On ϕ_0 -boundedness for the comparison differential system

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Abstract We investigate various ϕ_0 -boundedness and ϕ_0 -Lagrange stability of the trivial solution of comparison differential system. We also investigated the corresponding boundedness concepts of the trivial solution of the differential system using the theory of differential inequalities through cones and the method of cone valued Lyapunov functions.

Key Words : ϕ_0 -boundedness, ϕ_0 -uniform bounded, quasimonotone nondecreasing, ϕ_0 -Lagrange stability, cone valued Lyapunov functions.

1. Intoduction

Let R^n denote the n-dimensional Euclidean space with any convenient norm $\|\cdot\|$, and inner product(,).

$R_+ = [0, \infty)$, $R_+^n = \{u \in R^n \mid u_i \geq 0,$
$i = 1, 2, \cdots, n\}$, $C[R_+ \times R^n, R^n]$
denotes the space of continuous functions mapping $R_+ \times R^n$ into R^n.

Consider the differential system.

$$x' = f(t, x), x(t_0) = x_0, t \geq 0 \qquad \text{(A)}$$

where $f \in C[R_+ \times R^N, R^N]$.

Define $S\rho = \{x \in R^n \mid \|x\| < \rho, \rho > 0\}$.

Let $K \subset R^n$ be a cone in R^n, $n \leq N$, and $V \in C[R_+ \times S\rho, K]$.

Define for $(t, x) \in R_+ \times S\rho, h > 0$, the function $D^+V(t, x)$ by

$$D^+V(t, x) = \lim_{h \to 0^+} sup\left(\frac{1}{h}\right)[V(t + h, x + hf(t, x)) - V(t, x)]$$

Consider the comparison differential system

$$u' = g(t, u), \quad u(t_0) = u_0, \quad t_0 \geq 0 \qquad \text{(B).}$$

where $g \in C[R_+ \times K, R^n]$, and K is a cone in R^n.

Let , $S(\rho) = \{u \in K \mid \|u\| < \rho, \rho > 0\}$, $v \in C[R_+ \times S(\rho), K]$ and define for $(t, u) \in R_+ \times S(\rho), h > 0$, the function $D^+v(t, u)$ by $D^+v(t, u) =$
$(t + h, u + hg(t, u)) - v(t, u)]$

Definition 1.1 ([9]) A function $g : D \rightarrow R^n, D \subset R^n$, is said to be quasimonotone nondecreasing relative to the cone K when it satisfies that $x, y \in D$ with $x \leq_k y$ and $(\phi_0, y - x) = 0$ for some $\phi_0 \in K_0^*$, then

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$(\phi_0, g(y)-g(x)) \geq 0$, where K_0^* is adjoint cone of $K_0 = K-\{0\}$.

Definition 1.2([6]) The trivial solution $x=0$ of (A) is said to be

(B_1) equibounded if for each $\alpha \geq 0, t_0 \in R_+$, where exists $\beta = \beta(t_0, \alpha)$ such that the inequlity $\| x_0 \| \leq \alpha$ implies. $\| x(t, t_0, x_0) \| \leq \beta, \ t \geq t_0$ Other boundednessnotions $(B_2) \sim (B_8)$ can be similarly defined [6,8].

Now we give cone valued ϕ_0-boundedness definitions of the trivial solution of (B).

Definition 1.3 The differential system (B) is said to be

(B_1^*) ϕ_0-equibounded if for each $\alpha \geq 0$, $t_0 \in R_+$ there exist $\beta = \beta(t_0, \alpha)$ such that the inequlity $(\phi_0, u_0) \leq \alpha$ implies $(\phi_0, r(t)) < \beta$ for all $t \geq t_0$, where $r(t)$ is maximal solution of (B) and $\phi_0 \in K_0^*$;

(B_2^*) ϕ_0-uniform bounded if the β in (B_1^*) is independent of t_0;

(B_3^*) ϕ_0-quasi-equi-ultimately bounded if for each $\alpha \geq 0$ and $t_0 \in R_+$, there exist positive numbers N and $T = T(t_0, \alpha)$ such that the inequlity $(\phi, u_0) \leq \alpha$ implies $(\phi_0, r(t)) < N, t \geq t_0 + T$;

(B_4^*) ϕ_0-quasi-uniform ultimately bounded if the T in (B_3^*) is independent of to ;

