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Some Results About NBU_{mgf} Class of Life Distributions

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Abstract. A new class of life distributions is studied. This class is defined based on comparing the residual life time to the whole life in the moment generating function order giving "the new better than used in the moment generating function order ageing class (NBU_{mgf})". Some new results of this class are given including some closure properties and characterizations. Finally testing exponentiality against the NBU_{mgf} class is also addressed.

Key Words : *Life distributions, NBU_{mgf} ageing class, characterization of life distributions, life testing, efficiency.*

1. INTRODUCTION AND PRELIMINARIES

In reliability theory and life testing, ageing is often characterized via nonparametric classes of distributions drawn from the behavior of the random remaining life at a certain age either by itself or in relation to the whole new life (the random remaining life at time $t = 0$). Such classes are derived via several notions of comparison between random variables. Of the most commonly used comparisons we find, cf. *Muller* and *Stoyan* (2002) and *Shaked* and *Shanthikumar* (1994), stochastic comparison and increasing concave comparison. Formally, if X and Y are two random variables then we say that X is smaller than Y in the:

(i) stochastic order (denoted by $X \leq_{st} Y$), if

$$E\phi(X) \leq E\phi(Y) \text{ for all increasing functions } \phi;$$

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(ii) increasing concave order (denoted by $X \leq_{icv} Y$), if

$$E[\phi(X)] \leq E[\phi(Y)] \text{ for all increasing concave functions } \phi.$$

In the context of lifetime distributions, the above orderings have been used to give characterizations and new definitions of ageing classes. One of the most important approaches to the study of ageing is based on the concept of additional residual life. Let X be a lifetime random variable such that distribution function F with $F(0) = 0$. Given a unit which has survived up to time t, its *additional residual life* (*Barlow* and *Proschan*, 1981) is given by $X_t = [X - t | X > t]$, with $t \in (\alpha, \beta)$, where $\alpha = \sup\{x : F(x) = 0\}$ and $\beta = \inf\{x : F(x) = 1\}$. The survival function of X_t is given by $\overline{F}_t(x) = \overline{F}(t + x) \ / \ \overline{F}(t)$.

The comparison of the additional residual life at different times has been used to produce several notions of ageing. In particular, X is said to be

(i) new better than used (denoted by $X \in NBU_{st}$) if

$$X_t \leq_{st} X \text{ for all } t \geq 0,$$

(ii) new better than used in the increasing concave order (denoted by $X \in NBU_{icv}$) if

$$X_t \leq_{icv} X \text{ for all } t \geq 0.$$

For more details about the above ageing notions, one may refer to *Bryson* and *Siddiqui* (1969), *Barlow* and *Proschan* (1981) and *Deshpande, Kochar* and *Singh* (1986).

Another ordering that has come to use in reliability and life testing lately is the following:

Given two non-negative random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, respectively, X is said to be smaller than Y in the moment generating function ordering (denoted by $X \leq_{mgf} Y$) if, and only if,

$$\int_0^\infty e^{sx}\overline{F}(x)dx \leq \int_0^\infty e^{sx}\overline{G}(x)dx, \text{ for all } s > 0.$$

Recently, based on this notion, *Li* (2004) introduced a new ageing class of life distributions. Given a non-negative random variable X, we say that X is new better than used in the moment generating function order (denoted by $X \in NBU_{mgf}$) if

$$X_t \leq_{mgf} X \text{ for all } t \geq 0.$$

Equivalently, $X \in NBU_{mgf}$ if, and only if,

$$\int_0^\infty e^{sx}\overline{F}(x + t)dx \leq \overline{F}(t) \int_0^\infty e^{sx}\overline{F}(x)dx, \text{ for all } s \geq 0, \quad t \geq 0.1.1 \tag{1.1}$$

The relations between some of the above orderings and classes are as follows:

$$X \leq_{st} Y \Rightarrow X \leq_{icv} Y \Rightarrow X \leq_{mgf} Y, 1.2 \tag{1.2}$$

and

$$NBU_{st} \subset NBU_{icv} \subset NBU_{mgf}.$$

Some properties of the NBU_{mgf} class including some preservations properties have been discussed by *Li* (2004)), while *Zhang* and *Li* (2004) showed that the NBU_{mgf} class is preserved under both the non-homogeneous Poisson shock model and the general shock model.

In the current investigation, we further develop the NBU_{mgf} class. Some preservation and characterization properties are discussed in *Section* 2. In *Section* 3 we present a procedure to test that X is exponential against that it is NBU_{mgf} and not exponential.

2. CHARACTERIZATIONS RESULTS

Some results of interest in reliability theory are the preservation of ageing classes under parallel and/or series systems (see *Barlow* and *Proschan,* 1981). Next, we establish a preservation theorem under formation of a series system. Recall the following result (see *Shaked* and *Shanthikumar*, 1994) about the preservation of the moment generating function order.

Theorem 2.1.

