EIGENVALUE APPROACH FOR UNSTEADY FRICTION WATER HAMMER MODEL

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Abstract: This paper introduces an eigenvalue method of transforming the hyperbolic partial differential equations of a particular unsteady friction water hammer model into characteristic form. This method is based on the solution of the corresponding one-dimensional Riemann problem that transforms hyperbolic quasi-linear equations into ordinary differential equations along the characteristic directions, which in this case arises as the eigenvalues of the system. A mathematical justification and generalization of the eigenvalues method is provided and this approach is compared to the traditional characteristic method.

Keywords: Eigenvalue, Water Hammer, Unsteady Friction, Hyperbolic Partial Differential Equation

1. INTRODUCTION

Water hammer analysis is crucially important for estimating a variety of worst-case and challenging transient events in a Water Distribution Systems (WDS). In essence, transients occur whenever flow conditions are changed, but they are generally most important when rapid changes occur, say associated with power failure events, valve operations or fire fighting. Transient events are generally characterized by fluctuating pressures and velocities and are important precisely because these fluctuations can be of high magnitude, possibly large enough to break or damage pipes or other equipment, or to disrupt delivery conditions (Boulos *et al.*, 2004).

Transient flow in pipes is described by nonlinear hyperbolic partial differential equations which are derived from the continuity and momentum equations. A general solution of these equations is impossible due to the nonlinearity of the momentum equation and the complexity of both pipe networks and the associated boundary conditions. Various methods have been developed for analyzing transient flow in pressurized conduits. By linearizing the friction term and dropping other nonlinear terms in the equations of momentum and continuity, an approximate analytical solution to the equations may be found for sine-wave oscillations (Wylie and Streeter, 1993; Chaudhry, 1987). Any periodic forcing function can be handled by the so-called Impedance method (Suo and Wylie, 1989) or Transfer matrix method (Chaudhry, 1987); the forcing function is decomposed into various harmonics by Fourier analysis, and each harmonic is analyzed separately. Since all the equations and relationships are linear, the system response can be determined by superposition of individual responses, although there is, of course, an error introduced through the original linearization of the friction term.

The method of characteristics is the most popular approach for solving hydraulic transient problems; it transforms the partial differential equations of the continuity and momentum relations into ordinary differential equations (Abbott, 1966; Wylie and Streeter, 1993). These ordinary differential equations are then integrated to obtain a finite difference representation of the variables. The method of characteristics has several advantages including a firmly established stability criterion, an explicit solution so that different elements that are physically removed from one another in a system are handled independently, a procedure that is relatively simple, an approximation that is readily recognized, a high numerical accuracy when executed properly along the characteristic curves, and, for elementary systems, an implementation that includes a physical interpretation that is simple, yet precise (Wylie and Streeter, 1993). The traditional derivation of the characteristic equations is to combine the governing partial differential equations using an unknown multiplier. Values of this multiplier are determined by setting combinations of the terms involving partial derivatives to be equivalent to total derivatives (Wylie and Streeter, 1993; Chaudhry, 1987). Another approach, presented by Ghidaoui and Karney (1995), is to introduce a modified transformation of the hyperbolic partial differential equations into characteristic form using the total derivative concept for both open-channel and water-hammer applications. This modified differential equation shows the same characteristic form as the traditional method of characteristics and also transforms the governing hyperbolic equations.

Although this basic form of the governing equations is well known, a few challenging questions arise in the context of transient analysis work, such as the unsteady friction problem. Classical water hammer theory based on the assumptions of linear elastic behavior of the pipe-walls and quasi-steady friction losses is used to predict the maximum and minimum pressure surges. This approach is relatively accurate for simulating hydraulic transients in metal pipe, but it is considerably less precise for plastic pipes, particularly when the surge is generated by rapid changes in flow conditions. The characterization of unsteady friction is a challenging question and research is on-going (Brunone et al., 1991; Vitkovsky, 2001).

This paper presents an alternative method of transforming the hyperbolic partial different- tial equations of the unsteady friction water hammer model into characteristic form. This method is based on the solution of the corresponding one-dimensional Riemann problem and has been applied to several applications such as finding exact or approximate solutions of the shallow water equations (Kulikovskii et al., 2001; Weiyan, 1992); however, this method has not been, despite its wide applications for many kinds of quasi-linear hyperbolic systems, explicitly applied to the water hammer equations. Although the quasi-linear hyperbolic system of unsteady friction water hammer equations cannot be obtained as an exact solution with Riemann invariants (Kulikovskii *et al.*, 2001), the transformed equations can be conveniently handled numerically. The transformation provides crucial insight how the information is propagated along characteristic lines as well as the role of the simplified characteristic equations. A mathematical generalization of the eigenvalue approach is, for the first time, applied to the unsteady friction water hammer model. Although the method of characteristics is also applied to the same hyperbolic system and results in the same final system of equations as the eigenvalue approach, additional insight is achieved through the new approach.

