IE Interfaces
Vol. 17, No. 1, pp. 78-83, March 2004.

A Study on EOQ Model Involving Estimate Errors

Gyutae Kim[1] · Hark-Chin Hwang[1†] · Chang-Hyun Kim[2]

[1]Department of Industrial Engineering, Chosun University, 501-759
[2]Division of Transportation and Logistics Systems Engineering, Yosu University, 550-749

수요, 주문 및 재고비용이 불확실한 상황에서의 EOQ 모형에 관한 연구

김규태[1] · 황학진[1] · 김창현[2]

[1]조선대학교 산업공학과 / [2]여수대학교 교통물류시스템 공학부

We consider the sensitivity of average inventory cost rate when true values of the parameters in the EOQ model are unknown over known ranges. In particular, in the case that the valid range on the true economic lot size are known, we provide a formula for estimating the lot size under minimax criterion. Moreover, to estimate the valid range, we apply the propagation of errors technique. Then, we present a scheme to find a (valid) lot size, based on the estimated range of the true lot size from the propagation of errors technique.

Keyword: EOQ, sensitivity, propagation of errors technique

1. Introduction

In the basic Economic Order Quantity (EOQ) model, the optimal order quantity or lot size is determined by the three parameters of average demand rate, order (setup) cost, and inventory holding cost. When we know the exact values of the parameters, we can get the true(and optimal) lot size using the well-known EOQ formula. The values of the parameters are mainly measured in manufacturing or accounting departments. However, they often do not have the precise values but instead ranges for the estimated values of the parameters: the ranges to which they thought the true values might belong. Hence, in practice we unavoidably have additional costs from the estimation errors, i.e., the difference between the average cost rate with precise values and the average cost rate with values of some errors.

In this situation, one methodology for making a decision on order quantity is to deploy the minimax criterion, likely to be used by risk-averse managers who desire to select alternatives that avoid the worst possible outcome. Many studies have been done for the sensitivity analysis of average cost rate to errors in parameter estimation (Groff and Muth, 1972, Lowe and Schwarz, 1983). In particular, Lowe and Schwarz provided an objective function to measure the effect of errors in parameter estimation: the ratio of the average cost rate with imprecise values to the average cost rate with true values, denoted by $R(Q)$. Then, the policy for decision making is to determine the lot size which minimizes the maximum of $R(Q)$.

† Corresponding author : Hark-Chin Hwang, Department of Industrial Engineering, Chosun University, 501-759, Fax : +82-62-230-7128;
E-mail : hchwang@chosun.ac.kr
Received February 2003, accepted December 2003 after 1 revision.

However, it is questionable whether the lot size generated from the policy is within the valid range of the true size. Given a range to which the true optimal lot size belongs, it is trivial to check any suggested lot size, including the lot size from the policy, is valid or not. In case that no valid range is provided, we need to estimate the range of lot size using the estimated parameters of the EOQ model.

In this paper, we modify the policy to satisfy the validity constraint on lot sizes. To this end, a new formula for lot sizing has been derived to deal with the case when we are given a valid range of the true size. Moreover, in order to provide an estimate of the valid range, we deploy the propagation of errors technique. Illustrative examples will be presented to show the applicability of the technique. Finally, we present the scheme that finds a (valid) lot size, based on the estimated range of the true lot size from the propagation of errors technique.

2. Sensitivity of the EOQ Model

We are given ranges of estimations for the parameters, demand rate D, setup cost S, and inventory holding cost h as follows:

$$\begin{array}{ccccc} \underline{D} & \leq & D & \leq & \overline{D} \\ \underline{S} & \leq & S & \leq & \overline{S} \\ \underline{h} & \leq & h & \leq & \overline{h} \end{array} \tag{1}$$

When one requires the amount of Q in each time of order, the average cost rate, $ACR(Q)$, is

$$ACR(Q) = (SD/Q) + (hQ/2).$$

The minimum of the average cost rate is attained at the lot size of $Q^* = \sqrt{2SD/h}$ with the cost rate

$$ACR(Q^*) = \sqrt{2SDh}\,. \tag{2}$$

If one uses $\hat{Q}$ instead of Q due to estimation errors, additional costs incurs. To measure how much cost is increased, we use the ratio of the average cost ratio for $\hat{Q}$ to the average cost ratio for Q^*:

$$R(\hat{Q}) = \frac{ACR(\hat{Q})}{ACR(Q^*)}\,.$$

Let $\overline{K}$ be the set of triples (S, D, h) which satisfy the constraint (1). Though each parameter can have any value in its range, but all the triples in $\overline{K}$ might not be valid because of the interactions between the parameters. In general, there are two kinds of interactions between parameters: positive interaction that the value of one parameter is likely to increase as the value of the other parameter increases, and negative interaction that the value of one parameter has a tendancy to decrease as the value of the other parameter increases. Suppose that the parameter h has negative interaction against with the other parameters S and D. In this case, it is likely in real production that the three values $\overline{S}, \overline{D}, \underline{h}$ might appear at the same time with high probability whereas the values of $\overline{S}, \overline{D}, \underline{h}$ might not coexist.