Let the independent non-negative random variables X_1, X_2,...,X_n, $Y_1, Y_2, ..., Y_n$ have the survival functions $\overline{F}_1, \overline{F}_2, ..., \overline{F}_n$, $\overline{G}_1, \overline{G}_2, ..., \overline{G}_n$, respectively, and $\overline{F}_i$ and $\overline{G}_i$ are all completely monotone. If $X_i \leq_{mgf} Y_i$, $i = 1, 2, ..., n$, then $\min\{X_1, ..., X_n\} \leq_{mgf} \min\{Y_1, ..., Y_n\}$.

Also, we observe (see *Pellerey* and *Petakos,* 2002) that, given a set X_1, X_2,...,X_n of independent components and letting $T_n = \tau(X_1, X_2,...,X_n)$ be the random lifetime of coherent system with components X_1, X_2,...,X_n, we have

$$[T_n - t|T_n > t] \leq_{st} \tau\left(\{X_1 - t|X_1 > t\}, ..., \{X_n - t|X_n > t\}\right).2.2 \tag{2.1}$$

Now we give the following result.

Theorem 2.2.

Let X_1, X_2,...,X_n, be a set of NBU_{mgf} independent components, with completely monotone survival functions and consider $T_n = min\{X_1,..., X_n\}$. Then $T_n \in NBU_{mgf}$.

Proof.

By (2.2) and (1.2) we have that

$$[T_n - t|T_n > t] \leq_{mgf} \min\left\{[X_1 - t|X_1 > t, ..., X_n - t|X_n > t]\right\},$$

and by *Theorem* 2.1 we get

$$\min\left\{\{X_1 - t|X_1 > t\}, ..., \{X_n - t|X_n > t\}\right\} \leq_{mgf} T_n = \min\{X_1, ..., X_n\},$$

and then the result follows.

Let X and Y be two independent random variables with respective distribution functions F and G and with $P(X > Y) > 0$, and denote by $X_Y = [X - Y|X > Y]$ the residual life at random time (*RLRT*). Then the survival function of X_Y can be represented as

$$\begin{aligned} \overline{F}_Y(t) &= P(X - Y > t|X > Y) \\ &= \frac{\int_0^\infty \overline{F}(t+y)dG(y)}{\int_0^\infty \overline{F}(y)dG(y)}.2.3 \end{aligned} \quad (2.2)$$

RLRT *'s* appear in queueing theory. For example, in the classical $GI/G/1$ system, the idle time of the server can be expressed as $[T-(S+W)|T > S+W]$, where T, S and W are the interarrival time, the service time and the waiting time in the queue, respectively (cf. *Stoyan,* 1983). In the following, we establish characterization result for the NBU_{mgf} ageing notion.

Theorem 2.3.

Let X and Y be nonnegative independent random variables with distributions F and G, respectively. Then $F \in NBU_{mgf}$ if and only if $X_Y \leq_{mgf} X$ for any distribution function G.

Proof.

Suppose first that $F \in NBU_{mgf}$. From (1.1) and (2.3), we have

$$\begin{aligned} \int_0^\infty e^{sx}\overline{F}_Y(x)dx &= \frac{\int_0^\infty \int_0^\infty e^{sx}\overline{F}(x+y)dxdG(y)}{\int_0^\infty \overline{F}(y)dG(y)} \\ &\leq \frac{\int_0^\infty \overline{F}(y)\left[\int_0^\infty e^{sx}\overline{F}(x)dx\right]dG(y)}{\int_0^\infty \overline{F}(y)dG(y)} \\ &= \int_0^\infty e^{sx}\overline{F}(x). \end{aligned}$$

Hence, $X_Y \leq_{mgf} X$ follows.

To get the other way, suppose that $X_Y \leq_{mgf} X$ holds for any non-negative random variable Y. Then, $F \in NBU_{mgf}$ follows by taking Y as a degenerate variable.

3. TESTING AGAINST NBU_{mgf} ALTERNATIVES

In the context of reliability and life testing, the hazard rate of a life distribution plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distribution and exhibits different monotonic behaviors. The concept of the *ageless* notion is equivalent to the phenomenon that age has no effect on the hazard rate. Thus the *ageless* property is equal to constant hazard rate, that is, the distribution is exponential. Hence testing non-parametric classes is done by testing exponentially versus some kind of classes. This applies to many non-parametric classes such as NBU_{st},NBU_{icv}, $NBUT$, $DMRL$ and $IMIT$, among many others. For a recent literature on testing the above classes as well as others we refer the readers to *Ahmad* (2001), *Ahmad* and *Mugdadi* (2004), *Mugdadi* and *Ahmad* (2004), *Kayid* and *Ahmad* (2004), *Ahmad, Kayid* and *Li* (2005) and *Ahmad, Kayid and Pellerey* (2005). Much of the earlier literature is cited in those papers where definitions, inter-relations and discussion of above classes are presented.

According to (1.1) we take as a measure of departure from H_0,

$$\delta(s) = \int_0^\infty \overline{F}(t)dt \int_0^\infty e^{su}\overline{F}(u)du - \int_0^\infty \int_0^\infty e^{su}\overline{F}(u+t)dudt.$$

Lemma 3.1.

