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Superior Julia Set

Mamta Rani
Department of Computer Science, Gurukula Kangri University, Hardwar 249 404, India;
Mailing address: 21 Govind Nagar, Rishikesh 249 201, India;
Email: mamtarsingh@rediffmail.com

Vinod Kumar
Department of Computer Science, Gurukula Kangri University, Hardwar 249 404, India

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Julia sets, their variants and generalizations have been studied extensively by using the Picard iterations. The purpose of this paper is to introduce Mann iterative procedure in the study of Julia sets. Escape criterions with respect to this process are obtained for polynomials in the complex plane. New escape criterions are significantly much superior to their corresponding cousins. Further, new algorithms are devised to compute filled Julia sets. Some beautiful and exciting figures of new filled Julia sets are included to show the power and fascination of our new venture.

Keywords: filled superior Julia set, superior Julia set, filled Julia set, Julia set, Mandelbrot set, topologically conjugate, escape criterion, superior escape criterion, function/Picard iteration, Mann iteration, orbit, Mann orbit, one-step-machine, two-step machine.
MSC2000 Classification: 37F50, 28A80, 68U05.

1. INTRODUCTION

Perhaps no computer experiments exceed in excitement, fascination and wonderment the graphical representations of Julia and Mandelbrot sets in a complex plane. These are the parts of iteration dynamics. Julia sets live in complex plane and are non-empty (see Steinmnetz (1993, p. 28). For a detailed analysis of these sets, one may refer to Barnsley (1988), Beardon (1991), Branner & Hubbard (1988), Carleson & Gamelin (1993), Crilly, Earnshaw & Jones (1991), Crownover (1995), Devaney (1992), Edgar (1990), Kigami (2001), Peitgen, Jurgens & Saupe (1992a, 1992b, 1992c) and Barnsley et al. (1988).

Julia sets are the striking examples of computational experiments that were far ahead of its time. These mathematical objects were seen when computer graphics became available. Julia set is the place where all of the chaotic behavior of a complex function occurs (*cf.* Devaney 1992, p. 221).

Consider the quadratic family of the form $Q_c(z) = z^2 + c$. All real quadratic functions are topologically conjugate to the real polynomials Q_c for some c. This fact extends to the complex quadratic functions. All complex quadratic functions are topologically conjugate to the complex polynomial $Q_c(z)$ for some c.

The simplest example of a Julia set occurs for $Q_c(z)$ with $c = 0$. Here the Julia set is a circle. This particular Julia set is not a fractal, but usually Julia sets are fractals. Every Julia set for $Q_c(z) = z^2 + c$ is either connected or totally disconnected. Some of the connected Julia sets are simple closed curves. These closed curves are fractals if $0 < |c| < \frac{1}{4}$. There are also connected Julia sets that are not closed curves (for example, see Figure 3 when $c = -1$). All of the totally disconnected Julia sets have the property of being Cantor dust.

To iterate $z^2 + c$, we choose some value for z and apply function iteration (Picard iteration). Thus we get a Julia set for c. A commutative generalization of complex numbers called bicomplex numbers have also been used to generate Julia sets in three and four dimensions (Peitgen, Jurgens & Saupe 1992c). Rochon (2000) used bicomplex numbers to introduce bicomplex dynamics. In particular, he gave generalizations of the filled Julia sets in dimension three and four.

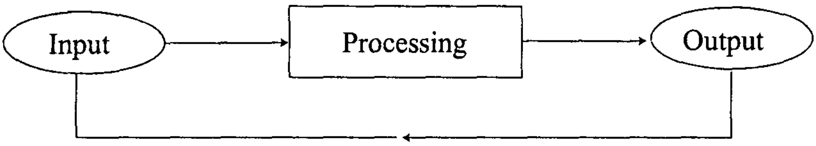
In this paper, we generate superior Julia sets in the complex plane by applying Mann iterations, which are more general than that of the Picard. We discuss a new escape criterion for quadratic functions to generate Julia sets with respect to Mann iterations in Section 3. We also discuss a similar escape criterion for cubic functions and finally a general escape criterion for polynomials having only two terms: highest degree term and a constant. The intent of the last section is to present some of the beautiful figures obtained while computing algorithms of filled superior Julia sets for quadratic, cubic and biquadratic polynomials.