Let $\phi(S, D, h)$ be the joint probability distribution function for the triples $(S, D, h) \in \overline{K}$. Then, for the aboue supposition on the interactions between the parameters, the probability of $(\overline{S}, \overline{D}, \underline{h})$ will be high, while the probability of $(\overline{S}, \overline{D}, \overline{h})$ is zero, i.e., $\phi(\overline{S}, \overline{D}, \overline{h}) = 0$. When $\phi(S, D, h) = 0$ for some $(S, D, h) \in \overline{K}$, we say that the triple (S, D, h) is not *valid*. Then, we can fromally define the valid set K as follows:

$$K = \{(S, D, h) : (S, D, h) \text{ is valid, } (S, D, h) \in \overline{K}\}.$$

Note that the triple $(S, D, h) \in \overline{K}$ is not valid if $(S, D, h) \notin K$.

When the given parameters are unknown, it is natural to choose the alternative that minimizes the worst-case outcome. This is called *minimax* criterion. In this criterion, we find the $\hat{Q}$ satisfying:

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in K} R(\hat{Q}) \tag{3}$$

As it is not easy to get the set K in practice, the set $\overline{K}$ was used instead of K in the Lowe and Schwarz's model.

2.1 Lowe and Schwarz's Model

Lowe and Schwarz(1983) considered the problem

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in \overline{K}} R(\hat{Q}) \tag{4}$$

and showed that the lot size minimizing the maximum risk is $\hat{Q}^* = \left(4\,\overline{S}\overline{D}\,\underline{S}\underline{D}/\left(\overline{h}\underline{h}\right)\right)^{1/4}$. However, we note that the order quantity $\hat{Q}^*$ may not be valid, that is, no $(\hat{S},\hat{D},\hat{h}) \in K$ may exist with $\sqrt{2\hat{S}\hat{D}/\hat{h}} = \hat{Q}^*$. The following example shows some $\hat{Q}^*$ is not valid.

Example 1.

Let H^+ be the half space which includes all the points above or on the hyperplane crossing the three points $(\overline{S},\underline{D},\underline{h})$, $(\underline{S},\overline{D},\underline{h})$, $(\underline{S},\underline{D},\overline{h})$ (the point $(\overline{S},\overline{D},\overline{h})$ belongs to H^+). Then, the true set K is defined as $K = \overline{K} \cap H^+$. Let V be the volume of the hexahedron $\overline{K}$. Then, the volume of K is 5/6 times V. The variables (S,D,h) are uniformly distributed with joint distribution function $\phi(\cdot)$:

$$\phi(S,D,h) = \begin{cases} \dfrac{1}{5V/6}, & \text{if } (S,D,h) \in K \\ 0, & \text{otherwise.} \end{cases}$$

When the ranges on the parameters, S, D, and h are $[60, 90]$, $[100000, 200000]$ and $[7, 9]$, respectively, H^+ is the set

$$H^+ = \{(S,D,h) : 200000S + 60D + 3000000h \geq 45000000\}.$$

In this case, the lot size satisfying (4) is $\hat{Q}^* = 1,618$. As the lot size Q is a function of S, D and h, i.e., $Q = \sqrt{2SD/h}$, we can get possible values (true region) of Q from the true set K. Hence, we consider the minimum and maximum value of Q in the set K. Using an optimization tool MATLAB (The MathWorks, Inc. (2000)), we can get the minimum and maximum values, 1,630 and 2,267, respectively. We note here that the lot size $\hat{Q}^*$ is less than the minimum value. Hence, this lot size is not valid.

