Performance Analysis of a Loss Retrial *BMAP/PH/N* System†

Che-Soong Kim*, Young-Jin Oh*

Abstract This paper investigates the mathematical model of multi-server retrial queueing system with the Batch Markovian Arrival Process (*BMAP*), the Phase type (*PH)* service distribution and the finite buffer. The sufficient condition for the steady state distribution existence and the algorithm for calculating this distribution are presented. Finally, a formula to solve loss probability in the case of complete admission discipline is derived.

Key Words: Retrial Queueing, *BMAP, PH* type, Markov Chain, Loss Probability

1. Introduction

Quick progress telecommunications implies of new queueing models interesting for research. One of the first investigated queueing models is Erlang loss model-the queue of the *M/M/N/0* type. It is used as a background for decision making in telephone systems until now. In this model, the arriving customer who finds all servers are busy upon arrival leaves the system forever without the service. It is considered
Because the behavior of the users of telephone networks is different from the assumed in Erlang loss model(the user can try to initiate the call a little bit later), retrial queueing models, which are characterized by the fact that the rejected call does not leave the system forever, but try the luck after some random time, are investigated intensively.

If the queueing model has N, $N \geq 2$ servers, finite buffer (or the buffer is absent at all) and the batch arrivals are possible, situations can occur when the number of free servers at the arrival epoch is less than the number of customers in the arriving batch. Different strategies of customers admission can be exploited. Here we analyze the following variant. If the arriving batch meets all servers be busy, it leaves the system forever without effecting on the system behavior. However, if at least one server is idle, the batch is admitted into the system. If the number of available servers is sufficient to serve all customers of a batch, the service starts immediately. In opposite case, a part of the customers start the service while the rest goes to so called orbit and try to get service later on. So, the considered model combines features of retrial and loss models. To the best of our knowledge, such models were not considered yet, at least in the context of the *BMAP/PH/N* type model.

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2. The Mathematical Model

The service device consists of N parallel identical servers. Service time distribution is of PH type. It means the following. The service process is directed by the continuous time Markov process m_t, $t \geq 0$. The state of this process at the service beginning epoch is defined according to the probabilistic row-vector $\vec{\beta} = (\beta_1, \cdots, \beta_M)$. Further, transitions of the process $m_t, t \geq 0$ are defined by the matrix S of dimension $M \times M$. The diagonal entries of the matrix are negative and $-S_{m,m}$ defines the parameter of the exponentially distributed sojourn time of the process in the state $m, |S_{m,m}| < \infty, m = \overline{1,M}$. The non-diagonal entries of the matrix S define the intensities of transitions of the process $m_t, t \geq 0$ in the state space $\{1, \ldots, M\}$. The value $-\sum_{m'=1}^{M} S_{m,m'}$ defines the intensity of the transition of the process $m_t, t \geq 0$ from the state m into the absorbing state. The epoch of the transition of the process $m_t, t \geq 0$ into the absorbing state defines the service completion epoch. Denote $S_0 = -Se$. Here and below eis a column-vector of appropriate size consisting of units. It is assumed that all the entries of the column-vector S_0 are non-negative and at least one of them is positive. The mean service time b_1 is calculated as $b_1 = \vec{\beta}(-S)^{-1}e$. The primary customers arrive to the system according to a *BMAP* (Batch Markovian Arrival Process). The notion of the *BMAP* and its detailed description is given by D.Lucantoni in [5]. Overview of related papers can be found in [2]. We denote the directing process of the *BMAP* by $\nu_t, t \geq 0$. The state space of the irreducible continuous time Markov chain ν_t is $\{0,1,\ldots,W\}$. As follows from [5], the behavior of the *BMAP* is characterized completely by the matrix generating function $D(z) = \sum_{k=0}^{\infty} D_k z^k, |z| < 1$. The matrix D_k characterizes the intensities of transitions of the process $\nu_t, t \geq 0$ which are accompanied by generating a batch of k customers, $k \geq 0$. The matrix $D(1)$ represents the generator of the process $\nu_t, t \geq 0$. The average arrival rate λ is defined as $\lambda = \vec{\theta} D'(1) e$ where $\vec{\theta}$ is the invariant vector of the stationary distribution of ν_t, $t \geq 0$. The vector $\vec{\theta}$ is the unique solution to the system $\vec{\theta} D(1) = \vec{0}, \vec{\theta} e = 1$. Here $\vec{0}$ is the row-vector of appropriate size consisting of zeroes. If all servers and busy at the epoch of a batch arrival, the batch is not admitted into the system and is considered be lost. If the number of idle servers is greater than the batch size, all arrived customers start the service and leave the system after its completion. If the batch size is bigger than the number of available servers, only a part of customers corresponding to a number of free servers starts processing while the rest moves to the orbit. Concerning the retrial process, we suppose that the inter-retrial times are exponentially distributed with the rate α_i which depend on the current number i of customers on the orbit. We assume that α_i approaches to infinity when $i \to \infty$. As a special case linear repeated requests can be handled.

