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On the Relationship between ϵ-sensitivity Analysis and Sensitivity Analysis using an Optimal Basis

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ABSTRACT

ϵ-sensitivity analysis is a kind of methods for performing sensitivity analysis for linear programming. Its main advantage is that it can be directly applied for interior-point methods with a little computation. Although ϵ-sensitivity analysis was proposed several years ago, there have been no studies on its relationship with other sensitivity analysis methods. In this paper, we discuss the relationship between ϵ-sensitivity analysis and sensitivity analysis using an optimal basis.

First, we present a property of ϵ-sensitivity analysis, from which we derive a simplified formula for finding the characteristic region of ϵ-sensitivity analysis. Next, using the simplified formula, we examine the relationship between ϵ-sensitivity analysis and sensitivity analysis using optimal basis when an ϵ-optimal solution is sufficiently close to an optimal extreme solution. We show that under primal nondegeneracy or dual nondegeneracy of an optimal extreme solution, the characteristic region of ϵ-sensitivity analysis converges to that of sensitivity analysis using an optimal basis. However, for the case of both primal and dual degeneracy, we present an example in which the characteristic region of ϵ-sensitivity analysis is different from that of sensitivity analysis using an optimal basis.

Keywords: Linear programming, Sensitivity analysis, Interior-point method, Optimal basis

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1. INTRODUCTION

In linear programming, it is important to know how the optimal solutions are affected when the input data are changed. Sensitivity analysis enables us to understand the implications of changing the input data on the optimal solution by a small number of computations.

Methods for sensitivity analysis in the simplex method have been well developed and given in numerous papers and textbooks (For example, see [3] and [6]). They are based on the concept of optimal bases, and require just a little more computation. However, according to Jansen, Jong, Roos, and Terlarky [8], the methods based on an optimal basis may yield incomplete information in the case of degeneracy because of alternative optimal bases.

Other methods of sensitivity analysis are presented by Yang [13], Adler and Monteiro [1], and Park *et al.* [4]. Yang [13] first suggested positive sensitivity analysis method which can be performed for an optimal non-extreme solution in linear programming. Park *et al.* [5] studied the properties of positive sensitivity analysis and its relationship with other sensitivity analysis. Adler and Monteiro [1] developed a method of parametric analysis on the right-hand side by introducing the optimal partition. To use positive sensitivity analysis or Adler and Monteiro's method, we need an optimal solution or the optimal partition, which requires additional computations in interior-point methods for linear programming ([2, 10]). Moreover, since the worst-case computational complexity of finding the characteristic region of the two sensitivity analysis methods is the same with that of solving linear programming, other practical sensitivity analysis methods were needed. For the case, Kim, Park, and Park [9] developed ϵ-sensitivity analysis which can be directly applied to interior-point solutions produced by interior-point methods. The main idea of ϵ-sensitivity analysis is that the final solution of interior point methods is linked with the scaling vector and the input data of linear programming. The characteristic region of ϵ-sensitivity analysis can be found with a small number of additional computations.

Although ϵ-sensitivity analysis was proposed several years ago, there have been no studies on its relationship with other sensitivity analysis methods. Although Kim, Park, and Park [9] showed that the characteristic region of ϵ-sensitivity analysis converges to that of sensitivity analysis using an optimal basis under the assumption of nondegeneracy, the relationship between ϵ-sensitivity analysis and sensitivity analysis using an optimal basis under degeneracy has not been discussed. The purpose of this paper is to examine the limiting behavior of ϵ-

sensitivity analysis applied to an ϵ-optimal solution converging to an optimal solution. First, we obtain a property of ϵ-sensitivity analysis that leads to a simpler formula for calculating the characteristic region of ϵ-sensitivity analysis (Section 2). Next, we study the relationship between ϵ-sensitivity analysis at an ϵ-optimal solution and sensitivity analysis using an optimal basis when the ϵ-optimal solution converges to an optimal extreme point solution (Section 3,4). The results obtained from this study will lead to a better understanding of the behavior of ϵ-sensitivity analysis for interior-point methods.

