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L_p ERROR ESTIMATES AND SUPERCONVERGENCE FOR FINITE ELEMENT APPROXIMATIONS FOR NONLINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL PROBLEMS

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ABSTRACT. In this paper we consider finite element methods for nonlinear hyperbolic integro-differential problems defined in $\Omega \subset R^d (d \le 4)$. A new initial approximation of $u_t(0)$ is taken. Optimal order error estimates in L_p for $2 \le p \le \infty$ are established for arbitrary order finite element. One order superconvergence in $W^{1,p}$ for $2 \le p \le \infty$ are demonstrated as well.

1. INTRODUCTION

Consider the following initial boundary value problem for the nonlinear hyperbolic problem with memory:

$$\begin{aligned}
&(a)\quad u_{tt} - \nabla\cdot\{a(x,u)\nabla u + \int_0^t b(u,t,\tau)\nabla u(x,\tau)d\tau = f(x,u), && (x,t) \in \Omega \times J,\\
&(b)\quad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(0), && x \in \Omega,\\
&(c)\quad u(x,t) = 0, && (x,t) \in \partial\Omega \times J,
\end{aligned} \tag{1.1}$$

where $J = [0,T]$, Ω is a bounded domain in $R^d(d \le 4)$ with smooth boundary. We assume data a, b, f, u_0, u_1 together with their derivatives to be bounded on $\Omega \times R$ and

$$0 < a^* \le a(x,s), \qquad (x,s) \in \Omega \times R.$$

The global nature of those assumptions is not restrictive, as we shall show below that the approximate solutions are uniformly close to the exact solution u of (1.1).

The object of this paper is to demonstrate optimal error estimates of finite element approximation in L_p for $2 \le p \le \infty$ and to derive the superconvergence in $W^{1,p}$ for

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$2 \leq p \leq \infty$ between the numerical solution and the Ritz-Volterra projection of the exact solution of (1.1).

For this purpose, let $\{S_h\}_{0<h\leq 1}$ be a family of finite-dimensional subspace of $H_0^1(\Omega)\bigcap W^{1,\infty}(\Omega)$, with the following approximation properties : for some $r \geq 2, 1 \leq s \leq r, 2 \leq p \leq \infty$ and C a positive constant

$$\inf_{\chi\in S_h}\{\|\chi - w\|_{0,p} + h\|\chi - w\|_{1,p}\} \leq Ch^s\|w\|_{s,p}, \qquad w \in W^{s,p}(\Omega)\bigcap H_0^1(\Omega). \tag{1.2}$$

In addition,we assume that $\{S_h\}$ satisfies the standard inverse properties in finite element spaces[10].

First we define the Ritz projection operator $R_h : H_0^1(\Omega) \to S_h$ for $0 \leq t \leq T$ by

$$(a(u)\nabla(R_h w - w), \nabla\chi) = 0, \quad \chi \in S_h. \tag{1.3}$$

Now we define the Ritz-Volterra projection operator $V_h : H_0^1(\Omega) \to S_h$ for $0 \leq t \leq T$ by

$$(a(u)\nabla(V_h w - w), \nabla\chi) + (\int_0^t b(u,t,\tau)\nabla(V_h w - w)d\tau, \nabla\chi) = 0, \quad \chi \in S_h. \tag{1.4}$$

Obviously, when $t = 0$, V_h is the same as Ritz projection operator R_h.

The semidiscrete finite-element approximation to the solution u of (1.1) is to find a map $U(t) : J \to S_h$ such that

$$\begin{aligned}&(a) \quad (U_{tt},\chi) + (a(U)\nabla U, \nabla\chi) + (\int_0^t b(U)\nabla U d\tau, \nabla\chi) = (f(U),\chi), \qquad \chi \in S_h,\\ &(b) \quad U(0) = U_0, \qquad U_t(0) = U_1,\end{aligned} \tag{1.5}$$

and $U_0, U_1 \in S_h$ is the approximation to u_0 and u_1, respectively, which will be given in (1.7), (1.8) below, respectively.

Let the error

$$U - u = (U - V_h u) + (V_h u - u) = \xi + \eta. \tag{1.6}$$

Then we choose the initial approximation $U(0)$ to satisfy

$$\begin{aligned}A(\xi(0),\chi) &\equiv (a(u_0)\nabla\xi(0), \nabla\chi) + (a_u(u_0)\xi(0)\nabla u_0, \nabla\chi) + \lambda(\xi(0),\chi)\\ &= -(a_u(u_0)\eta(0)\nabla u_0, \nabla\chi), \qquad \chi \in S_h,\end{aligned} \tag{1.7}$$

where λ is selected large enough to ensure the coerciveness of the bilinear form $A(\cdot,\cdot)$ over H^1.

