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A Moment Inequality for Exponential Better (Worse) Than Used EBU (EWU) Life Distributions with Hypothesis Testing Application

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Abstract. The exponential better (worse) than used EBU (EWU) class of life distributions is considered. A moment inequality is derived for EBU (EWU) distributions which demonstrate that if the mean life is finite, then all moments exist. Based on this inequality, a new test statistic for testing exponentiality against EBU (EWE) is introduced . It is shown that the proposed test is simple, enjoys good power and has high relative efficiency for some commonly used alternatives . Critical values are tabulated for sample sizes $n = 5(1)40$. A set of real data is used as a practical application of the proposed test in the medical science.

Key Words : *EBU (EWE), exponentiality, efficiency, moments, asymptotic normality.*

1. INTRODUCTION

Ever since the works of Barlow et al (1963) and Bryson and Siddiqui (1969), various classes of life distribution have been introduced in reliability. Currently the application of these classes of life distribution can be seen in engineering, social and biological science, maintenance and biometrics. Therefore, statisticans and reliability analysts have shown a grown interest in modeling survival data using classification of life distributions based on some aspects of aging, see for example Barlow and Proschan (1981)and Zacks (1992). The most well known families of life distributions are the classes of increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), decreasing mean residual life

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(DMRL), new better than used in expectation (NBUE) and harmonic new better than used in expectation (HNBUE). For some properties and interrelationships of these criteria we refer to Barlow and Proschan (1981), Bryson and Siddiqui (1969) and Klefsjo (1982).

The problem of testing exponentiality versus the classes (like IFR, IFRA, NBU, DMRL, NBUE and HNBUE) of life distributions has seen a good deal of literature for examles: Proschan and Pyke (1967), Ahmad (1994), Hollander and Proschan (1972 and 19975) , Kanjo (1993) and Abu-Youssef (2002).

Definition 1.1. A life distribution F, with $F(0) = 0$, survival function $\overline{F}$ and finite mean μ is said to be EBU if

$$\overline{F}(x+t) \leq \overline{F}(t)e^{\frac{-x}{\mu}}, \qquad x, t > 0 \tag{1.1}$$

The dual class of life distributions that is EWU is defined by reversing the inequality sign of relation (1.1).

Note that, the above defination is motivated by comparing the life length X_t of a component of age t with another new component of life length Y which is exponential with the same mean as X, this leads to X is EBU if and only if $X_t \leq_{st} Y$ for all $t \geq 0$. El-Batal (2002) introduced the above class of life distribution. He investigated their relationship to other classes of life distribution, closure properties under reliability operations, moment inequality and heritage property under shock model. The implication among EBU, NBUE, and HNBUE classes of life distribution are

$$EBU \rightarrow NBUE \rightarrow HNBUE$$

The thread that connects most work mentioned here is that a measure of departure from H_0, which is often some weighted function of F, is developed which is strictly positive under H_1 and is zero under H_0. Then, a sample version of this measure is used as test statistics and its properties are studied. In this sprit, the moment inequality developed in section 2 can be used to construct test statistic for EBU (EWU). In section 3 this test statistic is based on sample moments of aging distribution. This test statistic is simple to drive, and has exponentially high efficiencies and power for some of the well known alternatives relative to other tests. Montecarlo null distribution critical points obtained for sample sizes 5(1)40. Finaly we apply the proposed test to real practical data in medical science given in Aboummah et al. (1994).

2. MOMENT INEQUALITY

We state and prove the following result.

Theorem 2.1. If F is EBU (EWU), then

$$\frac{1}{(2r)!}\mu_{(2r)} \leq (\geq) \frac{1}{r!}\mu^r \mu_{(r)}, \qquad r \geq 1. \tag{2.1}$$

where

$$\mu_{(r+1)} = (r+1)\int_0^\infty x^r \overline{F}(x)dx. \tag{2.2}$$

Proof. Since F is EBU (EWU), then

$$\overline{F}(x+t) \leq (\geq) \overline{F}(t)e^{\frac{-x}{\mu}}$$

Multiplying both sides by $x^{r_1}t^{r_2}$, $r_1 > 0, r_2 > 0$ and integrating over $(0,\infty)$, w.r.t. x and t, then

$$\int_0^\infty \int_0^\infty x^{r_1}t^{r_2}\overline{F}(x+t)dxdt \leq (\geq) \int_0^\infty \int_0^\infty x^{r_1}t^{r_2}\overline{F}(t)e^{\frac{-x}{\mu}}dxdt \tag{2.3}$$