2.2 The Extended Model

Suppose that we are given a valid range for the true lot size,

$$\underline{Q} \leq Q \leq \overline{Q}, \tag{5}$$

as well as the ranges for the parameters of (1). Since the range (5) is valid, for each Q, $\underline{Q} \leq Q \leq \overline{Q}$, there exists at least one triple $(S,D,h) \in K$ such that $\sqrt{2SD/h} = Q$. We define another set of triples $\tilde{K}$ as $\tilde{K} = \left\{(S,D,h) : \underline{Q} \leq \sqrt{2SD/h} \leq \overline{Q},\ (S,D,h) \in \overline{K}\right\}$. In order to accommodate the validity information (5), the problem (4) is modified to

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in\tilde{K}} R(\hat{Q}). \tag{6}$$

Note that $ACR(\hat{Q}) = (SD/\hat{Q}) + (h\hat{Q}/2)$, which can be written as

$$ACR(\hat{Q}) = \sqrt{2SDh}\left[\frac{1}{\hat{Q}}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{SD}{h}}\right) + \hat{Q}\left(\frac{1}{2\sqrt{2}}\sqrt{\frac{h}{SD}}\right)\right].$$

Since $ACR(Q^*) = \sqrt{2SDh}$ by (2), from the definition of $R(\hat{Q})$, we have

$$R(\hat{Q}) = \left[\frac{1}{\hat{Q}}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{SD}{h}}\right) + \hat{Q}\left(\frac{1}{2\sqrt{2}}\sqrt{\frac{h}{SD}}\right)\right]$$

Then, the problem is equivalent to

$$\min_{\hat{Q}>0} \max_{(S,D,h)\in\tilde{K}} \left\{\left[\frac{1}{\hat{Q}}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{SD}{h}}\right) + \hat{Q}\left(\frac{1}{2\sqrt{2}}\sqrt{\frac{h}{SD}}\right)\right]\right\} \tag{7}$$

We let $y = SD/h$ with feasible region

$$\underline{y} \equiv \underline{Q}^2/2 \leq y \leq \overline{Q}^2/2 \equiv \overline{y}. \tag{8}$$

Then, we rewrite the problem (7) so that we consider only the variables $\hat{Q}$ and y:

$$\min_{\hat{Q}>0} \max_{\underline{y}\leq y\leq\overline{y}} \left\{2^{-1/2} y^{1/2} \hat{Q}^{-1} + 2^{-3/2} y^{-1/2} \hat{Q}\right\}. \tag{9}$$

From the same arguments that Lowe and Schwarz (1983) used, we can easily show that the optimal solution to (9) is $\hat{Q}_R^* = (4\underline{y}\overline{y})^{1/4}$. Thus, from this together with (8), we have $\hat{Q}_R^* = \sqrt{\underline{Q}\,\overline{Q}}$. In the following proposition, the final result is summarized.

Proposition 1.

The optimal solution to problem (6) is

$$\hat{Q}_R^* = \sqrt{\underline{Q}\,\overline{Q}}.$$

Then, the question remaining is how to get the estimated (valid) range of Q as in (5), which is the topic of the next section.

3. Error Analysis

Consider a function $y = f(x_1, \dots, x_n)$ where each variable x_j is defined for interval $(\underline{x}_j, \overline{x}_j)$ (in the EOQ model, $f(\cdot)$ is $f(S,D,h) = \sqrt{2SD/h}$). Then, we would like to get the valid range $(\underline{y}, \overline{y})$ of the decision variable y. Since it is often the case that each mid value $x_j = (\underline{x}_j + \overline{x}_j)/2$ is thought of as the most likely one, we set the most likely value of y as $y = f(x_1, \dots, x_n)$. Let $\Delta y = \overline{y} - y = y - \underline{y}$, which is called the composite error from the estimation errors of the variables x_j. To estimate the *composite error*, two methods of the total differential and propagation of errors are often used. The composite error Δy in total differential method is defined as

$$\Delta y = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \Delta x_j \tag{10}$$

In this equality, however, it is uncertain whether the effect of each individual error is to increase or decrease the combined error, which is a matter of randomness (Yoon, 1990). Hence, the range from the total differential method is so wide that it does not give us useful information.

In the propagation of errors technique, the error of y is not understood in terms of the approximate change to the disturbances of the variables x_j as in (10), but in terms of statistical deviation. From the statistical analyses in (Pugh and Winslow, 1996), for the standard deviation of y, we have

$$\sigma_y^2 = \left(\frac{\partial f}{\partial x_1}\sigma_2\right)^2 + \left(\frac{\partial f}{\partial x_2}\sigma_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\sigma_n\right)^2,$$

where σ_j is the standard deviation of variable x_j. When we replace σ_y and σ_j's by Δy and Δx_j's, we have for the composite error

$$(\Delta y)^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_j}\Delta x_j\right)^2. \tag{11}$$