3. Description of the System Behavior in Terms of Continuous Time Markov Chain

Let i_t be the number of calls on the orbit, $i_t \geq 0$, n_t be the number of busy servers, $n_t = \overline{0,N}, m_t^{(j)}$ be the state of the directing process of the service on the j^{th} busy server, $m_t^{(j)} = \overline{1,M}, j = \overline{1,n_t}$ (we assume here that the busy servers are numerated in order of their occupying, i.e. the server, which begins service, is appointed the maximal number among all busy servers; when some server finishes the service, the servers are correspondingly enumerated), v_t be the state of the directing process of the BMAP, $v_t = \overline{0,W}$, at the epoch $t, t \geq 0$.

Consider the multi-dimensional process $\xi_t = (i_t, n_t, v_t, m_t^{(1)}, \ldots, m^{(n_t)}), t \geq 0$. It is easy to see that this process is an irreducible Markov chain. Denote the stationary probabilities of this process as

$$p(i,n,v,m^{(1)},\ldots,m^{(n)}) = \lim_{t\to\infty} P\{i_t = i, n_t = n, v_t = v, m_t^{(1)} = m^{(1)}, \ldots, m_t^{(n)} = m^{(n)}\} \qquad (1)$$

for $i \geq 0, v = \overline{0,W}, m^{(j)} = \overline{1,M}, j = \overline{1,n}$, and $n = \overline{0,N}$. Enumerate the states of the chain $\xi_t, t \geq 0$, in lexicographic order and form the row-vectors $\vec{p}_i$ of the stationary state probabilities $p(i,n,v,m^{(1)},\ldots,m^{(n)})$, having dimensionality $K = (W+1)\frac{1-M^{N+1}}{1-M}, i \geq 0$. Define also the infinite-dimensional probability vector $\vec{p} = (\vec{p}_0, \vec{p}_1, \cdots)$.

Lemma. *If the in vector $\vec{p}$ exists then it satisfies the equilibrium equation*

$$\vec{p}A = \vec{0}, \qquad (2)$$

where $\vec{0}$ is the infinite row-vector consisting of zeroes and the matrix A is the infinitesimal generator of the chain $\xi_t, t \geq 0$, and has the following structure:

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & \cdots \\ A_{10} & A_{11} & A_{12} & A_{13} & \cdots \\ 0 & A_{21} & A_{22} & A_{23} & \cdots \\ 0 & 0 & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (3)$$

where the blocks A_{ij} of size $K \times K$ have the following form:

$$A_{i,i-1} = \alpha_i \begin{pmatrix} 0 & I_{\overline{W}} \otimes \beta & 0 & \cdots & 0 \\ 0 & 0 & I_{\overline{W}M} \otimes \beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{\overline{W}M^{N-1}} \otimes \beta \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad (4)$$

$$A_{i,i+k} = \begin{pmatrix} 0 & \cdots & 0 & D_{k+N} \otimes \beta^{\otimes N} \\ 0 & \cdots & 0 & D_{k+N-1} \otimes I_M \otimes \beta^{\otimes(N-1)} \\ 0 & \cdots & 0 & D_{k+N-2} \otimes I_M \otimes \beta^{\otimes(N-2)} \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & D_{k+1} \otimes I_{M^{N-1}} \otimes \beta^{\otimes 1} \\ 0 & \cdots & 0 & 0 \end{pmatrix}, k \geq 1 \qquad (5)$$

$$(A_{i,i})_{r,r'} = \begin{cases} 0, & r' < r-1, r = \overline{2,N} \\ I_{\overline{W}} \otimes S_0^{\oplus r}, & r' = r-1, r = \overline{1,N} \\ D_0 \oplus S^{\oplus r} - \alpha_i I_{\overline{W}M^r}, & r' = r, r = \overline{0,N,-1} \\ D_0 \oplus S^{\oplus N} + \sum_{k=1}^{\infty} D_k \otimes I_{\overline{W}M^N}, & r' = r = N \\ D_l \otimes I_{M^r} \otimes \beta^{\otimes l}, & r' = r+l, l = \overline{1,N-r}, \\ & r = \overline{0,N-1} \end{cases} \qquad (6)$$