We choose the initial approximation $U_t(0)$ to satisfy

$$\begin{aligned}(a(u_0)\nabla U_1, \nabla\chi) &= (a(u_0)\nabla u_1, \nabla\chi) - (a_u(u_0)u_1\nabla\eta(0), \nabla\chi)\\ &\quad -(b(u_0,0,0)\nabla\eta(0), \nabla\chi), \qquad \chi \in S_h.\end{aligned} \tag{1.8}$$

Finite element methods for parabolic integro-differential equations have been studied by several authors. For example, Cannon and Y,Lin[2] obtained optimal L_2 error estimates. In [8,11], optimal H^1 norm and maximum norm error estimates was obtained. In [3], several superconvergence results for the error between the approximate solution and the Ritz-Volterra projection of the exact solution is derived. For hyperbolic equations, Yirang Yuan,Hong Wang[13] get optimal L_2 and H^1 norms error estimates. But for nonlinear hyperbolic integro-differential equations, the theory is not perfect,the L_p and $W^{1,p}(2 \le p \le \infty)$ norms error estimates as well as some superconvergence estimates have not been derived before. In this paper,we shall use a initial approximation of $u(0)$ similar to [1] and take a new initial approximation of $u_t(0)$, this initial conditions enable us to obtain one order superconvergence of $U - V_h u$ in $W^{1,p}(2 \le p \le \infty)$ and optimal $L_p(2 \le p \le \infty)$ error estimates.

The rest of this paper is organized as follows. In section 2, some nesessary lemmas will be proved which are essential in the analysis. In section 3, optimal L_p error estimates and superconvergence in $W^{1,p}$ for $2 \le p < \infty$ will be presented. In section 4, maximun norm error estimates and superconvergence of gradients will be demonstrated.

2. Lemmas

In this section we shall given the error estimates of Ritz-Volterra projection and prove the estimates for the initial value error. In addition, we shall also establish L_2 estimates for $\xi_t, \xi_{tt}, \nabla\xi$ and $\nabla\xi_t$.

The following lemma is contained in [10, 12, 14].

Lemma 1. *For $t \in J, 0 \le l \le 3$ and $1 \le s \le r$ we have*

$$(a) \quad \|D_t^l(w - V_h w)\|_{0,p} + h\|D_t^l(w - V_h w)\|_{1,p} \le Ch^s, \qquad 2 \le p < \infty, r \ge 2,$$

$$(b) \quad \|D_t^l(w - V_h w)\|_{0,\infty} \le \begin{cases} Ch^s, & r > 2, \\ Ch^s \log h^{-1}, & r = 2, \end{cases} \tag{2.1}$$

$$(c) \quad \|D_t^l(w - V_h w)\|_{1,\infty} \le Ch^{s-1}, \qquad r \ge 2.$$

The following lemma is contained in [1, 14].

Lemma 2. *If $u \in L_\infty(0,t; W^{1,\infty}(\Omega) \bigcap W^{2,d}(\Omega))$, $u_t, u_{tt} \in L_\infty(0,t; W^{1,\infty}(\Omega))$, then $\nabla V_h u, \nabla(V_h u)_t$ and $\nabla(V_h u)_{tt}$ are uniformly bounded on $[0,t]$.*

Now, let us establish the estimates for $\nabla\xi(0), \xi_{tt}(0)$.

Lemma 3. *If* $u(0) \in W^{s,4}(\Omega)$, $u_t(0), u_{tt}(0) \in H^s(\Omega)$, *for* $2 \le s \le r$ *and* $r \ge 2$, *then*

$$\begin{aligned} &(a)\quad \|\xi(0)\|_1 \le Ch^s,\\ &(b)\quad \xi_t(0) = 0,\\ &(c)\quad \|\xi_{tt}(0)\| \le Ch^s. \end{aligned} \tag{2.2}$$

Proof. (a) Take $\chi = \xi(0)$ in (1.7) to get

$$\|\xi(0)\|_1^2 \le C\|\eta(0)\|\|\nabla\xi(0)\|,$$

which, by (2.1a), implies (2.2a).
(b) By differentiating (1.4) with respect to t we obtain

$$(a_u(u)u_t\nabla\eta, \nabla\chi) + (a(u)\nabla\eta_t, \nabla\chi) + (\int_0^t b_t(u)\nabla\eta d\tau, \nabla\chi) + (b(u,t,t)\nabla\eta, \nabla\chi) = 0, \quad \chi \in S_h, \tag{2.3}$$

from (1.8) we get

$$\begin{aligned}(a(u_0)\nabla\xi_t(0), \nabla\chi) = &-(a(u_0)\nabla\eta_t(0), \nabla\chi) - (a_u(u_0)u_1\nabla\eta(0), \nabla\chi)\\ &-(b(u_0,0,0)\nabla\eta(0), \nabla\chi), \qquad \chi \in S_h.\end{aligned} \tag{2.4}$$