Now, let's see how to apply the propagation of errors technique for the EOQ model, where the function $f(S,D,h) = \sqrt{2SD/h}$. In the EOQ model, for the corresponding composite error of (11), we have

$$\begin{aligned}\Delta Q^2 &= \left[\frac{\partial Q}{\partial S}\Delta S\right]^2 + \left[\frac{\partial Q}{\partial D}\Delta D\right]^2 + \left[\frac{\partial Q}{\partial h}\Delta h\right]^2 \\ &= \left[\sqrt{\frac{D}{2Sh}}\Delta S\right]^2 + \left[\sqrt{\frac{S}{2Dh}}\Delta D\right]^2 + \left[\sqrt{\frac{SD}{2h^3}}\Delta h\right]^2\end{aligned} \tag{12}$$

In Example 2, we take a look at how to compute the range of a lot size.

Example 2. Deployment of Propagation of Errors Technique

A materials management department tries to find a lot size for a new part. It is best estimated that yearly demand is D=(10,000±1,000), ordering cost $S = \$(100 \pm 10)$, and inventory carrying cost is $h = \$(10 \pm 1)$. Then, by (12), $\Delta Q = 39$. We estimate Q as f(100,10000,10)=447. Thus, we obtain the range of Q as $(Q \pm \Delta Q) = (447 \pm 39)$ or [408, 486].

The next example shows how well the propagation of errors technique works as opposed to actual statistics.

Example 3. Comparison of Propagation of Errors Technique with Actual Statistics

The yearly demand is fixed with value of 10,000 and the other two estimates are given as follows:

S : S350, S400, S450 with equal probabilities and therefore a standard deviation of 40.8

F : S11, S13, S15 with equal probabilities and therefore a standard deviation of 1.6

Each of the nine joint probabilities from the two distributions has equal probability of 1/9. Thus, we have the following lot sizes with equal probabilities:

2763	2542	2366
2954	2717	2530
3133	2882	2683

The actual mean and standard deviation of these values are compared with those obtained by propagation errors:

Actual Statistic	Propagation of Errors
$\mu = 2,730$	$Q = 2,717$
$\sigma = 223$	$\Delta Q = 269$

As the comparison shows, estimations of propagation of errors technique is almost close to the actual statistics.

4. EOQ Decision with Propagation of Errors

Since the estimated range from the propagations of errors technique is somewhat quite accurate with the actual range, it is worthwhile to deploy the result of propagations of errors when making decisions. Thus, under minimax criterion, our scheme for lot sizing can be described as follows:

(1) Estimate the range on Q, $\underline{Q} \le Q \le \overline{Q}$, using the propagation of errors technique

(2) Calculate the lot size $\hat{Q}_R^*$ using the formula in Proposition 1.

To compare the result of our scheme ($\hat{Q}_R^*$) with that of Lowe and Schwarz's model ($\hat{Q}^*$), we provide the following example.

Example 4. Comparison of $\hat{Q}_R^*$ and $\hat{Q}^*$

Consider Example 1 again with ten various instances of ranges of S, D and h. Recall that each triple (S, D, h) $\in K$ has probability greater than zero while each (S, D, $\not\in h$) K has probability zero, where $K = \overline{K} \cap H^+$ and H^+ is the set of points above or on the hyperplane defined by the three triples $(\overline{S}, \underline{D}, \underline{h})$, $(\underline{S}, \overline{D}, \underline{h})$, $(\underline{S}, \underline{D}, \overline{h})$. As the lot size Q is a function of S, D and h, i.e., $Q = \sqrt{2SD/h}$, we can get possible values (true range) of Q from the true set K. In Table 1, $(\underline{Q}, \overline{Q})$ is the true range found by an optimization tool MATLAB(The MathWorks, Inc. (2000)) and $(\hat{\underline{Q}}, \hat{\overline{Q}})$ is the estimated range by the propagation of errors technique. For each instance, we compared the lot size $\hat{Q}^*$ from Lowe & Schwarz's model with the lot size $\hat{Q}_R^*$ from our scheme based on the estimated valid range by propagation of errors.

Let's look at $\hat{Q}^*$, $\hat{Q}_R^*$ and the trure range $(\underline{Q}, \overline{Q})$. Note that all the ten $\hat{Q}_R^*$ belong to their corresponding $(\underline{Q}, \overline{Q})$, while four of $\hat{Q}^*$ are not in the range $(\underline{Q}, \overline{Q})$. In other words, we see that all the ten $\hat{Q}_R^*$'s are valid while the four of $\hat{Q}^*$ are invalid.