Set $x+t = u_1, u_2 = t$ in (2.3). Hence, the left hand side of (2.3) becomes

$$\int_0^\infty \int_0^{u_1} (u_1-u_2)^{r_1}u_2^{r_2}\overline{F}(u_1)du_2du_1 = \int_0^\infty u_1^{r_1+r_2+1}\overline{F}(u_1)\int_0^{u_1}(\frac{u_2}{u_1})^{r_2}$$

$$(1-\frac{u_2}{u_1})^{r_1}d(\frac{u_2}{u_1}) = \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{\Gamma(r_1+r_2+2)}\int_0^\infty u_1^{r_1+r_2+1}\overline{F}(u_1)du_1. \tag{2.4}$$

Using (2.2),(2.4) becomes

$$\int_0^\infty \int_0^{u_2} (u_1-u_2)^{r_1}u_2^{r_2}\overline{F}(u_1)du_2du_1 = \frac{r_1!r_2!}{(r_1+r_2+2)!}\mu_{(r_1+r_2+2)}. \tag{2.5}$$

The right hand side of (2.3) is given by

$$\int_0^\infty \int_0^\infty x^{r_1}t^{r_2}\overline{F}(x)e^{\frac{-x}{\mu}}dtdx = \frac{r_1!}{(r_2+1)}\mu^{r_1+1}\mu_{(r_2+1)}. \tag{2.6}$$

By using (2.5) and (2.6), (2.3) becomes

$$\frac{1}{(r_1+r_2+2)!}\mu_{(r_1+r_2+2)} \leq (\geq) \frac{1}{(r_2+1)!}\mu^{r_1+1}\mu_{(r_2+1)}. \tag{2.7}$$

Putting $r_1+1 = r_2+1 = r$ in (2.7), the theorm follows.

3. APPLICATIONS TO HYPOTHESES TESTING

3.1 Testing against EBU (EWE) alternatives

Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population with distribution F. We wish to test the null hypothesis H_0 : $\overline{F}$ is exponential with mean μ against H_1 : $\overline{F}$ is EBU (EWU) and not exponential. Using theorem (2.1), we may use the following as a measure of departure from H_0 in favor of H_1:

$$\delta_E = \frac{1}{r!}\mu^r \mu_{(r)} - \frac{1}{(2r)!}\mu_{(2r)} \geq (\leq) 0. \tag{3.1}$$

Note that under $H_0 : \delta_E = 0$, while under $H_1 : \delta_E > (<) 0$. Thus to estimate δ_E by $\hat{\delta}_{E_n}$, let $X_1, X_2, \ldots, X_n$ be a random sample from F and μ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n}\sum X_i$ is the usual sample mean . Then $\hat{\delta}_{E_n}$ is given by using (3.1) as

$$\hat{\delta}_{E_n} = \frac{1}{n(n-1)...(n-r+1)} \sum_i \sum_k \left\{ \frac{1}{r!}\Pi_{j=1}^r X_{1_j} X_k^r - \frac{1}{(2r)!} X_{1_i}^{2r} \right\}, \tag{3.2}$$

where $\sum_c$ extends over all indices $1 > i_1 \neq i_2 ...... \neq i_r \neq n$. Thus to make the test statistic scale invariant, we take

$$\hat{\Delta}_{E_n} = \frac{\hat{\delta}_{E_n}}{\overline{X}^{2r}}. \tag{3.3}$$

Setting $\phi(X_{1_1}, X_{1_2}, ..., X_{1_r}, X_2) = \frac{1}{r!}\Pi_{i=1}^r X_{1_i} X_2^r - \frac{1}{(2r)!} X_1^{2r}$, then $\hat{\Delta}_{E_n}$ in (3.3) is a clasical U-statistic, cf. Lee (1990). The following theorem summarizes the large sample properties of $\hat{\Delta}_{E_n}$.