Take $\chi = \xi_t(0)$ in (2.3) with $t = 0$, and combine (2.4) we have

$$(a(u_0)\nabla\xi_t(0), \nabla\xi_t(0)) = 0.$$

So we get (2.2b).
(c) Combine (1.1a), (1.4), (1.5) to yield the error equation

$$\begin{aligned}&(\xi_{tt}, \chi) + (a(U)\nabla\xi, \nabla\chi) + (\int_0^t b(U)\nabla\xi d\tau, \nabla\chi)\\ &= (f(U) - f(u) - \eta_{tt}, \chi) + ((a(u) - a(U))\nabla V_h u, \nabla\chi) + (\int_0^t (b(u) - b(U))\nabla V_h u d\tau, \nabla\chi),\\ &\qquad \chi \in S_h.\end{aligned} \tag{2.5}$$

Now subtract (1.7) from (2.5) with $t = 0$ and set $\chi = \xi_{tt}(0)$ to derive (in the sequal, t=0 will be omitted)

$$\begin{aligned}\|\xi_{tt}\|^2 &= (f(U) - f(u) - \eta_{tt} + \lambda\xi, \xi_{tt}) + (a_u(U)(U-u)\nabla u - (a(U) - a(u))\nabla u, \nabla\xi_{tt})\\ &\quad +((a(U) - a(u))\nabla(u - U), \nabla\xi_{tt})\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{2.6}$$

Then it follows from (2.1a) (2.2a), imbedding inequalities[4] and inverse properties

$$
\begin{aligned}
I_1 &\le (\|\xi\| + \|\eta\| + \|\eta_{tt}\|)\|\xi_{tt}\| \le Ch^s\|\xi_{tt}\|,\\
I_2 &= (\int_0^1 [a_u(u) - a_u(u + s(U-u))]ds(U-u)\nabla u, \nabla\xi_{tt})\\
&= (\int_0^1 [\int_0^1 a_{uu}(u + s(1-\tau)(U-u))d\tau](-s)ds(U-u)^2\nabla u, \nabla\xi_{tt})\\
&\le C(\|\xi\|_{0,4}^2 + \|\eta\|_{0,4}^2)\|\xi_{tt}\|_1\\
&\le C(\|\xi\|_1^2 + \|\eta\|_{0,4}^2)h^{-1}\|\xi_{tt}\|\\
&\le Ch^{2s-1}\|\xi_{tt}\|,\\
I_3 &\le C(\|\xi\|_{0,4} + \|\eta\|_{0,4})(\|\xi\|_{1,4} + \|\eta\|_{1,4})\|\xi_{tt}\|_1\\
&\le C(\|\xi\|_1 + \|\eta\|_{0,4})(h^{-1}\|\xi\|_1 + \|\eta\|_{1,4})h^{-1}\|\xi_{tt}\|\\
&\le Ch^{2s-2}\|\xi_{tt}\|.
\end{aligned}
$$

Collecting the estimates of $I_1 - I_3$ with (2.6) completes the proof.

Our next aim is to derive estimates for ξ_t and $\nabla\xi$.

Lemma 4. *Assume that* $u(0) \in W^{s,4}(\Omega), u_t(0), u_{tt}(0) \in H^s(\Omega).u, u_t, u_{tt} \in L_2(0,t; H^s(\Omega))$. $u \in L_\infty(0,t; W^{2,d}(\Omega) \bigcap W^{1,\infty}(\Omega))$ *and* $u_t \in L_\infty(0,t; W^{1,\infty}(\Omega))$, *then for* $t \in J$

$$\|\xi_t\| + \|\xi\|_1 \le Ch^s, \tag{2.7}$$

where $2 \le s \le r, r \ge 2$ when $d = 1,2$ or 3, $3 \le s \le r, r \ge 3$ when $d = 4$.

Proof. Setting $\chi = \xi_t$ in (2.5), by using $\epsilon-$inequality and Lemma 2.2, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}(a(U)\nabla\xi, \nabla\xi)\\
&\le \frac{d}{dt}((a(u) - a(U))\nabla V_h u, \nabla\xi) - \frac{d}{dt}\int_0^t ((b(u) - b(U))\nabla V_h u d\tau, \nabla\xi) + C\int_0^t (\|\xi\|_1^2 + \|\eta\|^2)d\tau.\\
&\quad -\frac{d}{dt}(\int_0^t b(U)\nabla\xi d\tau, \nabla\xi) + C(\|\xi_t\|_{0,\infty}^2 \cdot \|\xi\|_1^2 + \|\xi\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2 + \|\xi_t\|^2).
\end{aligned}
\tag{2.8}
$$