2. PRELIMINARIES

Basically there are two types of feedback machines (see, for instance, Mandelbrot 1998; Peitgen, Jurgens & Saupe 1992b).

- One-step machine,
- Two-step machine.

One step machines may be characterized by Picard iterations $x_{n+1} = f(x_n)$, where f is any function. It requires one number as input and returns a new number.



In two-step feedback machines, output is computed by the formula $x_{n+1} = g(x_n, x_{n-1})$. It requires two numbers as input and returns a new number. We introduce Mann iterations as an example of two-step feedback processes. In this iterative procedure, we use a parameter s, which lies between 0 and 1, and x_n and x_{n-1} are used as input and the output is denoted by x_{n+1}. Thus

$$x_{n+1} = g(f(x_n), x_n) = sf(x_n) + (1 - s)x_n.$$

Evidently, at $s = 0$, no change takes place in the output and, at $s = 1$, two-step machine works as a one-step machine.

Let $Q(z) := a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n, \quad a_0 \neq 0$ be a polynomial of degree n, where $n \geq 2$. The coefficients $a_0, a_1, \ldots, a_n$ are allowed to be complex numbers. In all that follows Q_c will stand for $z^2 + c$.

Definition 2.1. Let X be a non-empty set and $f : X \to X$. For a point x_0 in X, the Picard orbit (generally called *orbit of f or trajectory of f*) is the set of all iterates of a point x_0, that is:

$$O(f, x_0) := \{x_n : x_n = f(x_{n-1}), \quad n = 1, 2, \ldots\}.$$

Notice that the orbit $O(f, x_0)$ of f at the initial point x_0 is $\{f^n(x_0)\}$. In all that follows, by orbit we mean Picard orbit unless otherwise stated.

The collection of points that are bounded, *i. e.*, there exists an M such that $|\, Q^n(z) \,| < M$ for all n, is called the prisoner set, while the collection of points that are in the stable set of infinity is called the escape set. Thus the boundary of the prisoner set is simultaneously the boundary of the escape set and that is the Julia set for Q (see, for instance, Peitgen, Jurgens & Saupe 1992b).

Definition 2.2. The set of points K whose orbits are bounded under the function iteration of $Q(z)$ is called the *filled Julia set*. The Julia set of Q is the boundary of the filled Julia set K. The boundary of a set is the collection of points for which every neighborhood contains an element of the set as well as an element, which is not in the set (see, for instance, Crownover 1995; Devaney 1992; Holmgren 1994).

2.1. Escape Criterion for Quadratic Functions

There are different escape criterions for different types of functions. The following theorem gives the escape criterion for the quadratic function $Q_c = z^2 + c$, and its corollaries further refine the escape criterion for computational purposes (Beardon 1991; Crownover 1995; Devaney 1992; Peitgen, Jurgens & Saupe 1992a).

Theorem 2.1. *Suppose* $|z| \geq |c| > 2$, *where c is in the complex plane. Then we have* $|Q_c^n(z)| \to \infty$ *as* $n \to \infty$.

This theorem has following three corollaries.

Corollary 2.1. *Suppose* $|c| > 2$. *Then the orbit of 0 escapes to infinity under* Q_c.

Corollary 2.2. *Suppose* $|z| > \max\{|c|, 2\}$. *Then* $|Q_c^n(z)| > |(1+\lambda)^n z|$ *and so* $|Q_c^n(z)| \to \infty$ *as* $n \to \infty$, *where* λ *is a positive number.*

Corollary 2.3. *Suppose for some* $k \geq 0$ *we have* $|Q_c^k(z)| > \max\{|c|, 2\}$. *Then* $|Q_c^{k+1}(z)| > (1+\lambda)|Q_c^k(z)|$, *so* $|Q_c^n(z)| \to \infty$ *as* $n \to \infty$.

This corollary gives us an algorithm for computing the filled Julia set of Q_c, for some c.

2.2. Escape Criterion for Cubic Polynomials

The study of cubic polynomials is more complicated than the study of quadratics. A typical cubic polynomial has two critical points not just one unlike a quadratic function. It means several additional phenomena may occur in this case. For example, a cubic polynomial may have two distinct attracting fixed or periodic orbits. Unlike quadratics, where we have two distinct cases (the critical orbit either escapes or is bounded) as given in Theorem 2.1, there are three possibilities for cubics:

(i) Both critical orbits escape,
(ii) Both critical orbits are bounded.
(iii) One critical orbit escapes and one remains bounded.