The organization of this paper is as follows: In the next section, the definition of ϵ-sensitivity analysis is given, and a simplified formula for calculating its characteristic region is developed. In Section 3, we present the relationship between ϵ-sensitivity analysis and sensitivity analysis using an optimal basis under primal nondegeneracy or dual nondegeneracy. In Section 4, the relationship between ϵ-sensitivity analysis and sensitivity analysis using an optimal basis under both primal and dual degeneracy is discussed. Finally, some conclusions are given in Section 5.

2 . ϵ-SENSITIVITY ANALYSIS

Consider the following linear programming (LP):

$$(P): \begin{array}{ll} Min & c^T x \\ s.t & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad (D): \begin{array}{ll} Max & b^T y \\ s.t & A^T y + s = c \\ & s \geq 0 \end{array}$$

where $c \in \Re^n$, $b \in \Re^m$ and $A \in \Re^{m \times n}$ with $RANK(A) = m$. Throughout this paper, it is assumed that both (P) and (D) are feasible.

First of all, we define an ϵ-optimal solution.

Definition 1. (ϵ-optimal solution) *A solution (x, y, s) is called an ϵ-optimal solution if (x, y, s) satisfies the following three conditions:*

$$Ax = b, x \geq 0$$
$$A^T y + s = c, \quad s \geq 0$$
$$x^T s \leq \epsilon$$

where ϵ is a small positive number.

Moreover, an ϵ-optimal solution (x, y, s) is called an ϵ-optimal interior solution if $x > 0$ and $s > 0$.

Consider another linear programming problem (LP_θ) with a cost coefficient c_k that is perturbed by the amount of θ:

$$(P_\theta):\ \begin{array}{ll} Min & (c+\theta\, e_k)^T x \\ s.t & Ax = b \\ & x \geq 0 \end{array} \qquad (D_\theta):\ \begin{array}{ll} Max & b^T y \\ s.t & A^T y + s = c + \theta\, e_k \\ & s \geq 0 \end{array}$$

where $e_k \in \Re^n$ is a vector such that the k-th element is one and the others are zeros. Let I denote the identity matrix with the dimension that will be determined by the context of the problem in such a way that will avoid confusion. Given an ϵ-optimal interior solution $(\bar{x}, \bar{y}, \bar{s})$, we define ϵ-sensitivity analysis as the following:

Definition 2. (ϵ-sensitivity analysis) *ϵ-sensitivity analysis at an ϵ-optimal interior solution $(\bar{x}, \bar{y}, \bar{s})$ is to find the range of perturbation θ within which a solution $(\bar{x}, \hat{y}, \hat{s})$* remains an ϵ-optimal solution to (LP_θ) where $D = diag\{\sqrt{\bar{x}_j / \bar{s}_j}\}$,

$$\hat{y} = \bar{y} + \theta(AD^2A^T)^{-1}AD^2e_k,$$

and

$$\hat{s} = \bar{s} + \theta(I - A^T(AD^2A^T)^{-1}AD^2)e_k.$$

The range of perturbation θ is called the characteristic region of ϵ-sensitivity analysis at $(\bar{x}, \bar{y}, \bar{s})$.

In this paper, we consider only ϵ-sensitivity analysis where a cost coefficient is changed. See [9] for ϵ-sensitivity analysis where one of the right-hand sides is perturbed. The result presented at this paper can also be applied to ϵ-sensitivity where one of the right-hand sides is perturbed. The characteristic region of ϵ-sensitivity analysis at an ϵ-optimal solution $(\bar{x}, \bar{y}, \bar{s})$ is calculated by the following formulas:

(i) If $\bar{x}^T F e_k > 0$, $\max_j\{-\frac{\bar{s}_j}{F_{jk}} \mid F_{jk} > 0\} \leq \theta \leq 0$,

(ii) If $\bar{x}^T F e_k < 0$, $0 \leq \theta \leq \min_j\{-\frac{\bar{s}_j}{F_{jk}} \mid F_{jk} < 0\}$,

(iii) If $\bar{x}^T F e_k = 0$, $\max_j\{-\frac{\bar{s}_j}{F_{jk}} \mid F_{jk} > 0\} \leq \theta \leq \min_j\{-\frac{\bar{s}_j}{F_{jk}} \mid F_{jk} < 0\}$, where $F = I - A^T(AD^2A^T)^{-1}AD^2$ and $D = diag\{\sqrt{\bar{x}_j/\bar{s}_j}\}$. (For the details, see [9]).