2. EIGENVALUE APPROACH OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

The simplest hyperbolic equation is the linear scalar problem as follows:

$$u_t + au_x = 0 \tag{1}$$

with initial data $u(x, 0) = u_0(x)$, where u = dependent variable, t = time; x = distance; a = constant and partial derivatives are written as subscripted variables. The solution is

$$u(x,t) = u_0(x - at) \tag{2}$$

as is easily checked by differentiation. This form indicates that the solution at any time is a copy of the original function, but shifted by a distance at. In other words, the solution at (x, t) depends only on the value of $\xi = x - at$ which geometrically defines a characteristic curve. If the dependent variable is an n-vector, the hyperbolic equation is written as follows:

$$\widetilde{u}_t + A\widetilde{u}_x = 0 \tag{3}$$

where $\widetilde{u} = (u_1, u_2, \ldots, u_n)$ and A is $n \times n$ matrix. If A is diagonalizable, there must exist an invertible matrix P and a diagonal matrix $\Lambda = P^{-1}AP$ where P^{-1} = inverse of P. Using this identity, (3) can be rewritten as

$$\widetilde{v}_t + P^{-1}AP\widetilde{v}_x = 0 \tag{4}$$

where $\widetilde{v} = P^{-1}\widetilde{u}$. This is clearly equivalent to n independent scalar equations as follows:

$$(v_j)_t + \lambda_j (v_j)_x = 0, \quad j = 1, \ldots, n. \tag{5}$$

That is, the $n \times n$ linear system (3) has n characteristic speeds, given by the eigenvalues of A. If $A = A(x, t)$, then as above, the characteristic speeds $\lambda\ (x, t)$ also vary. The eigencomponents w_j propagate without interaction, except at boundaries where an incoming component can reflect energy into an outgoing component (Strikwerda, 1989).

3. APPLICATION FOR UNSTEADY FRICTION WATER HAMMER MODEL

The following water hammer model (Vitkovsky, 2001) is based on the model by Brunone *et al.* (1991) with improvements associated with using extended energy and momentum concepts. These improvements arise from derivations that include the velocity distribution to infer extra terms needed to describe unsteady friction in the familiar one dimensional water hammer model. The two hyperbolic PDEs representing the momentum equation and the relation of mass conservation of transient flow in closed conduits are written in this way as

$$L_1 = H_t + \frac{a^2}{g} V_x = 0 \tag{6}$$

$$L_2 = H_x + \frac{1}{g} V_t + \frac{fV|V|}{2gD} + \frac{k}{g}\left(V_t + a \cdot SGN(V) \cdot |V_x|\right) = 0 \tag{7}$$

where x = distance along the centerline of the conduit; t = time; H = piezometirc head; V = fluid velocity; D = inside pipe diameter; a = celerity of the shock wave; g = acceleration due to gravity; f = the steady state Darcy-Weisbach friction factor; k = unsteady friction coefficient and 'SGN' stands for the sign of the velocity. By defining $s = SGN(V \cdot V_x)$ and multiply (7) by $g/(1+k)$, the momentum equation can be rewritten as

$$\frac{g}{1+k}H_x + V_t + \frac{kas}{1+k}V_x + \frac{fV|V|}{2D(1+k)} = 0 \quad (8)$$

Combining (8) with (6) yields

$$\tilde{u}_t + A\tilde{u}_x = \tilde{b} \quad (9)$$

where $\tilde{u} = \begin{bmatrix} V \\ H \end{bmatrix}$, $A = \begin{bmatrix} \frac{kas}{1+k} & \frac{g}{1+k} \\ \frac{a^2}{g} & 0 \end{bmatrix}$ and

$$\tilde{b} = \begin{bmatrix} -\frac{fV|V|}{2D(1+k)} \\ 0 \end{bmatrix}$$

Equation (9) is of the same basic form as (3) with the key difference being the additional column vector that represents frictional dissipation. The eigenvalues of A are $\lambda_1 = as$ and $\lambda_2 = -as/(1+k)$, and the corresponding eigenvectors for λ_1 and λ_2 are $P_1 = [g, as]^T$ and $P_2 = [g, -as(1+k)]^T$, respectively. In order to transform matrix A into diagonal form, the identity $P^{-1}AP = \Lambda$ is used in which

$$P = [P_1 P_2] = \begin{bmatrix} g & g \\ as & -as(1+k) \end{bmatrix} \quad (10)$$

The matrix Λ is here a diagonal matrix and its entries are the eigenvalues of the matrix A. Introducing a new variable $\tilde{u} = P\tilde{v}$ into (9) and multiplying the resulting equation by P^{-1} produces

$$\tilde{v}_t + \Lambda\tilde{v}_x = P^{-1}\tilde{b} \quad (11)$$

where $\Lambda = P^{-1}AP = \begin{bmatrix} as & 0 \\ 0 & -as/(1+k) \end{bmatrix}$ and

$$P^{-1}b = \begin{bmatrix} \frac{-fV|V|}{2Dg(2+k)} & \frac{-fV|V|}{2Dg(2+k)(1+k)} \end{bmatrix}^T$$