Besides, as there are two complex parameters, this means that the natural parameter space for cubics is four-dimensional. One may refer to the work Branner & Hubbard (1988) who are leading experts in the study of dynamics of cubic polynomials. As in the case of quadratic polynomials, any cubic polynomial is conjugate to one of its special form (Devaney 1992, p. 266):

$$Q_{a,b}(z) = z^3 + az + b,$$

where a and b are complex numbers. The following result is needed in the construction of filled Julia sets (*cf.* Branner & Hubard 1988; Devaney 1992).

Theorem 2.2. *Let $Q_{a,b}(z) = z^3 + az + b$, where a, b are complex numbers. Suppose*

$$|\,Q^n_{a,b}(z)\,| > \max\left\{|\,b\,|, (|\,a\,| + 2)\frac{1}{2}\right\},$$

for some n. Then the orbit of z escapes to infinity.

3. MANN ITERATION AND SUPERIOR JULIA SET

Let A be a subset of real or complex numbers and $f : A \to A$. For $x_0 \in A$, construct a sequence $\{xn\}$ in A in the following manner.

$$\begin{aligned} x_1 &= s_1 f(x_0) + (1 - s_1)x_0, \\ x_2 &= s_2 f(x_1) + (1 - s_2)x_1, \ldots, \\ x_n &= s_n f(x_{n-1}) + (1 - s_n)x_{n-1}, \ldots, \end{aligned}$$

where $0 < s_n \leq 1$ and $\{sn\}$ is convergent to a non-zero number.

Definition 3.1. The sequence $\{xn\}$ constructed above is called *Mann sequence of iterates or superior sequence of iterates*. We may denote it by $SO(f, x_0, s_n)$.

Notice that $SO(f, x_0, s_n)$ with $s_n = 1$ is $O(f, x_0)$ (*cf.* Definition 2.1).

This procedure is essentially due to Mann (1953). We remark that the Mann or superior orbit $SO(f, x_0, s_n)$ with $s_n = \frac{1}{2}$ was first discussed by Krasnosel'skii (1955).

Now we define the Julia set for a function with respect to Mann iterates (superior iterates). We call it *superior Julia set* (sJ set in brief).

Definition 3.2. The set of points SK whose orbits are bounded under superior iteration of a function $Q(z)$ will be called the *filled sJ sets*. The sJ set of Q is the boundary of filled sJ set SK.

Escape criterions play a crucial role in the construction of filled Julia sets of a function. Now we obtain new and considerably improved escape criterions for these sets. We prefer to call the same Mann or superior escape criterions.

3.1. Superior Escape Criterions for Quadratics

The following theorem gives an escape criterion for the function $Q_c = z^2 + c$ in respect of the Mann iterative procedure.

Theorem 3.1. *Suppose* $|z| \geq |c| > \frac{2}{s}$ *where* $0 < s \leq 1$ *and* c *is a complex number. Define* $z_1 = (1-s)z + sQ_c(z), \ldots, zn = (1-s)z_{n-1} + sQ_c(z_{n-1}), n = 2, 3, \ldots$ *Then* $|z_n| \to \infty$ *as* $n \to \infty$.

Proof.

$$\begin{aligned} |z_1| &= |(1-s)z + sQ_c(z)| \\ &= |sz^2 + (1-s)z + sc)| \\ &\geq |sz^2 + (1-s)z| - |sc| \\ &\geq |z|(|sz + (1-s)|) - s|z| \\ &\geq |z|(|sz| - 1 + s) - s|z| \\ &= |z|(s|z| - 1). \end{aligned}$$

Since $s|z| > 2$, there is a $\lambda > 0$ such that $s|z| - 1 > 1 + \lambda$. Consequently

$$|z_1| > (1+\lambda)|z|.$$

Repeating this argument, we find $|zn| > (1+\lambda)^n |z|$.

Thus the Mann orbit of z under the quadratic function Q_c tends to infinity. This completes the proof. □

We derive the following interesting results.

Corollary 3.1. *Suppose that* $|c| > \frac{2}{s}$. *Then the superior orbit* $SO(Q_c, 0, s)$ *escapes to infinity.*

Notice that $|z| \geq |c|$ and $|z| > \frac{2}{s}$. So the following corollary is a refinement of the escape criterion discussed in the above theorem.