If $\phi(s) = \int_0^\infty e^{sx}dF(x)$ and $\mu = \int_0^\infty \overline{F}(x)dx$, then

$$\delta(s) = \frac{1}{s^2}\left[1 - (1-\mu s)\,\phi(s)\right], \; s \neq \frac{1}{\mu}.$$

Proof.

Note first that

$$\int_0^\infty e^{su}\overline{F}(u)du = E\int_0^X e^{su}du = \frac{1}{s}\left[\phi(s) - 1\right].$$

Next,

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{su}\overline{F}(u+t)dudt &= \int_0^\infty \int_t^\infty e^{s(w-t)}\overline{F}(w)dwdt \\ &= \frac{1}{s^2}\left[\phi(s) - 1\right] - \frac{\mu}{s}.\end{aligned}$$

Hence, the result follows.

To estimate $\delta(s)$, let $X_1, X_2, ..., X_n$ be a random sample from F. We estimate $\delta(s)$ by

$$\delta\ (s) = \frac{1}{n(n-1)} \sum\nolimits_i \sum\nolimits_{\neq j} \left[1 - (1 - sX_i)\, e^{sX_j}\right].$$

To find the limiting distribution of $\delta\ (s)$ we resort to the U-statistics theory (cf. *Lee*, 1989). Set

$$\phi_s\left(X_1, X_2\right) = 1 - \left(1 - sX_1\right) e^{sX_2}.$$

Thus

$$\begin{aligned} \phi_{1s}\left(X_1\right) &= E\left[\phi_s\left(X_1, X_2\right) | X_1\right] \\ &= 1 - \left(sX_1\right) \phi(s), \end{aligned}$$

and

$$\begin{aligned} \phi_{2s}\left(X_1\right) &= E\left[\phi_s\left(X_1, X_2\right) | X_1\right] \\ &= 1 - (1 - s\mu)\, e^{sX_1}. \end{aligned}$$

Thus set

$$\begin{aligned} \psi_s(X_1) &= \phi_{1s}\left(X_1\right) + \phi_{2s}\left(X_1\right) \\ &= 2 - (2 - sX_1)\phi(s) - (2 - s\mu)e^{sX_1}. \end{aligned}$$

Thus the variance of $\delta\ (s)$ is

$$\delta_s^2 = Var\left(\psi_s\left(X_1\right)\right).$$

Under H_0, we get that

$$\delta_{0,s}^2 = \frac{\mu^4 s^4}{(1 - s\mu)^2 (1 - 2s\mu)}, \quad s \neq \frac{1}{\mu}, \frac{1}{2\mu}.$$

Thus we arrive at:

Theorem 3.1.

As $n \to \infty$, $\sqrt{n}\left(\delta\ (s) - \delta(s)\right)$ is a symptomatically normal with zero mean and variance σ_s^2. Under H_0, the variance is $\sigma_{0,s}^2$.

Note that $\sigma_{0,s}^2$ can be easily estimated by:

$$\sigma_{0,s}^2 = \frac{\overline{X}^4 s^4}{\left(1 - s\overline{X}\right)^2 \left(2 - 2s\overline{X}\right)},$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Hence we reject H_0 if $\sqrt{n}\ \delta\ (s)/\ \sigma_{0,s} >> Z_\alpha$, the standard normal variate. To asses the quality of this procedure, we evaluate its *asymptotic Pittman efficacy* for three alternatives in the class (since they are in the new better than used in expectation class). These are:

(i) the Weibull Family:

$$\overline{F}_1(x) = e^{-X^\theta},\ \ x \geq 0,\ \ \theta \geq 0;$$

(ii) the Linear Failure Rate Family:

$$\overline{F}_2(x) = e^{-x-\frac{\theta}{2}x^2},\quad x \geq 0,\ \theta \geq 0;$$

(iii) the Makeham Family:

$$\overline{F}_3(x) = e^{-x-\theta(x+e^{-x}-1)},\quad x \geq 0,\ \theta \geq 0.$$

Note that the Pitman asymptotic efficacy (PAE) is defined by:

$$PAE\,(\delta(s)) = \left|d_\theta \delta(s)\right|_{\theta\to\theta_0} / \sigma_{s,0}.$$

In the above three cases we get:

(i) the Weibull:

$$s^{-1}(1-2s)^{\frac{1}{2}}\ln(1-s),\ \ s \geq 0, s \neq \frac{1}{2}, 1;$$

(ii) the Linear Failure Rate:

$$\frac{(1-2s)^{\frac{1}{2}}}{2(1-s)},\ \ s \geq 0, s \neq \frac{1}{2}, 1;$$

(iii) the Makeham Family:

$$\frac{(1-2s)^{\frac{1}{2}}}{2(1-s)},\ \ s \geq 0, s \neq \frac{1}{2}, 1.$$

From the above the following remarks are immediate:

Remark (1).

The linear failure rate has double the efficacy of the Makeham alternative.

Remark (2).

All these alternatives have efficacies decreasing in $s > 0$. Hence we should choose s small enough to get good efficacy of the test.

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