The definition of total derivative of $\tilde{v}$ is given by

$$\frac{d\tilde{v}}{dt} = \tilde{v}_t + \mathrm{I}\left(\frac{dx}{dt}\right)\tilde{v}_x \quad (12)$$

where I = identity matrix; and dx/dt = vector defining the path of transformation of $\tilde{v}$. Substituting $\tilde{v}_t$ of (11) into (12) and rearranging produces

$$\frac{d\tilde{v}}{dt} = \left(\Lambda - \mathrm{I}\frac{dx}{dt}\right)\tilde{v}_x + P^{-1}\tilde{b} \quad (13)$$

If dx/dt is defined as Λ, the first term of RHS is eliminated and (13) can be simplified as follows:

$$\frac{d\tilde{v}}{dt} = P^{-1}\tilde{b} \quad (14)$$

Equation (14) shows that the quasilinear hyperbolic system can be transformed into characteristic form along the characteristic lines given by the eigenvalues of the system. The rationale underlying the characteristic form is that, by an appropriate choice of coordinates, the original system of hyperbolic first-order equations can be replaced by a simplified system in the characteristic coordinates. Characteristic coordinates are the natural coordinates of the system in the sense they simplify the form and expression of the physical relations that govern the hyperbolic system.

Due to the non-constant term on the RHS, the hyperbolic system cannot be written exactly as Riemann invariants (Kulikovskii *et al.*, 2001); but the invariant-like term provides an important role in finding the numerical solution of the systems and in understanding how information is propagated along the characteristic directions. Equation (14) can be integrated to yield finite difference equations, which are conveniently handled numerically. To convert the new dependent variables to original ones, substituting $\tilde{v} = P^{-1}\tilde{u}$ into (14) yields

$$\frac{1+k}{g}\frac{dV}{dt}+\frac{1}{as}\frac{dH}{dt}=-\frac{fV|V|}{2Dg}$$
$$\text{along } \frac{dx}{dt}=as \qquad (15)$$

$$\frac{1}{g}\frac{dV}{dt}-\frac{1}{as}\frac{dH}{dt}=-\frac{fV|V|}{2Dg(1+k)}$$
$$\text{along } \frac{dx}{dt}=-\frac{as}{1+k} \qquad (16)$$

4. APPROACH WITH THE METHOD OF CHARACTERISTICS

Traditionally, the method of characteristics (MOC) has been one of the simplest and most computationally efficient techniques used to solve the unsteady state pipe flow equations (Wylie and Streeter, 1993; Vitkovsky, 2001). The basis for the method is the ability to transform the partial differential equations into ordinary differential equations that apply along specific lines called characteristics. The MOC, being an extremely flexible solution scheme, can be easily implemented in networks, at boundary conditions for non-pipe elements and with a variety of frictional forms. In addition, the MOC is well suited to handling discontinuities such as those caused by instantaneous changes in conditions. A linear combination of the equations (1) and (2) can be described by $\lambda L_1 + L_2$ and results in

$$\lambda\left(H_t+\frac{1}{\lambda}H_x\right)+\frac{1+k}{g}\left(V_t+\frac{a(ks+\lambda a)}{(1+k)}V_x\right)+\frac{fV|V|}{2gD}=0 \qquad (17)$$

The partial derivatives in (17) are transformed into total derivatives along the characteristic lines following

$$\frac{dx}{dt}=\frac{1}{\lambda}=\frac{a(ks+\lambda a)}{(1+k)} \qquad (18)$$

The solution of (18) from the multiplier λ can be found using the quadratic formula, producing the solutions

$$\lambda=\frac{1}{as} \text{ and } -\frac{(1+k)}{as} \qquad (19)$$

Therefore, (17) is now expressed in terms of total derivatives as

$$\lambda\frac{dH}{dt}+\frac{1+k}{g}\frac{dV}{dt}+\frac{fV|V|}{2gD}=0 \qquad (20)$$

The solutions for the multiplier λ are substituted into (20) forming two compatibility equations and the resulting equations are identical to those in (15) and (16).

5. CONCLUSION

The equations of continuity and momentum that govern the water hammer problem can be transformed into ordinary differential equations using an eigenvalue approach. The mathematical procedure to transform a one dimensional quasilinear partial differential system into characteristic form is straightforward and the simplified form of the system clearly shows how the information in hyperbolic systems is propagated along the characteristic lines defined by the eigenvalues of the system. In addition, it is shown that the method presents the same form as the traditional method of characteristics. Thus, the new approach has same flexibility and ease of implementation for the many different boundary conditions found in pipe network applications.

The basic rationale underlying both methods is that, by an appropriate choice of coordinates, the original system of hyperbolic equations can be replaced by a system whose coordinates are the characteristics. The simplification of the method of characteristics is particularly useful when applied to problems involving one or two first-order equations in two independent variables, but the eigenvalue approach is more widely applied for many kinds of quasi-linear hyperbolic systems.

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