(B_5^*) ϕ_0-equi-ultimately if (B_1^*) and (B_3^*) hold at the same time ;

(B_6^*) ϕ_0-uniform-ultimately bounded if (B_2^*) and (B_4^*) hold simultaneously ;

(B_7^*) ϕ_0-equi-Lagrange stable if (B_1^*) and (S_7^*) ([1]) hold simultaneously ;

(B_8^*) ϕ_0-uniform-ultimately bounded if (B_2^*) and (S_8^*) ([1]) hold simultaneously ;

Lemma 1.4([7]) Assume that (i) $V \in C[R_+ \times S\rho, K]$, $V(t,x)$ satisfies a Local Lipschitz condition in x relative to K and for $(t,x) \in R_+ \times S\rho, D^+V(t,x) \leq_k g(t, V(t,x))$;
(ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u with respect to K for each $t \in R_+$.

If $r(t, t_0, u_0)$ is a maximal solution of (B) relative to K and $x(t; t_0, x_0)$ is any solution of (A) with $V(t_0, x_0) \leq_k u_0$, then on the common interval of existence, we have
$V(t, x(t, t_0, x_0)) \leq_k r(t, t_0, u_0)$

2. Boundedness Theores

In this section, we investigate the corresponding boundedness concepts of the trivial solution $x=0$ of (A) using the theory of differential inequalities through cones and the method of cone-valued Lyapunov functions.

Let $H = \{a \in C[R_+, R_+] \mid a(t)$ is strictly increasing in t and $a(0)=0\}$.

Theorem 2.1 Assume that there exist functions $V(t,x)$ and $g(t,u)$ with the following properties;

(i) $g \in C[R_+ \times K, R^n]$, $g(t,0)=0$, $g(t,u)$ is quasimonotone in u relative to K

(ii) $V \in C[R_+ \times S\rho, K], K \subset R^n$, $V(t,0)=0$, $V(t,x)$ is locally Lipschitzion in x, and for $(t,x) \in R_+ \times S\rho$ and $\phi_0 \in K_0^*$,

$b(\|x\|) \leq (\phi_0, V(t,x)), \quad t \geq t_0 \geq 0$

(2.1) where $b \in H$ on the interval

$0 \le u < \infty$ and $b(u) \to \infty$ as $u \to \infty$.

(iii) $D^+ V(t,x) \le_k g(t, V(t,x))$,

$(t,x) \in R_+ \times S\rho$.

Then, the ϕ_0-equiboundedness of the system (B) implies the equiboundedness of the system (A)

Proof) Let $\alpha \ge 0$ and $t_0 \in R_+$ be given, and let $\|x_0\| \le \alpha$.

Since the equation (B) is ϕ-equibounded, given $\alpha_1 \ge 0$ and $t_0 \in R_+$, there exists a $\beta_1 = \beta_1(t_0, \alpha)$ that is continuous in t_0 for each α such that $(\phi_0, r(t, t_0, u_0)) < \beta_1$, (2.2)

$t \ge t_0$ provide $(\phi_0, u_0) \le \alpha_1$.

Moreover, as $b(u) \to \infty$ as $u \to \infty$, we can choose a $L = L(t_0, \alpha)$ verifying the relation

$\beta_1(t_0, \alpha) \le b(L)$ (2.3)

Now let $u_0 = V(t_0, x_0)$ in K.

Then, assumption (iii) and Lemma 1.4 show that

$V(t, x(t, t_0, x_0)) \le_k r(t, t_0, u_0),\ t \ge t_0$ (2.4)

where $r(t, t_0, u_0)$ is the maximal solution of the system (B)

Suppose, if possible, that there is a solution $x(t, t_0, x_0)$ with $\|x_0\| \le \alpha$ having that property that, for some $t_1 > t_0$, $\|x(t_1, t_0, x_0)\| = L$.

Then, because of relations (2.1),(2.2),(2.3), and (2.4), there results the following absurdity;

$b(L) = b(\|x\|) \le (\phi_0, V(t,x)) \le$

$(\phi_0, r(t, t_0, u_0)) < \beta_1(t_0, \alpha) \le b(L)$.

The proof is complete, since this contradiction implies that (B_1) holds.

