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Set Covering Problem and Reliability of the Covers

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Abstract. This work developed an algorithm for a set covering model when the reliability of covers is a concern. This model extended the usage of the set covering model.

Key Words : *set covering, reliability of cover, minimal cost, maximal reliability, integer programming.*

1. INTRODUCTION

A set covering problem considers the subsets I = {1, 2, ..., m} and J = {1, 2, ..., n} of integers. Let Pj $\subset$ I and $\wp$ = {Pj : j $\in$ J0}, J0 $\subset$ J. Clearly, $\wp$ is a collection of subsets, Pj, of I. A collection $\wp$ = {Pj : j $\in$ J0}, is a cover for I if $\bigcup_{j \in j_0} p_j = I$. The set covering problem determines a cover $\wp^*$, with minimum cost which is formulated as follows:

$$Min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t} \sum_{j=1}^{n} a_{ij} x_j \geq 1, i = 1, 2, \ldots, m,$$

$$x_j \in \{0, 1\}, j = 1, 2, \ldots, n,$$

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where

$$x_j = \begin{cases} 1, \text{if } P_j \in \wp^* \\ 0, \text{otherwise} \end{cases}$$

and

$$a_{ij} = \begin{cases} 1, & \text{if } i \in P_j, \\ 0, & \text{otherwise.} \end{cases}$$

This set covering problem is well known with many applications such as facility location, assigning customers to delivery routes, airline crews to flights, and workers to shifts, etc. (Beasley 1987; Fisher and Kedia 1990; Nemhauser and Wolsey, 1988).

In reality, $i \in P_j$ can fail sometimes. Thus, sometimes the probability of $i \in P_j$ can be less than 1; denoting probability of $i \in P_j$ as $\pi_j(i)$,

Then, the set covering problem is no longer straight forward. It is become probabilistic, and $\pi_j(i)$ can be considered as the reliability of $i \in P_j$.

Let

$$Q_j = \{(i, \pi_j(i) : i \in P_j\}$$
$$= (P_j, \pi_j), j \in J_0$$

We are finding a cover with minimal cost and maximal reliability. We call this problem as "the problem of set covering and reliability of covers"(SCRC).

Let $Q = \{Q_j : j \in J_0\}$, $J_0 \subset J$,and $Q_j = (P_j, \pi_j)$ and $\bigcup_{j \in J_0} P_j = I$.

Thus the (SCRC) is to simultaneously $\begin{cases} \text{minimize total cost} \\ \text{maximize cover reliability} \end{cases}$

$$\text{i.e.} \begin{cases} \min\limits_{J_0 \in J} \sum\limits_{j \in J_0} C_j \\ \max\limits_{Q \subset \Theta} R(Q) \end{cases}$$

$$\text{or } \min_{J_0 \subset J} \begin{cases} \sum\limits_{j \in J_0} C_j \\ -R(\bigcup\limits_{j \in J_0} Q_j) \end{cases}$$

2. RELIABILITY OF COVERS

Let

$$Q_j = (P_j, \pi_j) = \{(i, \pi_j(i)) : i \in P_j\}.$$

We define the operations among Q_j's as follows:

Definition.

1. $Q_l \cup Q_k = (P_l \cup P_k, \pi_{l\cup k})$
2. $\bigcup_{l \in J_0} Q_l = (\bigcup_{l \in J} P_l, \pi_{\bigcup l \in J_0})$

where

$$\pi_{j\cup k}(i) = 1 - (1 - \pi_j(i))(1 - \pi_k(i))$$
$$\pi_{\cup l \in J_0}(i) = 1 - \prod_{l \in J_0}(1 - \pi_l(i))$$

(SCRC) is a multi-objective 0-1 integer programming problem, which is formulated as follows:

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\max \begin{cases} 1 - \prod_{j=1}^{n}(1 - \pi_j(i) x_j) \\ \quad i = 1, 2, \ldots, m \end{cases}$$

$$s.t.\ \ x_j = 0,1 \quad j = 1, 2, \ldots, m$$

$$x_j = \begin{cases} 1, \text{if } P_j \in \wp^* \\ 0, \text{otherwise} \end{cases}$$

where

if $x_j \in \{0,1\}$, then $1 - \pi_j(i)x = (1 - \pi_j(i))^{x_j}$.