Integrating (2.8) with respect to t, combine the inequality

$$\|\xi\|^2 = \|\xi(0)\|^2 + \int_0^t \frac{d}{dt}\|\xi\|^2 d\tau \le \|\xi(0)\|^2 + \int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2)d\tau.$$

By using $\epsilon-$inequality, we get

$$
\begin{aligned}
\|\xi\|_1^2 + \|\xi_t\|^2 &\le C\{\|\xi(0)\|_1^2 + \|\xi_t(0)\|^2 + \|\eta(0)\|^2\\
&\quad + \int_0^t (\|\xi_t\|_{0,\infty}^2 \cdot \|\xi\|_1^2 + \|\xi\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2 + \|\xi_t\|^2)d\tau\} + \epsilon\|\xi\|_1^2.
\end{aligned}
$$

If we assume that

$$\|\xi_t\|_{0,\infty} \cdot \|\xi\|_1 \le Ch^s, \tag{2.9}$$

then it follows from (2.1a) and (2.2) that

$$\|\xi\|_1^2 + \|\xi_t\|^2 \le C\{h^{2s} + \int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2)d\tau\}. \tag{2.10}$$

Applying Gronwall Lemma now yields

$$\|\xi\|_1 + \|\xi_t\| \le Ch^s. \tag{2.11}$$

Finally it remains to verify the induction hypothesis (2.9),.First note that (2.9), by (2.2), holds for $t = 0$. When $t \in (0, T]$, it follows from inverse property that

$$\|\xi_t\|_{0,\infty} \cdot \|\xi\|_1 \le Ch^{-\frac{d}{2}}\|\xi\|_1\|\xi_t\| \le Ch^{2s-\frac{d}{2}} = o(h^s),$$

which implies that (2.9) is valid, then the proof has been completed.
Now we establish the estimates for ξ_{tt} and $\nabla\xi_t$.

Lemma 5. *Assume that* $u(0) \in W^{s,4}(\Omega), u_t(0), u_{tt}(0) \in H^s(\Omega), u, u_t, u_{tt} \in L_2(0, t; H^s(\Omega))$. $u \in L_\infty(0, t; W^{2,d}(\Omega) \bigcap W^{1,\infty}(\Omega))$ *and* $u_t, u_{tt} \in L_\infty(0, t; W^{1,\infty}(\Omega))$, *then for* $t \in J$

$$\|\xi_{tt}\| + \|\xi_t\|_1 \le Ch^s, \tag{2.12}$$

where $2 \le s \le r, r \ge 2$ when $d = 1, 2$ or 3, $3 \le s \le r, r \ge 3$ when $d = 4$.
Proof. By differentiating (2.5) with respect to t, we obtain

$$\begin{aligned}
&(\xi_{ttt}, \chi) + (a(U)\nabla\xi_t, \nabla\chi) + (\int_0^t b_t(U)\nabla\xi d\tau, \nabla\chi)\\
&= -(a_u(U)U_t\nabla\xi, \nabla\chi) - (b(U)\nabla\xi, \nabla\chi) + ((f(U) - f(u) - \eta_{tt})_t, \chi)\\
&\quad +((a(u) - a(U))_t\nabla V_hu, \nabla\chi) + ((a(u) - a(U))\nabla(V_hu)_t, \nabla\chi)\\
&\quad +(\int_0^t (b_t(u) - b_t(U))\nabla V_hud\tau, \nabla\chi) + ((b(u) - b(U))\nabla V_hu, \nabla\chi), \qquad \chi \in S_h.
\end{aligned} \tag{2.13}$$

Setting $\chi = \xi_{tt}$ in (2.13), using $\epsilon-$inequality and Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\xi_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}(a(U)\nabla\xi_t, \nabla\xi_t)\\
&\le \frac{d}{dt}((a(u) - a(U))_t\nabla V_hu, \nabla\xi_t) + \frac{d}{dt}((b(u) - b(U))\nabla V_hu, \nabla\xi_t)\\
&\quad +\frac{d}{dt}((a(u) - a(U))\nabla(V_hu)_t, \nabla\xi_t) - \frac{d}{dt}(a_u(U)U_t\nabla\xi, \nabla\xi_t)\\
&\quad -\frac{d}{dt}(b(U)\nabla\xi, \nabla\xi_t) + \frac{d}{dt}(\int_0^t (b_t(u) - b_t(U))\nabla V_hud\tau, \nabla\xi_t)\\
&\quad -\frac{d}{dt}(\int_0^t b_t(U)\nabla\xi d\tau, \nabla\xi_t) + C\{(\|\xi_t\|_{0,\infty}^2 + \|\xi_{tt}\|_{0,\infty}^2)\|\xi\|_1^2 + \|\xi\|_1^2 + \|\eta\|^2\\
&\quad +\|\eta_t\|^2 + \|\eta_{tt}\|^2 + \|\eta_{ttt}\|^2 + \|\xi_t\|_1^2 + \|\xi_{tt}\|^2\} + C\int_0^t (\|\xi\|_1^2 + \|\eta\|^2)d\tau.
\end{aligned} \tag{2.14}$$

