On uniform asymptotic stability of the nonlinear differential system

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Abstract We investigate various $\phi(t)$-stability of comparison differential equations and We obtain necessary and/or sufficient conditions for the asymptotic and uniform asymptotic stability of the differential equations $x' = f(t, x)$

Key Words : local Lipschitz condition, quasimonotone, $\phi(t)$-stability, cone-valued Lyapunov function, nonlinear differential equation

1. Preliminaries and Definitions

Lyapunov second methods are now well established subjects as the most powerful techniques of analysis for the stability and qualitative properties of nonlinear differential equations $x' = f(t, x)$, $x(t_0) = x_0 \in R^N$.

One of the original Lyapunov theorems is as follows:

Lyapunov Theorem. For $x' = f(t, x)$, assume that there exists a function $V: R_+ \times S_\rho \to R_+$ such that

(i) V is C^1-function and positive definite,

(ii) V is decresent,

(iii) $\frac{d}{dt} V(t, x) = V_t(t, x) + V_x \cdot f(t, x) \leq -a(\|x\|)$ for $t \geq 0$, $x \in S_\rho$, where $S_\rho = \{x \in R^N \| x \| < \rho\}$ for $\rho > 0$, $a(r)$ is strictly increasing function with $a(0) = 0$.

Then the trivial solution $x(t) \equiv 0$ is uniformly asymptotically stable.

The advantage of the method is that is that it does not require the knowledge of solutions to analyse the stability of the equations. However in practical sense, how to find suitable Lyapunov functions V for given equations are the most difficult questions. Hence weakening the conditions (i), (ii), and (iii), and enlarging the class of Lyapunov functions are basic trends in Lyapunov stability theory [2, 3, 4, 5, 6, 11].

In the unified comparison frameworks, Ladde [7] analysed the stability of comparison differential equations by using vector Lyapunov function methods.

Lakshmikantham and Leela [9] initiated the cone valued Lyapunov function methods to advoid the quasimonotonicity assumption of comparison equations. They obtained various useful differential inequalities with cone-valued Lyapunov functions. Akpan and Akinyele [1] extended and generalized the results of [7,8] to the ϕ_0-stabilities of the comparison differential equations by using the cone-valued Lyapunov functions.

Here we generalize, in some sense, the results of [1] to the $\phi(t)$-stabilities of comparison equations below.

Let R^n denote the n-dimensional Euclidean space with any equivalent norm $\|\cdot\|$, and scalar

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product (,). $R_+=[0,\infty)$. $C[R_+\times R^n, R^n]$ denotes the space of continuous functions from $R_+\times R^n$ into R^n.

Definition 1.1 ([11]). A proper subset K of R^n is called a cone if (i) $\lambda K\subset K$, $\lambda\geq 0$; (ii) $K+K\subset K$; (iii) $K=\overline{K}$; (iv) $K^\circ\neq\emptyset$; (v) $K\cap(K)=\{0\}$, where $\overline{K}$ and K° denote the closure and interior of K, respectively, and ∂K denotes the boundary of K. The order relation on R^n induced by the cone K is defined as follows:

For $x, y\in R^n$, $x\leq_k y$ iff $x-y\in K$, and $x\leq_{k^\circ} y$ iff $y-x\in K^\circ$.

Definition 1.2 ([11]). The set $K*=\{\phi\in R^n: (\phi, x)\geq 0$, for all $x\in K\}$ is called the *adjoint cone* of K if $K*$ itself satisfies Definition 1.1

Note that $x\in\partial K$ if and only if $(\phi, x)=0$ for some $\phi\in K_0*$, where $K_0=K-\{0\}$.

Consider the differential equation

$$x'=f(t,x),\ x(t_0)=x_0,\ t_0\geq 0 \quad (1)$$

where $f\in C[R_x\times R^N, R^N]$ and $f(t,0)=0$ for all $t\geq 0$. Let $S_\rho=\{x\in R^N: \|x\|<\rho\}$, $\rho>0$. Let $K\subset R^n$ be a cone in R^n, $n\leq N$. For $V\in C[R_+\times S_\rho, K]$ at $(t,x)\in R_+\times S_\rho$, let $D^+V(t,x)=\lim_{h\to 0^+}\left(\frac{1}{h}\right)[V(t+h, x+hf(t,x))-V(t,x)]$ be a Dini derivative of V along the solution curves of the equation (1).