Here $\delta_{r,N} = \begin{cases} 1, r = N, \\ 0, r \neq N, \end{cases}$ is *Kronecker's symbol,* $\otimes$ *is the sign of Kronecker's product, and* $\oplus$ *is the sign of Kronecker's sum.* $\beta^{\otimes l} \stackrel{def}{=} \underbrace{\beta \otimes \ldots \otimes \beta}_{l}, l \geq 1$, $S^{\oplus l} \stackrel{def}{=} \underbrace{S \oplus \ldots \oplus S}_{l}, l \geq 1$, $S^{\oplus 0} \stackrel{def}{=} 0$, $S_0^{\oplus l} \stackrel{def}{=} \sum_{m=0}^{l-1} I_{M^m} \otimes S_0 \otimes I_{M^{l-m-1}}, l \geq 1, \overline{W} = W+1$, I_L *and* 0_L *denote the identity matrix and zero matrix correspondingly of size* $L \times L, I_{M^0} = 1$.

It is easy to see that the generator A of the

Markov chain $\xi_t, t \geq 0$, differs from the corresponding generator of the analogous Markov chain for the *BMAP/PH/N* retrial system, which was investigated in [1], only with the last block entry of the last row of the matrix $A_{i,i+k}$ and the entry $(A_{i,i})_{N,N}$. So, technique of [1] can be effectively exploited to investigate the considered system. It means the following. Instead of investigating the continuous-time Markov chain $\xi_t, t \geq 0$, we deal with the discrete-time Markov chain has one-step of the Markov chain $\xi_t, t \geq 0$ transitions. This discrete-time Markov chain has one-step transition probability matrix which is obtained from the generator A by dividing entries of each its row by the modulus of the corresponding diagonal entry and adding 1 to the diagonal entry.

4. Stability Conditions

By analogy with [1], we can show that this discrete-time Markov chain belongs to the class of the so-called asymptotically quasi-toeplitz Markov chains introduced in [4].

Theorem 1. *Steady-state distribution of the considered queueing system (as well as the stationary distribution of both considered continuous and discrete tine Markov chains) exists for all values of the system parameters.*

The outline of the proof is the following. As follows from [4], stationary state distribution of the asymptotically quasi-toeplitz Markov chain exists if the stationary state distribution of its limiting quasi-toeplitz Markov chain exists. The limiting quasi-toeplitz Markov chain is characterized (see [3], [4]) by the matrix generating function of one-step transition probabilities $\hat{Y}(z)$ that here has the following form:

$$\hat{Y}(z) = \begin{pmatrix} 0 & I_{\overline{W}} \otimes \beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{\overline{W}} \otimes \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & O_{\overline{W}M^{N-1} \times \overline{W}M^{N-1}} & I_{\overline{W}M^{N-1}} \otimes \beta \\ 0 & 0 & 0 & \cdots & C^{-1}(I_{\overline{W}} \otimes S_0^{\oplus N})z & C^{-1}z(D(1) \oplus S^{\oplus N}) + zI \end{pmatrix}$$

where the diagonal matrix C has the diagonal entries coinciding with the modulus of the corresponding entries of the matrix $D(1) \oplus S^{\oplus N}$. It can be verified that the matrix $\hat{Y}(z)$ is reducible and the unique irreducible block of this matrix is the matrix $\hat{Y}(z)$ having the following form:

$$\hat{Y}(z) = \begin{pmatrix} C^{-1}z(D(1) \oplus S^{\oplus N}) + zI & C^{-1}(I_{\overline{W}} \otimes S_0^{\oplus N})z \\ I_{\overline{W}M^{N-1}} \otimes \beta & 0_{\overline{W}M^{N-1} \times \overline{W}M^{N-1}} \end{pmatrix}$$

Then, according the results from [4], sufficient condition of the stationary distribution existence for the Markov chains under consideration is the fulfillment of the condition:

$$(\det(zI - \hat{Y}(z)))'|_{z=1} > 0 \tag{7}$$

Taking into account the block structure of the matrix $\hat{Y}(z)$, the determinant in (7) can be transformed to the form:

$$\det(zI - \hat{Y}(z)) = (\det C^{-1})z^{\overline{W}M^{N-1}} \det T(z) \tag{8}$$

where $T(z) = -z(D(1) \oplus S^{\oplus N}) - (I_{\overline{W}} \otimes S_0^{\oplus N})(I_{\overline{W}M^{N-1}} \otimes \beta)$