From the above formulas, the value of $\bar{x}^T Fe_k$ plays an important role in the calculation of the characteristic region of ϵ-sensitivity analysis. It was shown that $\bar{x}^T Fe_k$ converges to zero as an ϵ-optimal interior solution converges to an optimal extreme solution which is both primal and dual nondegenerate ([9]). In this paper, we will show that $\bar{x}^T Fe_k$ converges to zero as an ϵ-optimal solution converges to any optimal solution that may be degenerate. Because of this property, only formula (iii) can be used to find the characteristic region of ϵ-sensitivity analysis if an ϵ-optimal solution is sufficiently close to an optimal solution. Given an index set σ of variables, A_σ denotes the submatrix of A with columns that correspond to indices in σ. Similarly, z_σ denotes the subvector of a vector z with components that correspond to indices in σ. Let $(\bar{x}, \bar{y}, \bar{s})$and (x^*, y^*, s^*) be an ϵ-optimal interior solution and an optimal solution to (LP), respectively. Let $B = \{j \mid x_j^* > 0\}$, and $N = \{j \mid x_j^* = 0\}$. In order to show that $\bar{x}^T Fe_k \to 0$ as $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$, we rewrite $\bar{x}^T Fe_k$ as the following:

$$\begin{aligned} \bar{x}^T Fe_k &= \bar{x}^T (e_k - A^T(AD^2A^T)^{-1}AD^2e_k) \\ &= \bar{x}_B^T((e_k)_B - A_B^T(AD^2A^T)^{-1}AD^2e_k) \\ &\quad + \bar{x}_N^T((e_k)_N - A_N^T(AD^2A^T)^{-1}AD^2e_k) \\ &= \bar{s}_B^T D_B^T((e_k)_B - A_B^T(AD^2A^T)^{-1}AD^2e_k) \\ &\quad + \bar{x}_N^T((e_k)_N - A_N^T(AD^2A^T)^{-1}AD^2e_k). \end{aligned} \tag{1}$$

Lemma 1 and 2 show that the first term of equation (1) converges to zero as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*). In Theorem 3, we prove that the second term of equation (1) also converges to zero as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*), and finally show that $\bar{x}^T Fe_k \to 0$ as $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$.

Lemma 1. *Let $(\bar{x}, \bar{y}, \bar{s})$ and (x^*, y^*, s^*) be an ϵ-optimal interior solution and an optimal solution to (LP), respectively. Let B and N denote the following index sets of variables*:

$$B = \{j \mid x_j^* > 0\}, \qquad N = \{j \mid x_j^* = 0\}.$$

Let $D = diag\{\sqrt{\bar{x}_j/\bar{s}_j}\}$ and $v^* = (AD^2A^T)^{-1}AD^2e_k$. Then, $A_B D_B^2((e_k)_B - A_B^T v^*) \to 0$ as $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$.

Proof. Consider the weighted least-square problem ($WL(D)$):

$$\min_{v \in \Re^m} \left\| De_k - DA^T v \right\|^2.$$

It is well known that if AD has full rank, then v^* is a unique optimal solution to ($WL(D)$) (See [7]). Moreover, v^* satisfies the following equations:

$$\begin{aligned} & AD(De_k - DA^T v^*) = 0 \\ \Leftrightarrow\ & AD^2 e_k - AD^2 A^T v^* = 0 \\ \Leftrightarrow\ & A_B D_B^2((e_k)_B - A_B^T v^*) = -A_N D_B^2((e_k)_N - A_N^T v^*)). \end{aligned} \tag{2}$$