Consider a comparison differential equation

$$u'=g(t,u),\ u(t_0)=u_0,\ t_0\geq 0 \quad (2)$$

where $g\in C[R_+\times K, R^n], g(t,0)=0$ for all $t\geq 0$ and K is a cone in R^n

Let $S(\rho)=\{u\in K: \|u\|<\rho\}$, $\rho>0$. for $v\in C[R_+\times S(p), K]$, at $(t,u)\in R_+\times S(p)$, let $D^+v(t,u)=\lim_{h\to 0^+}\left(\frac{1}{h}\right)[v(t+h, u+hg(t,u))-v(t,u)]$ be a Dini derivative of v along the solution curves of the equation (2).

Definition 1.3 ([11]). A function $g: D\to R^n$, $D\subset R^n$, is said to be *quasimonotone* nondecreasing relative to the cone K when it satisfies that if $x, y\in D$ with $x\leq_K y$ and $(\phi_0, y-x)=0$ for some $\phi_0\in K_0*$, then $(\phi_0, g(y)-g(x))\geq 0$.

Definition 1.4 ([8,10]). The trivial solution $x=0$ of (1) is *equistable* if for each $\varepsilon>0$, $t_0\in R_+$, there exists a positive function $\delta=\delta(t_0,\varepsilon)$ such that the inequality $\|x_0\|<\delta$ implies $\|x(t,t_0,x_0)\|<\varepsilon$, for all $t\geq t_0$.

Other stability notions can be similarly defined [8,10].

Now we give cone-valued $\phi(t)$-stability definations of the trivial solution of (2).

Definition 1.5 ([12]). Let $\phi: [0,\infty]\to K^*$ be a cone-valued function. The trivial solution $u=0$ of (2) is

(a) $\phi(t)$-equistable if for each $\varepsilon>0, t_0\in R_+$, there exists a positive function $\delta=\delta(t_0,\varepsilon)$ such that the inequality $(\phi(t_0), u_0)<\delta$ implies $(\phi(t_0), r(t))<\varepsilon$, for all $t\geq t_0$ where $r(t)$ is a maximal solution of (2);

(b) *uniformly* $\phi(t)$-stable if the δ in (a) is independent of t_0;

(c) *quasi-equi asymptotically* $\phi(t)$-stable if, for each $\varepsilon>0, t_0\in R^+$, there exist positive numbers $\delta=\delta(t_0)$ and $T=T(t_0,\varepsilon)$ such that $(\phi(t_0), u_0)<\delta$ implies $(\phi(t), r(t))<\varepsilon$ for all $t\geq t_0+T$;

(d) quasi uniformly asymptotically $\phi(t)$-stable if the numbers δ and T in (c) are independent of t_0;

(f) uniformly asymptotically $\phi(t)$-stable if (b) and (d) hold together.

In Definition 1.5, and for the rest of this paper, $r(t)$ denotes the maximal solution of (2) relative to the cone $K \subset R^n$[11].

Remark. When $\phi(t)$ is a constant in K_0*, the $\phi(t)$-stabilities are the same as in[1,11].

Let following comparison property plays a key role in our main theorem.

Lemma 1.6 ([9,110]). Assume that (i) $V \in C[R_+ \times S_\rho, K]$, $V(t,x)$ satisfies a local Lipschitz condition in x relative to K and for $(t,x) \in R^+ \times S_\rho$, $D^+V(t,x) \leq_K g(t, V(t,x))$; (ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u with respect to K for each $t \in R_+$.

If $r(t, t_0, u_0)$ is a maximal solution of (2) relative to K and (t, t_0, u_0) is any solution of (1) with $V(t_0, x_0) \leq_K u_0$, then on the common interval of existence, we have $V(t, x(t, t_0, u_0)) \leq_K r(t, t_0, u_0)$.

2. Stability Theorems

In this section, we investigate sufficient conditions for $\phi(t)$-stability, uniform $\phi(t)$-stability and asymptotic $\phi(t)$-stability of the trivial solution $u=0$ of the comparison equation (2). We also investigate the corresponding stability concepts of the trivial solution $x=0$ of (1) using differential inequalities with the method of cone-valued Lyapunov functions.

Theorem 2.1. Assume that

(i) $v \in C[R_+ \times S(\rho), K]$, $v(t,0)=0$, $v(t,u)$ is locally Lipschitzian in u relative to K and for each $(t,u) \in R_+ \times S(\rho)$, $D^+v(t,u) \leq_K 0$,

(ii) $g \in C[R_+ \times K, R^n]$, $g(t,0)=0$, $g(t,u)$ is quasimonotone in u relative to K

(iii) $\phi(t) \in K*$is a bounded continuous function on $[0,\infty)$ and $a[(\phi(t), r(t))] \leq (\phi(t), v(t, u(t)))$ for some function $a \in K$, $t \geq t_0$.