Differentiating (8) at the point $z=1$, we can show that condition (7) is equivalent to inequality

$$(\det T(z))'|_{z=1} > 0 \tag{9}$$

Decomposing the determinant of T (z) in the entries of ant column, it can be shown that inequality (9) is equivalent to inequality

$$\vec{x}(I_{\overline{W}} \otimes S^{\oplus N})e < 0 \tag{10}$$

where the vector $\vec{x}$ is the unique solution to the following system of linear algebraic equations

$$\vec{x}T(1) = \vec{0},\ \vec{x}e = 1 \tag{11}$$

By direct substitution, it can be shown that solution of system (11) has a form

$$\vec{x} = \vec{\theta} \otimes \vec{y} \tag{12}$$

where the vector $\vec{y}$ is the unique positive solution to the following system of linear algebraic equations:

$$\vec{y}\left(S^{\oplus N} + S_0^{\oplus N}(I_{\overline{W}M^{N-1}} \otimes \beta)\right) = \vec{0},\ \vec{y}e = 1 \tag{13}$$

Substituting (12), into (10), we get inequality

$$\vec{y}S^{\oplus N}e < 0 \tag{14}$$

It follows from (13) that (14) is equivalent to

$$\vec{y}S_0^{\oplus N}(I_{\overline{W}M^{N-1}} \otimes \beta)e > 0 \tag{15}$$

Because vector $\vec{y}$ is positive while the matrix $S_0^{\oplus N}(I_{\overline{W}M^{N-1}} \otimes \beta)$ is non-negative and non-zero, we conclude that inequality (15) is fulfilled any values of the system parameters. Theorem 1 is proven.

5. Algorithm for Calculation of the Stationary Distribution and Loss Probability

The algorithm for calculation of the stationary probability vectors $\vec{p}_i, i \geq 0$ is the same as one elaborated in [1]. One of the most important characteristics of the considered model is the probability P_{loss} that arbitrary customer is lost in the system.

Theorem2. *Loss probability P_{loss} in the case of completion admission discipline is calculated as follows:*

$$P_{loss} = 1 - \lambda^{-1}\sum_{i=0}^{N-1}\vec{p}_i\sum_{k=1}^{\infty}k\tilde{D}_k^{(i)}\vec{e} \tag{16}$$

where $\tilde{D}_k^{(i)} = D_k \otimes I_{M^i}, k \geq 1, i \geq 0.$

The outline of the proof is the following. According to a formula of the total probability, the loss probability P_{loss} is calculated as

$$P_{loss} = 1 - \sum_{i=0}^{N-1}\sum_{k=1}^{\infty}P_k P_i^{(k)}R^{(i,k)} \tag{17}$$

where P_k is a probability that an arbitrary customers arrives in a batch consisting of k customers, $P_i^{(k)}$ is a probability to see i servers being busy at the epoch of the k size batch arrival, $R^{(i,k)}$ is a probability that an arbitrary customer will not be loss conditional it arrives in a batch consisting of k customers and i servers are busy at the arrival epoch. It can be shown that

$$P_i^{(k)} = \frac{\vec{p}_i\,\tilde{D}_k^{(i)}\,\vec{e}}{\vec{\theta}\,D_k\,\vec{e}},\ i = \overline{0, N-1},\ k \geq 1 \tag{18}$$

$$P_k = \frac{k\vec{\theta}D_k\vec{e}}{\vec{\theta}\sum_{l=1}^{\infty}lD_l\vec{e}} = k\frac{\vec{\theta}\,D_k\,\vec{e}}{\lambda},\ k \geq 1 \tag{19}$$

$$R^{(i,k)} = \begin{cases} 1, & i \leq N-1 \\ 0, & i > N-1 \end{cases} \tag{20}$$

By substituting (18)-(20) into (17) after some algebra we get (16). Theorem 2 is proven.

6. Conclusions

This paper investigates the mathematical model of multi-server retrial queueing system with the Batch Markovian Arrival Process, the Phase type service distribution and the finite buffer. The Sufficient condition for the steady distribution existence and the algorithm for calculating this distribution are presented. The presented results give a straightforward algorithmic way for calculation of performance measures of the considered *BMAP* model. Also the considered model combines features of retrial and loss models. The results can be extended to the case of another disciplines. For example, the following disciplines, which occur in different real life systems, can be accounted.

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