Integrating (2.14) with respect to t, using $\epsilon-$inequality, we get

$$\begin{aligned}&\|\xi_{tt}\|^2+\|\xi_t\|_1^2\\&\le C\{\|\xi(0)\|_1^2+\|\xi_t(0)\|_1^2+\|\eta(0)\|^2+\|\eta_t(0)\|^2+\|\xi_{tt}(0)\|^2+\|\xi\|_1^2+\|\eta\|^2+\|\eta_t\|^2\\&+\|\xi_t\|_1^2+\|\xi_t\|_{0,\infty}^2\cdot\|\xi\|_1^2+\int_0^t[(\|\xi_t\|_{0,\infty}^2+\|\xi_{tt}\|_{0,\infty}^2)\|\xi\|_1^2+\|\xi\|_1^2+\|\eta\|^2+\|\eta_t\|^2\\&+\|\eta_{tt}\|^2+\|\eta_{ttt}\|^2+\|\xi_t\|_1^2+\|\xi_{tt}\|^2]\}+\epsilon\|\xi_t\|_1^2.\end{aligned}\tag{2.15}$$

If we assume that

$$(\|\xi_t\|_{0,\infty}+\|\xi_{tt}\|_{0,\infty})\|\xi\|_1\le Ch^s,\tag{2.16}$$

then it follows from (2.1a), (2.2) and (2.6) that

$$\|\xi_{tt}\|^2+\|\xi_t\|_1^2\le C\{h^{2s}+\int_0^t(\|\xi_{tt}\|^2+\|\xi_t\|_1^2)d\tau\}.\tag{2.17}$$

Applying Gronwall Lemma now yields

$$\|\xi_{tt}\|+\|\xi_t\|_1\le Ch^s.\tag{2.18}$$

Similar to Lemma 2.4, the hypothesis (2.16) can be proofed easily. Therefore the proof has been completed .

3. Error estimates and superconvergence for $2\le p<\infty$

In this section the optimal L_p error estimates for the semidiscrete finite element approximation for $2\le p<\infty$ will be proved in Theorem 3.1. In addition, superconvergence results in $W^{1,p}$ for $2\le p<\infty$ between the approximate solution and Ritz-Volterra projection of the exact solution of (1.1) will be derived in Theorem 3.2.

Theorem 6. *Let u and U be the solutions of* (1.1) *and* (1.5), *respectively. If, in addition to the hypotheses of Lemma* 2.2, $u\in W^{s,d}(\Omega)$ *for* $2\le s\le r$ *and* $r\ge 2$, *then for* $t\in J$

$$\|U-u\|_{0,p}\le Ch^s,\qquad 2\le p<\infty.\tag{3.1}$$

Proof. Write the error $U-u=(U-V_hu)+(V_hu-u)=\xi+\eta$ as before. To prove (3.1) we define the auxiliary problem. For $\phi\in L_{p'}(\Omega),p^{-1}+p'^{-1}=1$, let Φ be the solution of

$$(a(u)\nabla v,\nabla\Phi)=(v,\phi),\qquad v\in H_0^1(\Omega).\tag{3.2}$$

Thus

$$\|\Phi\|_{2,p'}\le C\|\phi\|_{0,p'}.\tag{3.3}$$

By (2.5) we have

$$
\begin{aligned}
(\xi,\phi) &= (a(u)\nabla\xi,\nabla\Phi)\\
&= ((a(u)-a(U))\nabla\xi,\nabla R_h\Phi)+(a(U)\nabla U,\nabla R_h\Phi)\\
&= (f(U)-f(u)-\eta_{tt}-\xi_{tt},R_h\Phi)+(\int_0^t(b(u)-b(U))\nabla V_h u d\tau,\nabla R_h\Phi)\\
&+((a(u)-a(U))\nabla U,\nabla R_h\Phi)-(\int_0^t b(U)\nabla\xi d\tau,\nabla(R_h\Phi-\Phi))-(\int_0^t b(U)\nabla\xi d\tau,\nabla\Phi)\\
&= I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{3.4}
$$

From the inverse property, Lemma 2.2 and (2.12) .

$$\|\nabla U\|_{0,\infty}\le\|\nabla\xi\|_{0,\infty}+\|\nabla V_h u\|_{0,\infty}\le C(h^{-\frac{d}{2}}\|\nabla\xi\|+1)\le C(h^{s-\frac{d}{2}}+1)\le C.$$