Each diagonal element of D_N converges to zero as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*), and each component of v^* is known to be bounded for any D with positive diagonal elements([11]). Therefore, the right-hand side of equation (2) converges to zero as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*), which implies that the left-hand side of equation (2) also converges to zero. □

Lemma 2. *Let* $(\bar{x}, \bar{y}, \bar{s})$ *and* (x^*, y^*, s^*) *be an* ϵ*-optimal interior solution and an optimal solution to* (LP), *respectively. Let B and N denote the following index sets* of *variables:*

$$B = \{j \mid x_j^* > 0\}, \qquad N = \{j \mid x_j^* = 0\}.$$

Let $D = diag\{\sqrt{\bar{x}_j / \bar{s}_j}\}$ and $v^* = (AD^2A^T)^{-1} AD^2 e_k$. Then, $\bar{s}_B^T D_B^2((e_k)_B - A_B^T v^*) \to 0$ as $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$.

Proof.

$$\begin{aligned} & \bar{s}_B^T D_B^2((e_k)_B - A_B^T v^*) \\ & = (c_B - A_B^T \bar{y})^T D_B^2((e_k)_B - A_B^T v^*) \\ & = c_B^T D_B^2((e_k)_B - A_B^T v^*) - \bar{y}^T A_B D_B^2((e_k)_B - A_B^T v^*) \\ & = (y^*)^T A_B D_B^2((e_k)_B - A_B^T v^*) - \bar{y}^T A_B D_B^2((e_k)_B - A_B^T v^*) \end{aligned} \tag{3}$$

Since $A_B D_B^2((e_k)_B - A_B^T v^*) \to 0$ by Lemma 1, both the first and second term of the right-hand side of equation (3) converge to zero as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*). □

Now, we will show that $\bar{x}^T F e_k \to 0$ as $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$.

Theorem 3. *Let* $(\bar{x}, \bar{y}, \bar{s})$ *and* (x^*, y^*, s^*) *be an* ϵ*-optimal interior solution and an opti-*

mal solution to (LP)*, respectively. Let* $D = diag\{\sqrt{\bar{x}_j / \bar{s}_j}\}$ *and* $F = I - A^T(AD^2A^T)^{-1}AD^2$. *Consequently,* $\bar{x}^T Fe_k \to 0$ *as* $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$.

Proof. Let B and N denote the following index sets of variables:

$$B = \{j \mid x_j^* > 0\}, \qquad N = \{j \mid x_j^* = 0\}.$$

Let $v^* = (AD^2A^T)^{-1}AD^2e_k$. From equation (1),

$$\bar{x}^T Fe_k = \bar{s}_B^T D_B^2((e_k)_B - A_B^T v^*) + \bar{x}_N^T((e_k)_N - A_N^T v^*). \tag{4}$$

From Lemma 2, the first term of equation (4) converges to zero as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*). Since each component of v^* is bounded, and each component of $\bar{x}_N$ converges to zero, the second term of equation (4) also converges to zero. Therefore, $\bar{x}^T Fe_k \to 0$ as $(\bar{x}, \bar{y}, \bar{s}) \to (x^*, y^*, s^*)$. □

3. THE RELATIONSHIP BETWEEN ϵ-SENSITIVITY ANALYSIS AND SENSITIVITY ANALYSIS USING AN OPTIMAL BASIS UNDER PRIMAL OR DUAL NONDEGENERACY

Let (x^*, y^*, s^*)be an optimal extreme solution to (LP). In addition, let $(\bar{x}, \bar{y}, \bar{s})$ be an ϵ-optimal interior solution. If both x^* and (y^*, s^*) are nondegenerate optimal extreme solutions to (P) and (D), respectively, the characteristic region of ϵ-sensitivity analysis at $(\bar{x}, \bar{y}, \bar{s})$ converges to that of sensitivity analysis using the optimal basis associated with (x^*, y^*, s^*) as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*) ([9]). However, when x^* or (y^*, s^*) is degenerate, the relationship between ϵ-sensitivity analysis and sensitivity analysis using an optimal basis has not been discussed. In Section 3, we show that if either x^* or (y^*, s^*) is nondegenerate, the characteristic region of ϵ-sensitivity analysis at $(\bar{x}, \bar{y}, \bar{s})$ converges to that of sensitivity analysis using the optimal basis associated with (x^*, y^*, s^*) as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*). In Section 4, we draw a conclusion from an example that if both x^* and (y^*, s^*) are degenerate, the characteristic region of ϵ-sensitivity analysis at $(\bar{x}, \bar{y}, \bar{s})$ may not converge to that of sensitivity analysis using an optimal basis associated with (x^*, y^*, s^*).