Therefore $1-\prod_{j=1}^{n}(1-\pi_j(i)x_j) = 1-\prod_{j=1}^{n}(1-\pi_j(i))^{x_j}$.

Then (SCRC) can be formulate as

$$\begin{cases} \min \sum_{j=1}^{n} c_j x_j \\ \max \ \alpha \\ \text{s.t. } \ 1-\prod_{j=1}^{n}(1-\pi_j(i))^{x_j} \geq \alpha, i = 1,2,\dots,n \\ \quad x_j \in \{0,1\} \end{cases}$$

Observe that

$$1-\prod_{j=1}^{n}(1-\pi_j(i))^{x_j} \geq \alpha$$

$$1-\alpha \geq \prod_{j=1}^{n}(1-\pi_j(i))^{x_j}$$

$$\ln(1-\alpha) \geq \sum_{j=1}^{n} x_i \ln(1-\pi_j(i))$$

$$-\sum_{j=1}^{n} x_i \ln(1-\pi_j(i)) \geq -\ln(1-\alpha)$$

$$\sum_{j=1}^{n} x_j \ln\frac{1}{1-\pi_j(i)} \geq \ln\frac{1}{1-\alpha}.$$

Thus, we obtain the following (SCRC) which is ready for solution.

$$(\text{SCRC}) \begin{cases} \min \sum_{j=1}^{n} c_j x_j \\ \max \ \alpha \\ s.t. \ \sum_{j=1}^{n} x_j \ln\frac{1}{1-\pi_j(i)} \geq \ln\frac{1}{1-\alpha}, i = 1,2,\dots,n \\ \quad x_j \in \{0,1\} \end{cases}$$

To "solve" this bi-objective 0-1 linear program problem, we apply the constrain method by controlling α and minimizing the cost. The following is the proposed solution method.

Step 0: Determine $\alpha^* = \max\{\alpha : \sum \frac{x_j}{\ln(1-\pi_j(i))} \geq \ln \frac{1}{1-\alpha}, x_j \in \{0,1\}\}$.

Choose α_* = minimal acceptable level chosen by DM.

Let $\Delta = \frac{\alpha^* - \alpha_*}{k}$, k is a given positive integer.

Step1:

$$\alpha_1 = \alpha_* + \Delta$$
$$\alpha_2 = \alpha_1 + \Delta$$
$$\vdots$$
$$\alpha_{k-1} = \alpha_{k-2} + \Delta$$
$$\alpha^* = \alpha_k = \alpha_{k-1} + \Delta$$

Step 2: For i =1,2,...,k. solve (P_i)

$$(P_i)\begin{cases} \min \sum_{j=1}^{n} c_j x_j \\ s.t. \sum_{j=1}^{n} \frac{x_j}{\ln(1-\pi_{j(i)})} \geq \ln \frac{1}{1-\alpha_i} \\ x_j \in \{0,1\} \end{cases}$$

obtain optimal objective value z_i^*.

Step3: Analyze $(\alpha_i, z_i^*), i = 1,2,...,k$.

Step4: DM choose i, such that (P_i) give the DM an optimal decision.

3. EXAMPLE

Let $I = \{1, 2, 3, 4, 5\}$. Suppose that there are four fuzzy subsets of I, and each is represented as follows: $\tilde{P}_1$ = {(1, 0.4), (2, 0.1), (3, 0.5), (4, 0.7), (5, 0.8)}, $\tilde{P}_2$ = {(1, 0.1), (2, 0.3), (3, 0.8), (4, 0.2), (5, 0.6)}, $\tilde{P}_3$ = {(1, 0.3), (2, 0.7), (3, 0.2), (4, 0.9), (5, 0.4)}, and $\tilde{P}_4$ = {(1, 0.5), (2, 0.9), (3, 0.4), (4, 0.1), (5, 0.2)}. Table 1 shows the matrix of $\mu_j(i)$, i =

1, 2, ..., 5, j = 1, 2, ..., 4. And we associate each $\tilde{P}_j$ with the corresponding cost c_j, j = 1, 2, ..., 4, shown as table 1.