By using the same way as in [7], we can select $\sigma>1$ such that

$$\|R_h\Phi\|\le C\|R_h\Phi\|_{1,\sigma}\le C\|\Phi\|_{1,\sigma}\le C\|\Phi\|_{2,p'}.$$

Then from imbedding inequalities, the inverse property and the stability of R_h in $W^{1,d'}$, $d^{-1}+d'^{-1}=1$,that

$$
\begin{aligned}
I_1 &\le C(\|\xi\|+\|\eta\|+\|\xi_{tt}\|+\|\eta_{tt}\|)\|R_h\Phi\|\\
&\le Ch^s\|R_h\Phi\|\le Ch^s\|\Phi\|_{2,p'},\\
I_2 &\le C(\int_0^t(\|\xi\|_{0,d}+\|\eta\|_{0,d})d\tau)\|R_h\Phi\|_{1,d'}\\
&\le C(\int_0^t(\|\xi\|_1+\|\eta\|_{0,d})d\tau)\|R_h\Phi\|_{1,d'}\le Ch^s\|\Phi\|_{2,p'},\\
I_3 &\le C(\|\xi\|_{0,d}+\|\eta\|_{0,d})\|R_h\Phi\|_{1,d'}\\
&\le C(\|\xi\|_1+\|\eta\|_{0,d})\|R_h\Phi\|_{1,d'}\le Ch^s\|\Phi\|_{2,p'},\\
|I_4| &\le C\|R_h\Phi-\Phi\|_{1,p'}\int_0^t\|\xi\|_{1,p}d\tau\\
&\le Ch\|\Phi\|_{2,p'}h^{-1}\int_0^t\|\xi\|_{0,p}d\tau\le C\|\Phi\|_{2,p'}\int_0^t\|\xi\|_{0,p}d\tau,\\
|I_5| &= \int_0^t(\xi,b_u(U)\nabla u\cdot\nabla\Phi)d\tau+\int_0^t(\xi,b(U)\nabla\cdot(\nabla\Phi))d\tau\\
&\le C\|\Phi\|_{2,p'}\int_0^t\|\xi\|_{0,p}d\tau.
\end{aligned}
$$

Combine our estimates with (3.4) and noting (3.3), we have

$$\|\xi\|_{0,p}=\sup_{\phi\in L_{p'}(\Omega)}\frac{(\xi,\phi)}{\|\phi\|_{0,p'}}\le C(h^s+\int_0^t\|\xi\|_{0,p}d\tau).$$

Applying Gronwall Lemma and noting (1.6), (2.1a), the inequality (3.1) follows .

Theorem 7. *Let u and U be the solutions of (1.1) and (1.5), respectively. If, in addition to the hypotheses of Lemma 2.2, $u \in W^{3,p}(\Omega)$ for $2 \leq s \leq r$ and $r \geq 2$, then for $t \in J$*

$$\|U - V_h u\|_{1,p} \leq Ch^s, \tag{3.5}$$

where $2 \leq p < \infty$ when $d = 1$ or 2, $2 \leq p \leq \dfrac{2d}{d-2}$ when $d = 3$ or 4.

Proof. We first introduce the other auxiliary problem. Denote ψ_x to be an arbitrary component of $\nabla\psi$ and let Ψ be the solution of

$$(a(u)\nabla v, \nabla\Psi) = -(v, \psi_x), \qquad v \in H_0^1(\Omega). \tag{3.6}$$

Thus

$$\|\Psi\|_{1,p'} \leq C\|\psi\|_{0,p'}, p^{-1} + p'^{-1} = 1. \tag{3.7}$$

Similar to (3.4) we have

$$\begin{aligned}(\xi_x, \psi) &= (f(U) - f(u) - \eta_{tt} - \xi_{tt}, R_h\Psi) + ((a(u) - a(U))\nabla U, \nabla R_h\Psi) \\ &\quad + (\int_0^t (b(u) - b(U))\nabla V_h u d\tau, \nabla R_h\Psi) - (\int_0^t b(U)\nabla\xi d\tau, \nabla R_h\Psi) \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.8}$$

Then similar to Theorem 3.1, we have

$$\begin{aligned}I_1 &\leq C(\|\xi\| + \|\eta\| + \|\xi_{tt}\| + \|\eta_{tt}\|)\|R_h\Psi\| \leq Ch^s\|R_h\Psi\| \leq Ch^s\|\Psi\|_{1,p'}, \\ I_2 &\leq C(\|\xi\|_{0,p} + \|\eta\|_{0,p})\|R_h\Psi\|_{1,p'} \leq C(\|\xi\|_1 + \|\eta\|_{0,p})\|R_h\Psi\|_{1,p'} \leq Ch^s\|\Psi\|_{1,p'}, \\ I_3 &\leq C(\int_0^t (\|\xi\|_{0,p} + \|\eta\|_{0,p})d\tau)\|R_h\Psi\|_{1,p'} \\ &\leq C(\int_0^t (\|\xi\|_1 + \|\eta\|_{0,p})d\tau)\|R_h\Psi\|_{1,p'} \\ &\leq Ch^s\|\Psi\|_{1,p'}, \\ I_4 &\leq C\|R_h\Psi\|_{1,p'}\int_0^t \|\xi\|_{1,p}d\tau \leq C\|\Psi\|_{1,p'}\int_0^t \|\xi\|_{1,p}d\tau.\end{aligned}$$