Then the trivial solution $u=0$ of (2) is $\phi(t)$-equistable.

Proof. Let $\varepsilon > 0$ be arbitrarily given and let $M = \text{sub}\{ \| \phi(t) \| \quad t \geq t_0\}$. Since $a^{-1}(Ma(\eta))$ is continuous and $a^{-1}(Ma(0))=0$, there exists $\varepsilon_1 > 0$ such that $a^{-1}(Ma(\eta)) \leq \varepsilon$ for $0 \leq \eta \leq \varepsilon_1$. Since $v(t,0)=0$ and $v(t,u)$ is continuous in u given $a(\varepsilon_1)>0$, $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t, a(\varepsilon_1))$ such that $\| u_0 \| < \delta_1$ implies $\| v(t_0, u_0) \| < a(\varepsilon_1)$. Now for the bounded continuous function $\phi(t) \in K_0*$, $(\phi(t_0), u_0) \leq \| \phi(t_0) \| \; \| u_0 \| < \| \phi(t_0) \| \delta_1$ implies $(\phi(t), v(t_0, u_0)) < \| \phi(t) \| \cdot a(\varepsilon_1)$. Put $\delta = \| \phi(t_0) \| \delta_1$. Then $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), v(t_0, u_0)) < \| \phi(t) \| \cdot \| v(t_0, u_0) \| < M \cdot a(\varepsilon_1)$. Let $u(t)$ be any solution of (2) such that $(\phi(t_0), u(t_0)) < \delta$. Then by (i), $v(t, u(t)) \leq_K v(t_0, u_0)$, $t \geq t_0$. Thus $(\phi(t_0), u(t_0)) < \delta$ implies $a[(\phi(t_0), u(t_0))] \leq (\phi(t), v(t, u(t))) \leq (\phi(t), v(t_0, u(t_0))) < M \cdot a(\varepsilon_1)$. Hence $(\phi(t), r(t)) \leq a^{-1}(M(a(\varepsilon_1))) \leq \varepsilon$ which completes the proof.

Theorem 2.2 Let the conditions (i) and (ii) of Theorem 2.1 hold. Assume further that for some condinuous function $\phi(t) \in K_0*$, for each $(t,u) \in R_+ \times S(\rho)$, $D^+(\phi(t), v(t,u)) \leq 0$ and

$a[(\phi(t), r(t))] \leq (\phi(t), v(t, u(t))) \leq b[(\phi(t), r(t))]$, $a, b \in K$.

Then the trivial solution $u=0$ of (2) is uniformly $\phi(t)$-stable.

Proof. For $\varepsilon > 0$, let $\delta = b^{-1}[a(\varepsilon)]$. Let $u(t)$ be any solution of (2) such that $(\phi(t_0), u_0) < \delta$. Then by the hypothesis, $(\phi(t), v(t, u(t)))$ is decreasing and so $(\phi(t), v(t, u(t))) \leq (\phi(t_0), v(t_0, u(t_0)))$ for all $t \geq t_0$. Thus $a[(\phi(t), r(t))] \leq (\phi(t), v(t, u(t)))$

$\leq(\phi(t_0), v(t_0, u(t_0)))\leq b[(\phi(t_0), r(t_0))]=b[(\phi(t_0), u_0)]<b(\delta)=b(b^{-1}(a(\varepsilon))=a(\varepsilon)$. Hence $(\phi(t_0), u(t_0))<\delta$ implies $(\phi(t), r(t))\leq\varepsilon$ for each $t\geq t_0$

Theorem 2.3 Let the conditions of Theorem 2.1 hold. Assume further that

$D^+(\phi(t), v(t, u))\leq -c[(\phi(t), v(t, u))]$ for each $t\geq t_0$, where $t_0\in R_+, c\in K$ (3)

Then the trivial solution $u=0$ of (2) is equi-asymptotically $\phi(t)$-stable.