Table 1. The matrix of $\mu_j(i)$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$\tilde{P}_1$ (c_1=4)	0.4	0.1	0.5	0.7	0.8
$\tilde{P}_2$ (c_2=3)	0.1	0.3	0.8	0.2	0.6
$\tilde{P}_3$ (c_3=5)	0.3	0.7	0.2	0.9	0.4
$\tilde{P}_4$ (c_4=2)	0.5	0.9	0.4	0.1	0.2

According to the (P1) model, we then have the following mathematical programming:

$$Min \sum_{j=1}^{n} c_j x_j = 4x_1 + 3x_2 + 5x_3 + 2x_4,$$

s.t.

$$1-[(1-0.4x_1)(1-0.1x_2)(1-0.3x_3)(1-0.5x_4)] \geq \alpha$$
$$1-[(1-0.1x_1)(1-0.3x_2)(1-0.7x_3)(1-0.9x_4)] \geq \alpha$$
$$1-[(1-0.5x_1)(1-0.8x_2)(1-0.2x_3)(1-0.4x_4)] \geq \alpha$$
$$1-[(1-0.7x_1)(1-0.2x_2)(1-0.9x_3)(1-0.1x_4)] \geq \alpha$$
$$1-[(1-0.8x_1)(1-0.6x_2)(1-0.4x_3)(1-0.2x_4)] \geq \alpha$$
$$x_j \in \{0,1\}, j = 1, 2, ..., 4.$$

Calculating the inequalities of constraint, we can rewrite the above formulation as P2's form:

$$Min\ 4x_1 + 3x_2 + 5x_3 + 2x_4$$

s.t.

$$0.4x_1 + 0.1x_2 + 0.3x_3 + 0.5x_4 - 0.04x_1x_2 - 0.12x_1x_3 - 0.02x_1x_4 - 0.03x_2x_3 - 0.05x_2x_4 - 0.15x_3x_4 + 0.012x_1x_2x_3 + 0.02x_1x_2x_4 + 0.06x_1x_3x_4 + 0.015x_2x_3x_4 - 0.006x_1x_2x_3x_4 \geq \alpha$$
$$0.1x_1 + 0.3x_2 + 0.7x_3 + 0.9x_4 - 0.03x_1x_2 - 0.07x_1x_3 - 0.09x_1x_4 - 0.021x_2x_3 - 0.27x_2x_4 - 0.63x_3x_4 + 0.021x_1x_2x_3 + 0.027x_1x_2x_4 + 0.063x_1x_3x_4 + 0.189x_2x_3x_4 - 0.0189x_1x_2x_3x_4 \geq \alpha$$
$$0.5x_1 + 0.8x_2 + 0.2x_3 + 0.4x_4 - 0.4x_1x_2 - 0.1x_1x_3 - 0.2x_1x_4 - 0.16x_2x_3 - 0.32x_2x_4 - 0.08x_3x_4 + 0.08x_1x_2x_3 + 0.16x_1x_2x_4 + 0.04x_1x_3x_4 + 0.064x_2x_3x_4 - 0.032x_1x_2x_3x_4 \geq \alpha$$

$0.7x_1 + 0.2x_2 + 0.9x_3 + 0.1x_4 - 0.14x_1x_2 - 0.63x_1x_3 - 0.07x_1x_4 - 0.18x_2x_3 - 0.02x_2x_4 - 0.09x_3x_4 + 0.126x_1x_2x_3 + 0.014x_1x_2x_4 + 0.063x_1x_3x_4 + 0.018x_2x_3x_4 - 0.0126x_1x_2x_3x_4 \geq \alpha$

$0.8x_1 + 0.6x_2 + 0.4x_3 + 0.2x_4 - 0.48x_1x_2 - 0.32x_1x_3 - 0.16x_1x_4 - 0.24x_2x_3 - 0.12x_2x_4 - 0.08x_3x_4 + 0.192x_1x_2x_3 + 0.096x_1x_2x_4 + 0.064x_1x_3x_4 + 0.048x_2x_3x_4 - 0.0384x_1x_2x_3x_4 \geq \alpha$