Corollary 3.2 (Escape Criterion). *Suppose that* $|z| > \max\{|c|, \frac{2}{s}\}$. *Then* $|z_n| > (1+\lambda)^n |z|$, *and* $|zn| \to \infty$ *as* $n \to \infty$.

Corollary 3.3. *Suppose that* $|zk| > \max\{|c|, \frac{2}{s}\}$ *for some* $k \geq 0$. *Then* $|z_{k+1}| > (1+\lambda)|zk|$, *and* $|zn| \to \infty$ *as* $n \to \infty$.

This corollary gives an algorithm for computing the filled sJ set of Q_c, for any c. Given any point z satisfying $|z| \leq |c|$, we compute the superior orbit of z. If, for some n,

$|zn|$ lies outside the circle of radius $\max\{|c|, \frac{2}{s}\}$, we are guaranteed that the orbit escapes. Hence z is not in the filled sJ set. On the other hand, if $|zn|$ never exceeds this bound, then z is by definition in the filled sJ set, denoted by SK_c. We shall make extensive use of this algorithm in the next section. It is better understood if we call it semi-algorithm, since in actual practice, we cannot determine whether a point actually remains in SK in finite time.

3.2. Superior Escape Criterions for Cubic Polynomials

First we prove the following theorem for the function $Q_{a,b} = z^3 + az + b$ with respect to the Mann iterative procedure.

Theorem 3.2. *Suppose* $|z| \geq |b| > (|a| + \frac{2}{s})^{\frac{1}{2}}$, *where* $0 < s \leq 1$ *and* a *and* b *are in the complex plane. Define* $z_1 = (1-s)z + sQ_{a,b}(z), \ldots, z_n = (1-s)z_{n-1} + sQ_{a,b}(z_{n-1})$, $n = 2, 3, \ldots$. *Then* $|zn| \to \infty$ *as* $n \to \infty$.

Proof.

$$\begin{aligned}
|z1| &= |(1-s)z + s(z^3 + az + b)| \\
&= |sz^3 + asz + z - sz + sb| \\
&\geq |sz3 + asz + z - sz| - |sb| \\
&\geq |z||sz^2 + as + 1 - s| - s|z| \\
&\geq |z|\{|sz^2 + as| - 1 + s\} - s|z| \\
&= |z|\{s|z^2 + a| - 1 + s - s\} \\
&= |z|\{s|z^2 + a| - 1\} \\
&= s|z|\left\{|z^2 + a| - \frac{1}{s}\right\} \\
&\geq s|z|\left\{|z^2| - |a| - \frac{1}{s}\right\} \\
&= s|z|\left\{|z^2| - \left(|a| + \frac{1}{s}\right)\right\}.
\end{aligned}$$

Since $|z| > (|a| + \frac{2}{s})^{\frac{1}{2}}$,

$$|z|2 - \left(|a| + \frac{1}{s}\right) > \frac{1}{s}, \quad i.e., \quad s\left\{|z|^2 - \left(|a| + \frac{1}{s}\right)\right\} > 1.$$

Hence there is a $\gamma > 1$ such that $|z_1| > \gamma|z|$. Repeating this argument, we find $|z_n| > \gamma^n|z|$. Therefore the superior orbit of z under the cubic polynomial $Q_{a,b}(z)$ tends to infinity. This completes the proof. □

Corollary 3.4. *Suppose that* $|b| > (|a| + \frac{2}{s})\frac{1}{2}$. *Then the superior orbit* $SO(Q_{a,b}, 0, s)$ *escapes to infinity.*

Corollary 3.5 (Escape Criterion). *Suppose* $|z| > \max\{|b|, (|a| + \frac{2}{s})^{\frac{1}{2}}\}$. *Then* $|z_n| > \gamma^n |z|$ *and so* $|zn| \to \infty$ *as* $n \to \infty$.

Corollary 3.6. *Assume* $|z_k| > \max\{|b|, (|a| + \frac{2}{s})^{\frac{1}{2}}\}$ *for some* $k \geq 0$, *Then* $|zk + 1| > \gamma |z_k|$, *and* $|zn| \to \infty$ *as* $n \to \infty$.

Corollary 3.5 gives an escape criterion for cubic polynomials. From Corollary 3.6, we find an algorithm for computing filled sJ sets of $Q_{a,b}$ for any a, b.