Proof. By Theorem 2.1, the trivial solution of (2) is $\phi(t)$-equistable. By the formula (3), $(\phi(t), v(t, u(t)))$ is monotone decreasing in t and hence the limit $v^*=\lim_{t\to\infty}(\varnothing(t), v(t, u(t)))$ exists. Suppose $v*\neq 0$. Then $c(v*)\neq 0, c\in K$. Since $c(r)$ is monotone, $c[(\phi(t), v(t, u(t)))]\geq c(v*)$, and so $D^+(\phi(t), v(t, u))\leq -c[(\phi(t), v(t, u))]\leq -c(v*)$. Then

$$\int_{t_0}^{t} D^+(\phi(s), v(s, u(s)))ds\leq \int_{t_0}^{t} -c(v*)ds$$

Thus $(\phi(t), v(t, u(t)))\leq -c(v*)(t-t_0)+(\phi(t_0), v(t_0,u_0))$. Accordingly, as $t\to\infty$, we have $(\phi(t), v(t, u(t)))\to -\infty$. This contradicts the condition $a[(\phi(t), r(t))]\leq(\phi(t), v(t, u(t)))$. It follows that $v*=0$ Thus $(\phi(t), v(t, u(t)))\to 0$ as $t\to\infty$. Hence given $\varepsilon>0$, and for each $t_0\in R_+$, there exist $\delta=\delta(t_0)$ and $T=T(t_0,\varepsilon)$ such that for all $t\geq t_0+T, (\phi(t_0), u_0)<\delta$ implies $(\phi(t), r(t))<\varepsilon$.

Theorem 2.4 Let the hypothesis of Theorem 2.2 hold with

$$D^+(\phi(t), v(t, u(t)))\leq -c[(\phi(t), r(t))]$$

for each $t\geq t_0$ where $t_0\in R_+$ and for some $c\in K$.

Then the trivial solution $u=0$ of (2) is uniformly asymptotically $\phi(t)$-stable. Let $\varepsilon>0$ be arbitrarily given. Choose $\delta=\delta(\varepsilon)$ which is independent of t_0. Let $u(t)$ be a solution of (2) such that $(\phi(t_0), u_0)<\delta$. Let $v*=\sup\{(\phi(t), v(t, u(t)))\ (\phi(t_0), u_0)<\delta\}$.

Set $T(\varepsilon)=v*/c(\varepsilon)$. we claim that

$(\phi(t_0), u_0)<\delta$ implies $(\phi(t), r(t))<\varepsilon,\ t\geq t_0+T(\varepsilon)$. (4)

Suppose that (4) is not true. Then there would exist at least one $t\geq t_0+T(\varepsilon)$ such that $(\phi(t_0), u_0)<\delta$ implies $(\phi(t), r(t))\geq\varepsilon$. Since $c\in K$, from the the condition $D^+(\phi(t), v(t, u(t)))\leq -c[(\phi(t), r(t))]$,we have $D^+(\phi(t), v(t, u(t)))\leq -c[(\phi(t), r(t))]\leq -c(\varepsilon)$. Integrating, $\int_{t_0}^{t} D^+(\phi(s), v(s, u(s)))ds\leq\int_{t_0}^{t} -c(\varepsilon)ds$ implies that $(\phi(t), v(t, u(t)))\leq(\phi(t_0), v(t_0, u_0))-c(\varepsilon)(t-t_0)$ for all $t\geq t_0+T(\varepsilon)$.

Then $\lim_{t\to\infty}(\phi(t), v(t, u(t)))=-\infty$ which is a contradiction.

Theorem 2.5 Assume that

(i) $V\in C[R_+\times S_p, K]$, $V(t,x)$ is locally Lipschitzian in x relative to K and for $(t,x)\in R_+\times S_\rho$, $D^+V(t,x)\leq_K g(t, V(t,x))$,

(ii) $g\in C[R_+\times K, R^n]$ and $g(t,u)$ is quasi-monotone in u relative to K for each $t\in R_+$,

(iii) there exist $a, b\in K$ such that for some $\phi(t)\in K_0$., for each $x\in S_P$, $b(|x|)\leq(\phi(t), V(t,x))\leq a(|x|)$, $t\geq t_0\geq 0$

Then the trivial solution $x=0$ of (1) *has the cor- responding one of the stability properties if the trivail solution* $u=0$ of (2) has each one of the $\phi(t)$-stability properties in Definition 1.5

Proof. Suppose that the trivial solution $u=0$of (2) is $\phi(t)$-equistable. Let $0<\varepsilon<\rho$ be arbitrarily given and $t_0 \in R_+$. Then there exists for all $\delta=\delta(t_0,\varepsilon)>0$ such that $(\phi(t_0),u_0)<\delta$ implies $(\phi(t),r(t))<b(\varepsilon)$ for all $t\geq t_0$ where $r(t)$ be a maximal solution of (2) relative to K For given $x_0=x(t_0)\in S_\rho$, we can take $u_0=u(t_0)$ in K such that

$a(|x(x_0)|)=(\phi(t_0),u(t_0))$ and

$V(t_0,x(t_0))\leq_K u_0$.