$x_j \in \{0,1\}, j = 1, 2, \ldots, 4.$

Suppose the desired level $\alpha = 0.5$. Let $y_{j_1 j_2 \cdots j_t} = x_{j_1} \cdot x_{j_2} \cdots x_{j_t}$. For instance, $y_{12} = x_1x_2$, $y_{123} = x_1x_2x_3$, $y_{1234} = x_1x_2x_3x_4$, etc. Then, we obtain the formulation of the form of (P3):

$Min\ 4x_1 + 3x_2 + 5x_3 + 2x_4$

s.t.

$0.4x_1 + 0.1x_2 + 0.3x_3 + 0.5x_4 - 0.04y_{12} - 0.12y_{13} - 0.2y_{14} - 0.03y_{23} - 0.05y_{24} - 0.15y_{34} + 0.012y_{123} + 0.02y_{124} + 0.06y_{134} + 0.015y_{234} - 0.006y_{1234} \geq 0.5$

$0.1x_1 + 0.3x_2 + 0.7x_3 + 0.9x_4 - 0.03\ y_{12} - 0.07\ y_{13} - 0.09\ y_{14} - 0.021\ y_{23} - 0.27\ y_{24} - 0.63\ y_{34} + 0.021\ y_{123} + 0.027\ y_{124} + 0.063\ y_{134} + 0.189\ y_{234} - 0.0189\ y_{1234} \geq 0.5$

$0.5x_1 + 0.8x_2 + 0.2x_3 + 0.4x_4 - 0.4\ y_{12} - 0.1\ y_{13} - 0.2\ y_{14} - 0.16\ y_{23} - 0.32\ y_{24} - 0.08\ y_{34} + 0.08\ y_{123} + 0.16\ y_{124} + 0.04\ y_{134} + 0.064\ y_{234} - 0.032\ y_{1234} \geq 0.5$

$0.7x_1 + 0.2x_2 + 0.9x_3 + 0.1x_4 - 0.14\ y_{12} - 0.63\ y_{13} - 0.07\ y_{14} - 0.18\ y_{23} - 0.02\ y_{24} - 0.09\ y_{34} + 0.126\ y_{123} + 0.014\ y_{124} + 0.063\ y_{134} + 0.018\ y_{234} - 0.0126\ y_{1234} \geq \alpha$

$0.8x_1 + 0.6x_2 + 0.4x_3 + 0.2x_4 - 0.48\ y_{12} - 0.32\ y_{13} - 0.16\ y_{14} - 0.24\ y_{23} - 0.12\ y_{24} - 0.08\ y_{34} + 0.192\ y_{123} + 0.096\ y_{124} + 0.064\ y_{134} + 0.048\ y_{234} - 0.0384\ y_{1234} \geq 0.5$

$2y_{12} \leq x_1 + x_2 \leq 1 + y_{12}$

$2y_{13} \leq x_1 + x_3 \leq 1 + y_{13}$

$2y_{14} \leq x_1 + x_4 \leq 1 + y_{14}$

$2y_{23} \leq x_2 + x_3 \leq 1 + y_{23}$

$2y_{24} \leq x_2 + x_4 \leq 1 + y_{24}$

$2y_{34} \leq x_3 + x_4 \leq 1 + y_{34}$

$2y_{123} \leq y_{12} + x_3 \leq 1 + y_{123}$

$2y_{124} \leq y_{12} + x_4 \leq 1 + y_{124}$

$2y_{134} \leq y_{13} + x_4 \leq 1 + y_{134}$

$2y_{234} \leq y_{23} + x_4 \leq 1 + y_{234}$

$2y_{1234} \leq y_{123} + x_4 \leq 1 + y_{1234}$

$x_j \in \{0,1\}, j = 1, 2, \ldots, 4.$

Solving the programming problem by LINGO software, we obtain the optimal solution $x_1^* = 1$, $x_2^* = 0$, $x_3^* = 0$, $x_4^* = 1$, and the total cost is 6.

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