Let B and N denote the index set of the basic variables and the nonbasic variables of an optimal basis of (LP), respectively. When $c_k$ck is perturbed, the characteristic region of sensitivity analysis using the optimal basis A_B is equal to the range of θ such that

$$\begin{bmatrix} A_B^T \\ A_N^T \end{bmatrix} A_B^{-T}(c_B + \theta(e_k)_B) \leq \begin{bmatrix} c_B + \theta(e_k)_B \\ c_N + \theta(e_k)_N \end{bmatrix} \tag{5}$$

Let (x^*, y^*, s^*) be the optimal extreme solution determined by the optimal basis A_B. Equation (5) can be rewritten as the following:

$$\begin{aligned} & s^* + \theta(e_k + A^T A_B^{-T}(e_k)) \geq 0 \\ \Leftrightarrow\ & s^* + \theta(e_k + A^T \pi) \geq 0 \end{aligned} \tag{6}$$

where $\pi = A_B^{-T}(e_k)_B$. On the other hand, ϵ-sensitivity analysis at an ϵ-optimal interior solution $(\bar{x}, \bar{y}, \bar{s})$ finds the range of θ such that

$$\bar{s} + \theta(e_k - A^T v^*) \geq 0 \tag{7}$$

where $v^* = (AD^2A^T)^{-1}AD^2e_k$ and $D = diag\{\sqrt{\bar{x}_j / \bar{s}_j}\}$. Recall that $\bar{x}^T F e_k$ can be regarded as zero if $(\bar{x}, \bar{y}, \bar{s})$ is sufficiently close to (x^*, y^*, s^*). Comparing equation (6) with equation (7), we find that ϵ-sensitivity analysis is analogous to sensitivity analysis using the basis A_B, except that π is replaced by v^*. If v^* converges to π as $(\bar{x}, \bar{y}, \bar{s})$ converges to (x^*, y^*, s^*), we can conclude that the characteristic region of ϵ-sensitivity analysis converges to that of sensitivity analysis using the optimal basis A_B. In the next lemma and theorem, we will show that under the appropriate assumption v^* converges to π as $(\bar{x}, \bar{y}, \bar{s})$converges to (x^*, y^*, s^*).

Consider the weighted least-square problem $(WL(D))$:

$$(WL(D)) : \min_{v \in \Re^m} \left\| De_k - DA^T v \right\|^2$$

where D is a diagonal matrix with positive diagonal elements. Note that the optimal solution to $(WL(D))$depends on the diagonal matrix D.

Lemma 4. *Let B and N denote the index set of the basic variables and the nonbasic variables of an optimal basis of (LP), respectively. Suppose that a sequence*

$\{D^1, D^2, \cdots\}$ *of diagonal matrices with positive diagonal elements satisfies the following conditions*:

(*i*) $D_{jj}^r \geq \xi$, $\forall_j \in B$ and $\forall_r \geq K$, where K is some sufficiently large integer and ξ *is some positive constant*,

(*ii*) $\lim_{r\to\infty} D_{jj}^r = 0$, $\forall_j \in N$,

Let v^r be an optimal solution to the weighted least square problem ($WL(D^r)$)and $\pi \to A_B^{-T}(e_k)_B$. Then, $v^r \to \pi$ as $r \to \infty$

Proof. The optimal solution v^r to ($WL(D^r)$) is rewritten as r

$$\begin{aligned} v^r &= (A(D^r)^2A^T)^{-1}A(D^r)^2e_k \\ &= (A_B(D_B^r)^2A_B^T + A_N(D_N^r)^2A_N^T)^{-1}(A_B(D_B^r)^2(e_k)_B \\ &\quad + A_N(D_N^r)^2(e_k)_N) \end{aligned} \tag{8}$$