Note that if $x(t,t_0,x_0)$ is any solution of (1) such that $V(t_0,x(t_0))\leq_K u_0$, then by Lemma 1.6, $V(t,X(t))\leq_K r(t)$.

From (iii), we may assume that $V(t,0)=0$. Su- ppose $u_0\in K^0$ and $(\phi(t_0),u_0)<\delta$. Since $V(t_0,x_0)$ is continuous in x , there exist $\bar{\delta}(u_0)>0$ such that $V(t_0,x_0)\leq_K u_0$ for any $\|x_0\|<\bar{\delta}$.

Now choose $\delta_1>0$ such that $a(\delta_1)<\delta$ and $\delta_1<\bar{\delta}$. Then the ineqalities $|x(t_0)|<\delta_1$ and $a(|x(t_0)|<\delta$ hold simultaneously. Since $b(|x(t)|)\leq(\phi(t),V(t,x(t))))\leq(\phi(t),r(t))<b(\varepsilon)$ for all $t\geq t_0$, $|x(t,t_0,x_0)|<\varepsilon$ whenever $|x(t_0)|<\delta_1$. Hence the trivail solution $x=0$ of (1) is equistable.

In the above, choosing $\delta=\delta(\varepsilon)$ which is independent of t_0 , the uniform stability follows from the same argument.

Suppose that the trivial solution $u=0$ of (2) is quasi-equi asymptotically $\phi(t)$- stable. Then, following the same arguments for all $t\geq t_0+T(\varepsilon)$ there exists a positive function $\delta=\delta(t_0,\varepsilon)<\varepsilon$ satisfying $\|x_o\|<\delta$ and $a(\|x_0\|)<\delta$simultaneously.

It follows that $\|x_0\|<\delta$ implies $\|x(t,t_o,x_0)\|<\varepsilon,\ t\geq t_0+T(\varepsilon)$.

If this is not true, then there exists, a divergent sequence $\{t_k\}$, $t_k\geq t_0+T$ such that $\|x(t_k,t_0,\ x_0)\}$ for some $\|x_0<\delta\|<\delta$, $k=1,2,3\cdots$. Using (iii) and Lemma 1.6 we have $b(\varepsilon)\leq(\phi(t_k),V(t_k,x(t_k,t_0,x_0)))\leq(\phi(t_k),r(t_k,t_0,u_0))<b(\varepsilon)$f or some $u_0\varepsilon K$ which is a contradiction. The other stability properties can be similarly proved.

Theorem 2.7 Assume that

(i) $g\in C[R_+\times K,R^n]$, $g(t,0)=0)$, $g(t,u)$ is quasimonotone in u relative to K for each $t\in R_+$

(ii) $(\phi(t),r(t))\leq\beta(\phi(t),u(t)),\beta\in K,)$, $u(t)$ is a solution of (2)

If the trivial solution u=0 of (2) is uniformly asymptotically $\phi(t)$-stable, If the trivial solution $u=0$ of (2) is uniformly asymptotically $\phi(t)$ -stable, then there exists a cnve valued Lyapunov function v with the following properties

(A) $v\in C[R_+\times S^*(\rho),K]$, $v(t,0)=0$, $v(t,u)$ is locally Lipschitzian in u relative to K for each $t\in R_+$.

(B) For some $\phi(t)\in K_0$

$(\phi(t),v(t,u(t)))\leq b[(\phi(t),r(t))]$, $a,b\in K$.

(C) For $(t,u)\in R_+\times S^*(\rho)$, and for $p(t)$ is increasing and bounded, $D^+(\phi(t),v(t,u))\leq k-p'(t)(\phi(t),v(t,u))$, where $p'(t)$ exists.

Proof. Let $u=u(t,0,u_0)$so that $u_0=u(0,t,u)$ Define a cone-valued function $v(t,u(t))$ by

$$v(t,u(t))=\ \exp(-p(t))C[(\phi(t),r(t))]u(t,,0,\sigma_\omega(u(0,t,u))) \qquad (11)$$

where $C[(\phi(t),r(t))]=(\frac{1}{D})[1-\exp[-D,(\phi(t),r(t),r(t)0]]$ $D>0,p'(t)$ exist and $\sigma_w(x)$is the function defined in (7) ; and

$u(t, u_0)$is any solution of (2).