Combine our estimates with (3.8) and noting (3.7), we have

$$\|\xi\|_{1,p} \leq C(h^s + \int_0^t \|\xi\|_{1,p}d\tau).$$

Applying Gronwall Lemma, the inequality (3.5) follows.

4. Error estimates and superconvergence for $p = \infty$

In this section we only consider two-dimensional space R^2. The optimal maximum norm error estimates and superconvergence of gradients will be established.

In order to derive the maximum norm error estimates, we need to define the Green functions associated with the bilinear form $(a(u)\nabla\cdot, \nabla\cdot)$(see[12]). Let $G_z^h \in S_h$ and

$G_z^* \in H_0^1(\Omega)$ be the discrete Green function and pre-Green function respectively, g_z^* the directional derivative of G_z^* along some direction with respect to z. Then G_z^h and g_z^h are the finite element approximations to G_z^* and g_z^*, respectively . From [12], we know that

$$\begin{aligned}
&\|G_z^h\| + \|G_z^h\|_{1,p'} + \|G_z^*\|_{1,1} + \|g_z^h - g_z^*\|_{1,1} \leq C, p^{-1} + p'^{-1} = 1, p > 2,\\
&\|g_z^*\|_{1,1} + \|g_z^h\|_{1,1} + \|g_z^h\|^2 \leq C \log h^{-1},\\
&\|G_z^h - G_z^*\| \leq \begin{cases} Ch, & r > 2,\\ Ch \log h^{-1}, & r = 2.\end{cases}
\end{aligned} \tag{4.1}$$

We shall first show the following L_∞ norm error estimates.

Theorem 8. *Let u and U be the solutions of (1.1) and (1.5), respectively. If, in addition to the hypotheses of Lemma 2.2, $u \in W^{s,\infty}(\Omega)$ for $2 \leq s \leq r$ and $r \geq 2$, then for $t \in J$*

$$\|U - u\|_{0,\infty} \leq \begin{cases} Ch^s, & r > 2,\\ Ch^s \log h^{-1}, & r = 2.\end{cases} \tag{4.2}$$

Proof. From (2.1*b*) we need to bound ξ only . When $r > 2$, similar to (3.4), for $z \in \Omega$ and $t \in J$

$$\begin{aligned}
\xi(z,t) &= (a(u)\nabla\xi, \nabla G_z^h)\\
&= (f(U) - f(u) - \eta_{tt} - \xi_{tt}, G_z^h) + ((a(u) - a(U))\nabla U, \nabla G_z^h)\\
&\quad + (\int_0^t (b(u) - b(U))\nabla V_h u d\tau, \nabla G_z^h) - (\int_0^t b(U)\nabla\xi d\tau, \nabla(G_z^h - G_z^*))\\
&\quad - (\int_0^t b(U)\nabla\xi d\tau, \nabla G_z^*)\\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{4.3}$$

Then from imbedding inequalities and (4.1)

$$\begin{aligned}
I_1 &\leq C(\|\xi\| + \|\eta\| + \|\xi_{tt}\| + \|\eta_{tt}\|)\|G_z^h\| \leq Ch^s,\\
I_2 &\leq C(\|\xi\|_{0,p} + \|\eta\|_{0,p})\|G_z^h\|_{1,p'} \leq C(\|\xi\|_1 + \|\eta\|_{0,p})\|G_z^h\|_{1,p'} \leq Ch^s,\\
I_3 &\leq C(\int_0^t (\|\xi\|_{0,p} + \|\eta\|_{0,p})d\tau)\|G_z^h\|_{1,p'}\\
&\leq C(\int_0^t (\|\xi\|_1 + \|\eta\|_{0,p})d\tau)\|G_z^h\|_{1,p'} \leq Ch^s,\\
|I_4| &\leq C\|G_z^h - G_z^*\|_{1,1} \int_0^t \|\xi\|_{1,\infty} d\tau \leq Ch \int_0^t \|\xi\|_{1,\infty} d\tau \leq C \int_0^t \|\xi\|_{0,\infty} d\tau\\
|I_5| &= |\int_0^t (a\nabla(\frac{b}{a}\xi), \nabla G_z^*)d\tau - \int_0^t (a\xi\nabla(\frac{b}{a}), \nabla G_z^*)d\tau|\\
&\leq C\int_0^t P_h(\frac{b}{a}\xi)d\tau + C\|G_z^*\|_{1,1} \int_0^t \|\xi\|_{0,\infty} d\tau \leq C \int_0^t \|\xi\|_{0,\infty} d\tau,
\end{aligned}$$