Theorem 3.1. As $n \to \infty, \sqrt{n}(\hat{\Delta}_{E_n} - \Delta_E)$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = \mu^{-4r} Var \left\{ \frac{rX_1\mu^{r-1}\mu_{(r)} + X_1^r \mu^r}{r!} - \frac{X_1^{2r} + r\mu_{(2r)}}{(2r)!} \right\}. \tag{3.4}$$

Under $H_0 : \Delta_E = 0$ and variance σ_0^2 is given by

$$\sigma_0^2 = \frac{(4r)!}{((2r)!)^2} + \frac{(2r)!}{(r!)^2} - \frac{2(3r)!}{r!(2r)!} + 3r^2 - 4r. \tag{3.5}$$

Proof: Since $\hat{\Delta}_{E_n}$ and $\frac{\hat{\delta}_{E_n}}{\mu^{4r}}$ have the same limiting distribution, we use $\sqrt{n}(\hat{\delta}_{E_n} - \delta_{E_n})$. Now this is asymptotically normal with mean 0 and variance $\sigma^2 = var[\phi(X_1)]$, where

$$\begin{aligned} \phi(X_1) &= E[\phi(X_1, X_2, ...X_{r+1})|X_1] + E[\phi(X_2, X_1, ...X_{r+1})|X_1] \\ &+ E[\phi(X_2, X_3, ...X_r, X_1)|X_1]. \end{aligned} \tag{3.6}$$

But

$$\phi(X_1) = \frac{rX_1\mu^{r-1}\mu_{(r)} + X_1^r\mu^r}{r!} - \frac{X_1^{2r} + r\mu_{(2r)}}{(2r)!}. \tag{3.7}$$

Then (3.4) follows.

Under H_0

$$\phi(X_1) = rX_1 + \frac{X_1^r}{r!} - \frac{X_1^{2r}}{(2r)!} - r. \tag{3.8}$$

Hence (3.5) follows. The Theorem is proved.

When $r = 1$,

$$\delta_{E_1} = \mu^2 - \frac{1}{2}\mu_{(2)}, \tag{3.9}$$

in this case $\sigma_0^2 = 1$ and the test statistic is

$$\hat{\delta}_{E_{1n}} = \frac{1}{n(n-1)}\sum\sum_{i \neq j}\left\{X_iX_j - \frac{1}{2}X_i^2\right\}. \tag{3.10}$$

and

$$\hat{\Delta}_{E_{1n}} = \frac{\hat{\delta}_{E1_n}}{\overline{X}^2}, \tag{3.11}$$

which is quite simple statistics.

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{E_{1n}}/\sigma_0$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$. To illustrate the test, we calculate, via Monte Carlo Method, the empirical critical points of $\hat{\Delta}_{E_{1n}}$ in (3.11) for sample sizes 5(1)40. Tables 3.1 gives the upper percentile points for 95%, 98%, 99% . The calculations are based on 5000 simulated samples sizes $n = 5(1)40$.

To asses how good this procedure is relative to others in the literatures, we use the concept of Pitman's asymptotic efficiency (PAE). To do this we need to evalute PAE of the proposed test and compare it with other tests. Since the test statistic $\hat{\Delta}_{E_n}$ in (3.3) is new and no other tests are known for these class EBU. We compare these to some other classes. Here we choose the tests K^* and $\hat{\Delta}_n$ presented by Hollander and Proschan (1975) and Kango (1993) respectively. Note that PAE of $\hat{\Delta}_{E_n}$ is given by

$$PAE(\Delta_E(\theta)) = \left\{\frac{d}{d\theta}\Delta_E(\theta)|_{\theta\to\theta_0}\right\}/\sigma_0. \tag{3.12}$$

Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i)	Linear failure rate family	: $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$,	$x > 0, \theta > 0$
(ii)	Weibull family	: $\bar{F}_\theta = e^{-x^\theta}$,	$x \geq 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate family and $\theta = 1$ for Weibull family. Direct calculations of PAE of K^*,$\hat{\Delta}_n$ and $\hat{\Delta}_{E_{1n}}$ are summarized in Table 3.2.