3.3. A Genaral Escape Criterion

Now we attempt to obtain a general escape criterion for polynomials of the form $G_c(z) = z^n + c$.

Theorem 3.3. *For functions of the form* $G_c(z) = z^n + c$, $n = 1, 2, \ldots,$ *where* $0 < s \leq 1$ *and* c *is in the complex plane, Define* $z_1 = (1-s)z + sG_c(z), \ldots, z_n = (1-s)z_{n-1} + sG_c(z_{n-1})$, $n = 2, 3, \ldots$. *Then the general escape criterion is* $\max\{|c|, (\frac{2}{s})^{\frac{1}{(n-1)}}\}$.

Proof. We shall prove this theorem by induction.

For $n = 1$, $G_c(z) = z + c$. So the escape criterion is $|c|$ which is obvious,*i. e.*, $|z| > \max\{|c|, 0\}$.

For $n = 2$, $G_c(z) = z^2 + c$, the escape criterion is $\max\{|c|, \frac{2}{s}\}$ (see Theorem 3.1).

For $n = 3$, $G_c(z) = z^3 + c$, and the result follows from Theorem 3.2 with $a = 0$ and $b = c$, *i. e.*, the escape criterion is $\max\{|c|, (\frac{2}{s})^{\frac{1}{2}}\}$. So the theorem is true for $n = 1, 2$ and 3.

Now suppose that the theorem is true for any n. Let

$$G_c(z) = z^{n+1} + c \quad \text{and} \quad |z| \geq |c| > \left(\frac{2}{s}\right)\frac{1}{n}.$$

Then

$$\begin{aligned} |z_1| &= |(1-s)z + s(z^{n+1} + c)| \\ &= |sz^{n+1} - sz + z + sc| \\ &\geq |sz^{n+1} - sz + z| - s|c| \\ &\geq\geq |z||sz^n - s + 1| - s|z| \\ &\geq |z|\{|szn| + s - 1\} - s|z| \\ &= |z|(s|z|^n - 1). \end{aligned}$$

Since $|z| > (2/s)\frac{1}{n}$, $s|z|^n - 1 > 1$. Hence for some $\lambda > 0$, we have

$$s|z|^n - 1 > 1 + \lambda.$$

Thus $|z_1| > (1+\lambda)|z|$.

Repeating this argument we get

$$|zn| > (1+\lambda)^n|z|.$$

Therefore, the Mann orbit of z under iteration of $z^{n+1}+c$ tends to infinity. Hence $\max\{|c|, (\frac{2}{s})^{\frac{1}{n}}\}$ is escape criterion. This proves the theorem. □

Corollary 3.7. *Suppose that* $|c| > (\frac{2}{s})\frac{1}{(n-1)}$. *Then the superior orbit* $SO(G_c, 0, s)$ *escapes to infinity.*

Corollary 3.8. *Suppose for some* $k \geq 0$ *we have* $|zk| > \max\{|c|, (\frac{2}{s})^{\frac{1}{(k-1)}}\}$. *Then* $|zk+1| > (1+\lambda)|zk|$, *so* $|zn| \to \infty$ *as* $n \to \infty$.

This corollary gives a general algorithm for computing filled sJ sets for the functions of the form $G_c(z) = z^n + c, n = 1, 2 \ldots$.

4. GENERATION OF FILLED SUPERIOR JULIA SETS

Graphically filled Julia sets are more appealing than the Julia sets. Therefore, we have written a program in C++ to generate filled Julia sets and superior filled Julia sets. Now we present some beautiful filled sJ sets for quadratic, cubic and biquadratic functions.

Figure 1. Filled superior Julia set for quadratic polynomial with $(s, c) = (0.8, -1354)$

4.1. Filled Superioir Julia Sets for $Q_c = z^2 + c$

For an algorithm to generate a filled Julia set, we refer to Devaney (1992), Holmgren (1994) and Peitgen, Jurgens & Saupe (1992a, 1992c), and one may devise an almost similar algorithm to generate filled sJ sets. We generate a few stiking filled sJ sets