When $u=0$, then the right hand side of (11) vanishes so that $v(t,0)=0$. Using (i) and Corollary 2.7.1 in [5] and for $u_1, u_2 \in S(\rho)$

$\| v(t, u_1) - v(t, u_2) \|_G = \| \exp(-p(t)) C[(\phi(t), r(t))] u_1(t, 0, \sigma_w(u_1(0, t, u_1)))$- $\exp(-p(t)) C[(\phi(t), r(t))] u_2(t, 0, \sigma_w(u_2(0, t, u_2))) \|_G$

$\leq_K ||N(t)|| \| \sigma_w(u_1(0, t, u_1)) - \sigma_w(u_2(0, t, u_2)) \|_G \times \exp \int_0^t L(s) ds \leq_K l(t) \| w \| \| N(t) \| \| u_1 - u_2 \|_G \exp \int_0^t L(s) ds$

$= \beta(t) \| u_1 - u_2 \|_G$,

where

$\beta(t) = l(t) \| w \| \exp(-p(t)) C[(\phi(t), r(t))] \| \exp \int_0^t L(s) ds \geq 0$

Now

$\| tv(t+\delta, u^*) - v(t, u) \| v(t+\delta, u^*) - v(t+\delta, u) \| + \| v(t+\delta, u) - v(t+\delta, u(t+\delta, u(u+\delta, t, u)) \| + \| v(t+\delta, u(t+\delta, u)) - v(t, u) \|$

Since $v(t, u)$ is locally Lipschitizan in u and v is continuous in δ, then the frist two terms in the right hand side of the inequality are small whenever $\| u - u^* \|$ and δ are samll Using (11), the third term tends to zero as δ tends to zero. Therefore (t, u) is continuous in all its arguments. Since $u=0$ is uniformly asymptotically $\phi(t)$-stable, them given $\varepsilon > 0$, there exist two numbers $\delta = \delta(\varepsilon)$ and $T = T(\varepsilon)$, independ- ent of t_0 such that

$(\phi(t_0), u(t_0)) < \delta \rightarrow (\phi(t), r(t)) < \varepsilon$ for $t > T(\varepsilon)$.

And so

$(\phi(t), v(t, u(t))) = \exp(-p(t)) C[(\phi(t), r(t))](\phi(t), u(t, 0, \sigma_w(u(0, t, u)) \leq \varepsilon \cdot C[(\phi(t), r(t))] = b[(\phi(t), r(t))], b \in K$

$(\phi(t), v(t, U(t))) = \exp(-p(t)) C[(\phi(t), r(t))](\phi(t), u(t, 0, \sigma_w(u(0, t, u))) \geq q C[(\phi(t), r(t))] \beta^{-1} [(\phi(t), r(t))]$ by condition (ii)

$= a[(\phi(t), r(t))], a \in K$ since $C, \beta^{-1} \in K$

where $q = \infty \exp(-p(t))$

Hence $a[(\phi(t), r(t))] \leq (\phi(t), v(t, u)) \leq b[(\phi(t), r(t))], a, b \in K$.

$v(t+h, u+hg(t, u)) - v(t, u) \leq_K \beta(t) \mid u + hg(t, u) - u(t+h, t, u) \mid \cdot e(t, z, h) + v(t+h, u(t+h, t, u)) - v(t, u)$,

where $\lim_{t \to \infty}$

Dividing both side by $h > 0$ and taking limsup as $h \to 0^+$, and using (11) and unoqueness of (2) we obtain

$D^+(\phi(t), v(t, u)) \leq \lim_{h \to 0^+} C[(\phi(t+h), r(t+h))] \times (\phi(t), u(t+h, 0, \sigma_w(u(0, t+h, u)))) - \exp(-p(t)) C[\phi(t), r(t)](\phi(t), u(t, 0, \sigma_w(u(0, t, u))))] = \exp(-p(t)) C [(\phi(t), r(t))](\phi(t), u(t, 0, \sigma_w(u(0, t, u)))) \times \lim_{h \to 0^+}$

$= -p'(t)(\phi(t), v(t, u))$.

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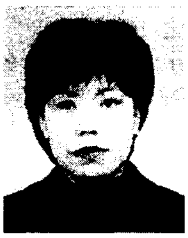

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