Another interesting point would be the comparison of expected total costs for the two lot sizes, $\hat{Q}^*$ and $\hat{Q}_R^*$. In general, given a lot size Q, the average total cost can be computed as follows:

Table 1. Comparison of $\hat{Q}_R^*$ with $\hat{Q}^*$

No.	S		D		h		$\hat{Q}^*$	$E[ACR(\hat{Q}^*)]$	Q		$\hat{Q}$		$\hat{Q}_R^*$	$E[ACR(\hat{Q}_R^*)]$
	$\underline{S}$	$\overline{S}$	$\underline{D}$	$\overline{D}$	$\underline{h}$	$\overline{h}$			$\underline{Q}$	$\overline{Q}$	$\hat{\underline{Q}}$	$\hat{\overline{Q}}$		
1	200	300	1100	2000	17	33	175	3118	156	266	134	218	171	3120
2	10	90	100	10000	4	9	100	2420	149	671	94	463	209	1524
3	10	90	100	10000	1	9	141	1814	149	1342	80	556	211	1426
4	320	562	20000	40000	70	100	535	33759	506	801	428	688	543	33721
5	30	120	70000	100000	10	20	843	10286	548	1549	595	1249	862	10255
6	3000	5000	20000	40000	120	300	1075	161297	889	1414	749	1389	1020	161662
7	60	90	100000	200000	7	9	1618	9435	1632	2267	1335	2019	1642	9426
8	2	400	10	80000	1	2	189	38325	400	8000	907	5641	2262	4372
9	5000	8000	100000	200000	200	400	2515	545611	2233	3162	1880	3219	2460	546222
10	160	230	100000	200000	7	9	2615	15173	2667	3625	2165	3243	2650	15161

$$E[ACR(Q)] = \iiint_{(S,D,h)\in K} \frac{1}{5V/6}(SD/Q + hQ/2)\, dS\, dD\, dh$$

For the lot sizes $\hat{Q}^*$ and $\hat{Q}_R^*$, we computed their expected total cost as shown in the Table 1. Although the expected total cost based on $\hat{Q}_R^*$ is not always smaller than that on $\hat{Q}^*$, we observe that the expected total cost based on valid $\hat{Q}_R^*$ is no worse than that on invalid $\hat{Q}^*$. Furthermore, the large difference between the expected total costs based on $\hat{Q}_R^*$ and on $\hat{Q}^*$ in the 8th instance of Table 1 indicates that our scheme might often be quite better than the Lowe and Schwarz's model.

5. Conclusions

In the EOQ model with unknown values of demand rate, setup and carrying costs but instead with known ranges of them, we took sensitivity analysis into consideration. In extending the Lowe and Schwarz's model, under the minimax criteria, we derived new formula for generating lot size in case that we are given a valid range on the true lot size. To estimate the valid range, the propagation of errors technique has been used and its applicability was tested by examples. Finally, we suggested the scheme for lot sizing that first estimates the valid range using propagation of errors technique and then calculates the lot size from the new formula. Experiments showed that the scheme is more likely to generate valid lot sizes with small expected total cost than the Lowe and Schwarz's model.

Acknowledgements

This study was supported by research grant of Chosun University, 2004.

References

Groff, G. K., and Muth, J. F. (1972), Operations Management Analysis for Decisions, D. Irwin, Homewood, IL.

Lowe, T. J. and Schwarz, L. B. (1983), Parameter Estimation for the EOQ Lot-Size Model: Minimax and Expected Value Choices, *Naval Research Logistics Quarterly*, 30, 367-376.

Pugh, E. M. and Winslow, G. H. (1966), The Analysis of Physical Measurements, Addision-Wesley.

The MathWorks, Inc. (2000), Optimization Toolbox, 2.0.

Yoon, K. P. (1990), Capital Investment Analysis Involving Estimate Error, *The Engineering Economist*, 36, 21-30.

김 규 태
성균관대학교 산업공학과 학사
Fairliegh Dickinson 대학교 산업공학과 석사
Auburn 대학교 산업공학과 박사
현재: 조선대학교 산업공학과 부교수
관심분야: 경제적 의사결정, Real Option 이론, ABC 시스템

김 창 현
한양대학교 산업공학과 학사
한국과학기술원 산업공학과 석사
포항공과대학교 산업공학과 박사
현재: 여수대학교 교통·물류시스템공학부
관심분야: 물류 및 생산정보시스템, e-Business, 인공지능 응용

황 학 진
포항공과대학교 산업공학과 학사
포항공과대학교 산업공학과 석사
포항공과대학교 산업공학과 박사
현재: 조선대학교 산업공학과
관심분야: 스케줄링, 물류최적화