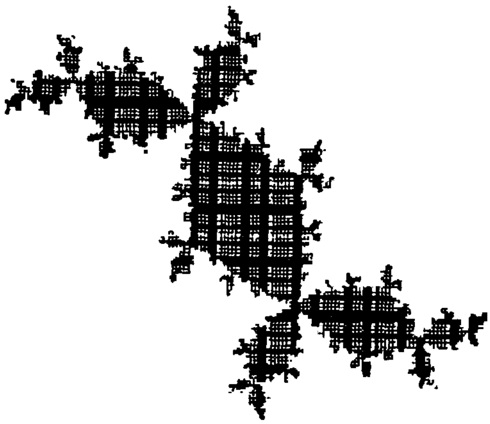


Figure 2. Filled superior Julia set for quadratic polynomial with $(s, c) = (0.7, -0.5 + 1.6i)$

for various values of s and c. Two such sJ sets are given in Figure 1 and Figure 2 for $(s, c = 0.8, -1.354)$ and for $(s, c) = (0.7, -0.5 + 1.6i)$ respectively.

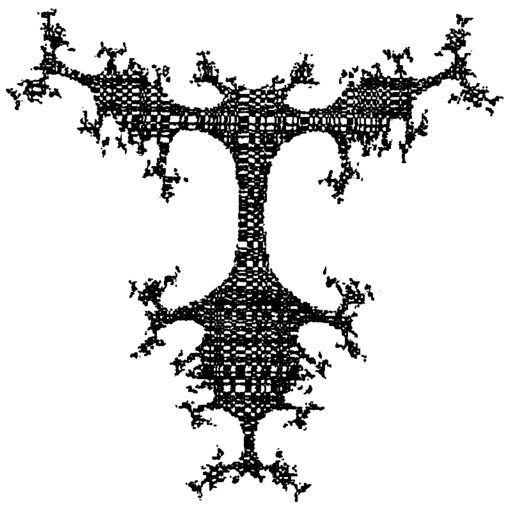


Figure 3. Flower Pot: filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, -i, -2.75i)$

4.2. Filled Superioir Julia Sets for $Q_{a,b} = z^3 + az + b$

We present some of the beautiful figures generated for $s = 0.5$. Our endeavour to give names to some of the figures is based on their resemblances with objects known to us.

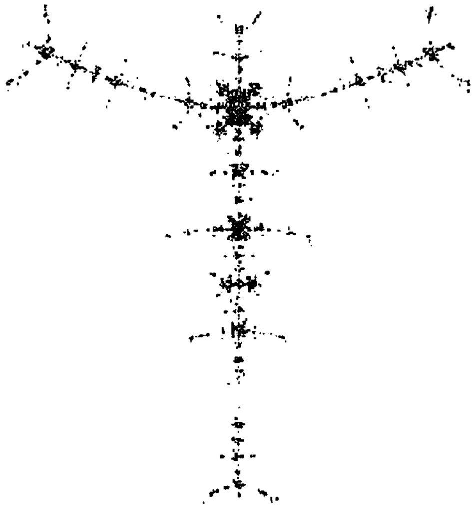


Figure 4. T-structure: filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, 0, -3.55i)$

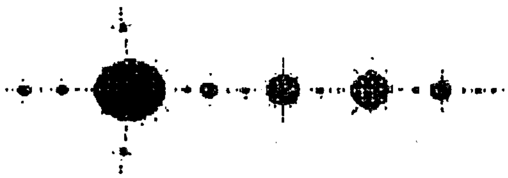


Figure 5. Filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, -5.3 - 5.3i, -1)$

We are excited to see a flowerpot in Figure 3, a T structure in Figure 4 for $(a, b) = (-i, -2.75i)$ and $(0, -3.55i)$ respectively. Also see Figure 5 when $(a, b) = (-5.3 - 5.3i, -1)$. Further a wall hanging, dumbbell and a beautiful painting are presented in Figure 6, Figure 7 and Figure 8 when $(a, b) = (-0.255 + 1.35i, -0.122 - 0.75i)$, $(-3.5 - 3.5i, 0)$ and $(-0.5 - 3.7i, -1 - 2i)$ respectively. We are fascinated to see beautiful shapes in Figure 9 and Figure 10 for $(a, b) = (0, -0.55)$ and $(-3 - 3i, -1)$ respectively.

