from the homogeneous identity (18) for $\mu^0(s)$ and $G^0_0, ..., G^0_{N_1-1}$ that

$$G^0 = (G^0_0, ..., G^0_{N_1-1})^T \equiv 0.$$

Hence, $\tilde{\mu}^0 \equiv 0$ and we arrive at the contradiction to the assumption that $\tilde{\mu}^0$ is a non-trivial solution of the homogeneous equation (24). Thus, the homogeneous Fredholm equation (24) has only a trivial solution in $C^0(\Gamma) \times E_{N_1}$.

We have proved the following assertion.

Theorem 3. *If* $\Gamma^1 \in C^{2,\lambda}$, $\lambda \in (0,1]$, $\Gamma^2 \in C^{2,0}$, *then (24) is a Fredholm equation of the second kind and index zero in the space* $C^0(\Gamma) \times E_{N_1}$. *Moreover, equation (24) has a unique solution* $\tilde{\mu} = \begin{pmatrix}\mu_*\\ G\end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$ *for any* $\tilde{\Phi} = \begin{pmatrix}\Phi\\ H\end{pmatrix} \in C^0(\Gamma) \times E_{N_1}$.

As a consequence of the theorem 3 and lemma 1 we obtain the corollary.

Corollary. *If* $\Gamma^1 \in C^{2,\lambda}$, $\lambda \in (0,1]$, $\Gamma^2 \in C^{2,0}$, *then equation (24) has a unique solution* $\tilde{\mu} = \begin{pmatrix}\mu_*\\ G\end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}$ *for any*

$$\tilde{\Phi} = \begin{pmatrix}\Phi\\ H\end{pmatrix} \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1},$$

where $p = \min\{\lambda, 1/2\}$.

We recall that $\tilde{\Phi}$ belongs to the class of smoothness required in the corollary if conditions (3) hold. Besides, equation (24) is equivalent to the system (19), (20). As mentioned above, if the function $\mu_*(s)$ from $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ and the constants $G_0, ..., G_{N_1-1}$ constitute a solution of system (19), (20), then

$$\mu(s) = \mu_*(s)Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$$

is a solution of system (15), (16), and therefore $\mu(s)$ satisfies equations (13). We obtain the following statement.

Lemma 2. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$ and the conditions (3) hold, then the equations (13) have a solution $\mu(s)$ from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$. This solution is expressed by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$, where $\mu_*(s)$ from $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ is found by solving the Fredholm equation (24), which is uniquely solvable.*

Remark. The solution of the equations (13) ensured by lemma 2 is unique in the space $C^{\omega_o}_{1/2}(\Gamma^1) \times C^0(\Gamma^2)$ for any $\omega_o \in (0, p]$. The proof can be given by a contradiction to the assumption that the homogeneous equations (13) have a nontrivial solution in this space. The proof is almost the same as the proof of Theorem 3. Consequently, the numerical solution of the equations (13) can be obtained by the direct numerical inversion of the integral operator of the system (13). In doing so, Hölder functions can be approximated by continuous piecewise linear functions, which also obey Hölder inequality. The simplification for numerical solving the equations (13) is suggested in the remark to the equation (15b) in the section 3.

On the basis of the theorem 2 we arrive at the final result.

Theorem 4. *If $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$ and conditions (3) hold, then the solution of the problem* **U** *exists and is given by (8), where $\nu(s)$ is defined in (11) and $\mu(s)$ is a solution of the equations (13) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$ ensured by lemma 2.*

It can be checked directly that the solution of the problem **U** satisfies condition (1) with $\epsilon = -1/2$. Explicit expressions for singularities of the solution gradient at the end-points of the open curves can be easily obtained with the help of formulae presented in [3].

Theorem 4 ensures existence of a classical solution of the problem **U** when $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$ and the conditions (3) hold. The uniqueness of the classical solution follows from the theorem 1. On the basis of our consideration we suggest the following scheme for solving the problem **U**. First, we find the unique solution of the Fredholm equation (24) from $C^0(\Gamma) \times E_{N_1}$. This solution automatically belongs to

$$C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2) \times E_{N_1}, \quad p = \min\{\lambda, 1/2\}.$$

Second, we construct the solution of the equations (13) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. Finally, by substituting $\nu(s)$ from (11) and $\mu(s)$ in (8) we obtain the solution of the problem **U**.

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