Table 3.1 Critical Values of $\hat{\Delta}_{E_{1n}}$

n	95%	98%	99%
5	0.3908	0.4310	0.4523
6	0.3592	0.3971	0.4240
7	0.3384	0.3769	0.3996
8	0.3358	0.3733	0.3968
9	0.3203	0.3619	0.3835
10	0.3088	0.3502	0.3733
11	0.3009	0.3394	0.3634
12	0.2877	0.3274	0.3509
13	0.2886	0.3282	0.3466
14	0.2822	0.3164	0.3439
15	0.2746	0.3157	0.3370
16	0.2726	0.3080	0.3332
17	0.2585	0.2959	0.3186
18	0.2608	0.2976	0.3153
19	0.2521	0.2897	0.3090
20	0.2463	0.2831	0.3016
21	0.2410	0.2768	0.3035
22	0.2413	0.2795	0.2981
23	0.2365	0.2703	0.2956
24	0.2326	0.2680	0.2862
25	0.2321	0.2690	0.2931
26	0.2281	0.2573	0.2763
27	0.2252	0.2574	0.2749
28	0.2204	0.2543	0.2658
29	0.2164	0.2521	0.2754
30	0.2212	0.2510	0.2714
31	0.2173	0.2458	0.2672
32	0.2071	0.2413	0.2610
33	0.2078	0.2407	0.2608
34	0.2101	0.2379	0.2540
35	0.1976	0.2288	0.2478
36	0.1999	0.2366	0.2555
37	0.1986	0.2333	0.2512
38	0.1945	0.2291	0.2504
39	0.1961	0.2252	0.2439
40	0.1956	0.2274	0.2436

Table 3.2 PAE of K^*,$\hat{\Delta}_n$ and $\hat{\Delta}_{E_{1n}}$

Distribution	K^*	$\hat{\Delta}_n$	$\hat{\Delta}_{E_{1n}}$
F_1 Linear failure rate	0.871	0.433	1.0000
F_2 Weibull	0.1.20	0.144	1.000

From Table 3.2, the test statistic $\hat{\Delta}_{E_{1n}}$ is more efficient than $\hat{\Delta}_n$ and K^* for linear failur rate family, but it is more efficient than $\hat{\Delta}_n$ only for Weibull family.

Finally, the power of the test statistics $\hat{\Delta}_{E_{1n}}$ is considerd for 95% percentile in Table 3.1 for three of the most commonly used alternatives [see Hollander and Proschan (1975)], they are

(i) Linear failure rate : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$, $x > 0, \theta > 0$
(ii) Makeham : $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$, $x \geq 0, \theta > 0$
(iii) Weibull : $\bar{F}_\theta = e^{-x^\theta}$, $x > 0, \theta > 0$.

These distributions are reduced to exponential distribution for appropriate values of θ.

Table 3.3 Power Estimate of $\hat{\Delta}_{E_{1n}}$

		Sample Size		
Distribution	θ	n=10	n=20	n=30
F_1	2	0.237	0.450	0.583
Linear failure	3	0.285	0.537	0.678
rate	4	0.320	0.588	0.753
F_2	2	0.169	0.288	0.358
Makham	3	0.208	0.375	0.472
	4	0.233	0.436	0.556
F_3	2	0.740	0.978	0.994
Weibull	3	0.994	1.000	1.000
	4	1.000	1.000	1.000

Note that: Since $\hat{\Delta}_{E_n}$ defines a class (with parameter) r of test statistic, we choose r that the maximizes the PAE of that alternatives. If we take $r = 1$ then our test will have more efficacy than others.

4. NUMERICAL EXAMPLES

Consider the data in Abouammoh et al (1994). These data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospital in Saudi

Arabia and the ordered life times (in days) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852.

Using equation (3.11), the value of test statistics, based on the above data is $\hat{\Delta}_{E_{1n}} = 0.408$. This value leads to the acceptance of H_1 at the signficance level $\alpha = 0.95$ see Table 3.1. Therefore the data has EBU Property.

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