Theorem 2.2 Let the conditions of Theorem 2.1 hold with $b(\|x\|) \le (\phi_0, V(t,x))$ being replaced by

$b(\|x\|) \le (\phi_0, V(t,x)) \le a(\|x\|)$ (2.5)

where $a \in H$.

Then, if the system (B) is ϕ_0-uniform bounded, the system (A) is likewise uniform bounded.

Poof) We choose $\alpha_1 = a(\alpha)$, which is independent of t_0. Since $\beta_1 = \beta_1(\alpha)$ in this case, it is easy to see from the choice of L that it is also independent of t_0.

Theorem 2.3 Under the assumptions of Theorem 2.1, the ϕ_0-quasi-equi-ultimate boundedness of the system B implies the quasi-equi-ultimate boundedness of the system A.

Poof) By hypotheses, given $\alpha_1 \ge 0$ and $t_0 \in R_+$, there exist positive numbers N_1 and $T = T(t_0, \alpha)$ such that

$(\phi_0, r(t, t_0, u_0)) < N_1, t \ge t_0 + T$ (2.6)

whenever $(\phi_0, u_0) \le \alpha_1$.

Since $b(u) \to \infty$ with u, it is possible to find a positive number N verifying

$b(N) \ge N_1$. (2.7)

We choose $u_0 = V(t_0, x_0)$ in K and obtain the estinate (2.4) as in Theorm 2.1. Now let there exist a sequence $\{t_k\}, t_k \ge t_0 + T, t_k \to \infty$ as $k \to \infty$ such that, for some solution $x(t, t_0, x_0)$ of (2.1) satisfying $\|x_0\| \le \alpha$,

We have $\|x(t_k, t_0, x_0)\| \ge N$.

We are led to a contradiction

$b(N) \le (\phi_0, V(t_k, x(t_k, t_0, x_0)) \le$

$(\phi_0, r(t_k, t_0, u_0)) < N_1 \le b(N)$

This completes the proof of Theorem 2.3.

Theorem 2.4 Under the ssumptions of Theorem 2.1, the equi-ultimately boundedness of the system (B) implies the equi-ultimate boundedness of the system (A).

This proof of this theorem can be constructed by combining the proofs of Theorem 2.1 and 2.3.

Theorem 2.5 Let the assumptions of Theorem 2.2 hold.

Then, the ϕ_0-equi-Lagrange stability of the system (B) assures the equi-Lagrange stability of the system (A).

Proof) By Theorem 2.1, equioundedness of the system (A) follows, and hence (S_7) remains to be proved.

Let $\epsilon > 0, \alpha \geq 0$, and $t_0 \in R_+$ be given, and let $(\phi_0, u_0) \leq \alpha$.

As in the proof of the theorem 2.1, there exists $\alpha_1 = \alpha_1(t_0, \alpha)$ satisfying $\|x_0\| \leq \alpha$, $a(\|x_0\|) \leq \alpha_1$ simulaneously.

Since (S_7^*) holds, given $\alpha_1 \geq 0, b(\epsilon) > 0$, and $t_0 \in R_+$, there exists a $T = T(t_0, \epsilon, \alpha)$ such that $(\phi_0, u_0) \leq \alpha_1$ implies

$$(\phi_0, r(t, t_0, u_0)) < b(\epsilon),\ t \geq t_0 + T \tag{2.8}$$

Choose $u_0 = U(t_0, x_0)$.

Then $U(t, x) \leq_k r(t)$.

Now choose $\delta_1 > 0$ such that.

If possible, let there exist a sequence $\{t_k\}$, $t_k \geq t_0 + T$, $t_k \to \infty$ as $k \to \infty$, such that for some solution $x(t, t_0, x_0)$ satisfying $\|x_0\| \leq \alpha$,, we have $\|x(t_k, t_0, x_0)\| \geq \varepsilon$.

Then $b(\epsilon) \leq (\phi_0, V(t_k, x(t_k, t_0, x_0)) \leq (\phi_0, r(t_k, t_0, u_0)) < b(\epsilon)$ which proves (S_7).

The proof is complete.

Theorem 2.6 Let the assumptions of Theorem 2.2 hold.

Then, the ϕ_0-uniform Lagrange stability of the system (B) implies uniform Lagrange stability of the system (A).

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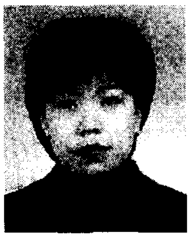

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