4.3. Filled Superioir Julia Sets for $G_c = z^4 + c$

Recall that the escape criterion for $z^4 + c$ is $\max(|c|, (\frac{2}{s})\frac{1}{3})$ (see Theorem 3.3). We get a figure symmetric in all four quadrants when $(s, c) = (1, -1)$, see Figure 11. It is

Figure 6. Wall Hanging: filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, -0.255 + 1.35i, -0.122 - 0.75i)$

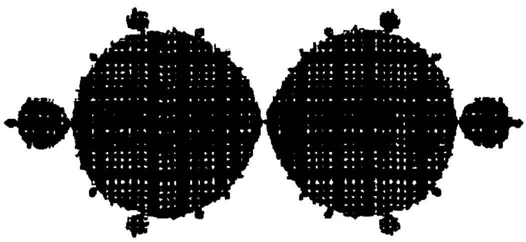


Figure 7. Dumbbell: filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, -3.5 - 3.5i, 0)$

interesting to see some other filled sJ sets in Figure 12, Figure 13 and Figure 14, when $(s, c) = (0.7, -0.5 - 0.55i)$, $(0.05, -0.5 - i)$ and $(0.1, -9.8)$ respectively.

5. CONCLUDING REMARKS

In the existing literature, Julia sets and their generalizations have been developed using one-step feedback process (Picard iterations). We have introduced two-step feedback process (Mann iterates or superior iterates) in the study of Julia sets and obtained sJ sets.

Figure 8. Painting: filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, -0.5 - 3.7i, -1 - 2i)$

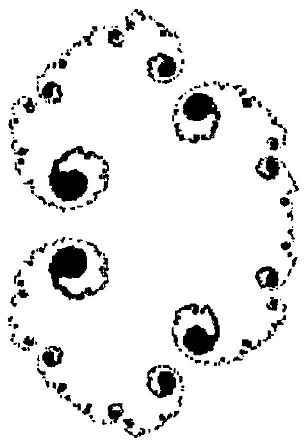


Figure 9. Filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, 0, -0.55)$

We have derived superior escape criterions for quadratic and cubic polynomials and a general escape criterion for a general polynomial of the form $z^n + c$, $n = 2, 3, 4, \ldots$ as well. Further, using these criterions, new algorithms have been devised to compute filled sJ sets for these functions in the complex plane. Some new fascinating filled superior Julia sets have been generated.

In the new escape criterions, the range of c in $z^2 + c$ and $z^4 + c$ increases. For example, the escape and the superior escape criterions for z^2+c are $|c| > 2$ and $|c| > \frac{2}{s}$ respectively (*cf.* Theorems 2.1 and 3.1). For a mild comparison, take $s = 0.1$ to see that these criterions

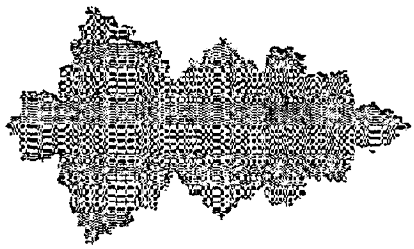


Figure 10. Filled superior Julia set for cubic polynomial with $(s, a, b) = (0.5, -3 - 3i, -1)$

Figure 11. A symmetric figure in all the four quadrants: filled superior Julia set for biquadratic polynomial with $(s, c) = (1, -1)$

Figure 12. Filled superior Julia set for cubic polynomial with $(s, c) = (7, -0.5 - 0.55i)$

differ by 18 for $z^2 + c$. Further, the range of a and b increases in $z^3 + az + b$ (see Theorem 3.2). We observe that the area of a filled superior Julia set is significantly larger than the corresponding Julia set.

Figure 13. Filled superior Julia set for biquadratic polynomial with $(s, c) = (0.05, -0.5 - i)$

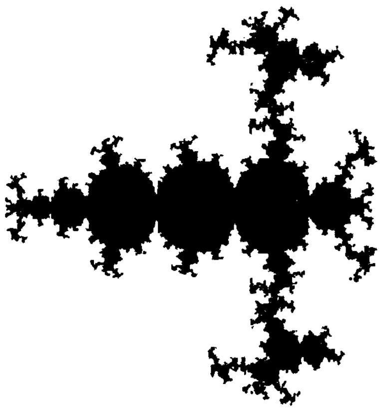


Figure 14. Filled superior Julia set for biquadratic polynomial with $(s, c) = (0.1, -9.8)$

The following question comes in a natural way: Let the parameter s in the superior orbit be replaced by a sequence $\{s_n\}$ of real numbers converging to a nonnegative number (< 1) in Section 3.1 and subsequent constructions. Shall we get effectively new constructions?

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