where $P_h : L_2(\Omega) \to S_h$ is the L_2 projection operator. Combine our estimates with (4.3) and applying Gronwall Lemma, we have

$$\|\xi\|_{0,\infty} \le Ch^s, \qquad 2 \le s \le r, r > 2. \tag{4.4}$$

When $r = 2$, recall that

$$\|\xi\|_{0,\infty} \le C(\log h^{-1})^{\frac{1}{2}} \|\xi\|_1,$$

from (2.7), we have

$$\|\xi\|_{0,\infty} \le Ch^2(\log h^{-1})^{\frac{1}{2}}, \qquad r = 2. \tag{4.5}$$

Noting (1.6), (2.1b), the conclusion of the theorem is now concluded.

Corollary 9. *Under the hypotheses of Lemma 2.2, assume that $u \in W^{s,p}(\Omega)$ for $2 \le s \le r, r \ge 2$ and $p > 2$. Then for $t \in J$*

$$\|U - V_h u\|_{0,\infty} \le \begin{cases} Ch^s, & r > 2, \\ Ch^s(\log h^{-1})^{\frac{1}{2}}, & r = 2. \end{cases} \tag{4.6}$$

We finally show $W^{1,\infty}$ superconvergence for $U - V_h u$.

Theorem 10. *Under the hypotheses of Theorem 4.1, for $2 \le s \le r$ and $r \ge 2$, then for $t \in J$*

$$\|U - V_h u\|_{1,\infty} \le \begin{cases} Ch^s \log h^{-1}, & r > 2, \\ Ch^s(\log h^{-1})^2, & r = 2. \end{cases} \tag{4.7}$$

Proof. Similar to (4.3), for $z \in \overline{\Omega}$ and $t \in J$

$$\begin{aligned} \partial_z \xi(z,t) &= (a(u)\nabla\xi, \nabla g_z^h) \\ &= (f(U) - f(u) - \eta_{tt} - \xi_{tt}, g_z^h) + ((a(u) - a(U))\nabla U, \nabla g_z^h) \\ &\quad + (\int_0^t (b(u) - b(U))\nabla V_h u d\tau, \nabla g_z^h) - (\int_0^t b(U)\nabla\xi d\tau, \nabla(g_z^h - g_z^*)) \\ &\quad - (\int_0^t b(U)\nabla\xi d\tau, \nabla g_z^*) \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{4.8}$$

Then from imbedding inequalities and (2.1b), (4.1), (4.2),

$$I_1 \le C(\|\xi\| + \|\eta\| + \|\xi_{tt}\| + \|\eta_{tt}\|)\|g_z^h\| \le Ch^s(\log h^{-1})^{\frac{1}{2}},$$

$$I_2 \le C(\|\xi\|_{0,\infty} + \|\eta\|_{0,\infty})\|g_z^h\|_{1,1} \le \begin{cases} Ch^s \log h^{-1}, & r > 2, \\ Ch^s(\log h^{-1})^2, & r = 2, \end{cases}$$

$$I_3 \leq C(\int_0^t (\|\xi\|_{0,\infty} + \|\eta\|_{0,\infty})d\tau)\|g_z^h\|_{1,1} \leq \begin{cases} Ch^s \log h^{-1}, & r > 2, \\ Ch^s(\log h^{-1})^2, & r = 2, \end{cases}$$

$$|I_4| \leq C\|g_z^h - g_z^*\|_{1,1} \int_0^t \|\xi\|_{1,\infty} d\tau \leq C \int_0^t \|\xi\|_{1,\infty} d\tau,$$

$$|I_5| = |\int_0^t (a\nabla(\frac{b}{a}\xi), \nabla g_z^*)d\tau - \int_0^t (a\xi\nabla(\frac{b}{a}), \nabla g_z^*)d\tau|$$

$$\leq C \int_0^t \partial_z P_h(\frac{b}{a}\xi)d\tau + C\|g_z^*\|_{1.1} \int_0^t \|\xi\|_{0,\infty} d\tau$$

$$\leq \begin{cases} C\int_0^t \|\xi\|_{1,\infty} d\tau + Ch^s \log h^{-1}, & r > 2, \\ C\int_0^t \|\xi\|_{1,\infty} d\tau + Ch^s(\log h^{-1})^{\frac{3}{2}}, & r = 2. \end{cases}$$

Combine our estimates with (4.5) and applying Gronwall Lemma , we obtain the result.

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