Since A_B is a basis matrix and each diagonal element of D_N^r converges to zero, we can obtain from equation (8) that $v^r \to A_B^{-T}(e_k)_B$ as $r \to \infty$. □

Lemma 4 implies that the optimal solution to ($WL(D^r)$) converges to the optimal solution to the weighted least square problem ($WL(D_B^r)$)as $r \to \infty$:

$$WL(D^r): \min_{v\in\Re^m} \left\| D^r e_k - D^r A^T v \right\|^2$$

$$WL(D_B^r): \min_{v\in\Re^m} \left\| D_B^r (e_k)_B - D_B^r A_B^T v \right\|^2$$

Theorem 5. Let B and N denote the index set of the basic variables and the nonbasic variables of an optimal basis of (LP), respectively. Let (x^*,y^*,s^*) be the *optimal extreme solution associated with the basis* A_B. *Let* $Q = \{(\bar{x}^r,\bar{y}^r,\bar{x}^r) \mid r = 1, 2, \cdots\}$ *be a sequence of ϵ-optimal interior solutions with* $\lim_{r\to\infty}(\bar{x}^r,\bar{y}^r,\bar{x}^r) = (x^*,y^*,s^*)$. *If the sequence Q satisfies condition (a),*

(*a*) as $r \to \infty$,

$$\frac{\max_{j\in N}\{\bar{x}_j^r/\bar{s}_j^r\}}{\min_{j\in B}\{\bar{x}_j^r/\bar{s}_j^r)\}} \to 0,$$

then the characteristic region of ϵ-sensitivity analysis at $(\bar{x}^r,\bar{y}^r,\bar{x}^r)$ *converges to* *that of sensitivity analysis using* A_B as $r \to \infty$.

Proof. Let $1/\alpha^r = \min_{j\in B}\{\sqrt{\bar{x}_j^r/\bar{s}_j^r}\}$ and $D^r = diag\{\sqrt{\bar{x}_j^r/\bar{s}_j^r}\}$. In addition, let $Z^r = \alpha^r D^r$ and $v^r = (A(D^r)^2A^T)^{-1}A(D^r)^2e_k$. Since the optimal solution to $(WL(D^r))$ is equal to the optimal solution to $(WL(Z^r))$, v^r is the optimal solution to both $(WL(D^r))$ and $(WL(Z^r))$. Moreover, since the sequence of diagonal matrices, $\{Z^1, Z^2, \cdots\}$, satisfies the condition (a), it also satisfies the assumption of Lemma 4. Consequently, by using Lemma 4 we obtain that $v^r \to \pi$ as $r \to \infty$. This, together with equations (6) and (7), implies that the characteristic region of ϵ-sensitivity analyzes at the ϵ-optimal solutions of Q converges to the characteristic region of sensitivity analysis using the optimal basis A_B. □

Condition (a) in the above theorem can be explained in connection with the weighted least square problem $(WL(D^r))$:

$$\min_{v\in\Re^m}\left\|\begin{matrix} D_B^r(e_k)_B - D_B^r A_B^T v \\ D_N^r(e_k)_N - D_N^r A_N^T v\end{matrix}\right\|^2$$

Condition (a) assumes that the weights of rows in B is infinitely larger than those of rows in N. If x^* is nondegenerate, condition (a) is obviously satisfied. Therefore, the characteristic region of ϵ-sensitivity analysis at $(\bar{x}^r, \bar{y}^r, \bar{x}^r)$ converges to that of sensitivity analysis using A_B as $r \to \infty$. However, if x^* is degenerate, condition (a) may not be satisfied because $x_j^r \to 0$ and $s_j^r \to 0$ for some $j \in B$ In this case, consider another condition (b):

(b) there exists positive constants μ_j *and* ϕ_j *such that for some large K,*

$$\mu_j s_j^r \le x_j^r \le \phi_j s_j^r,\ \forall r \ge K \text{ if } x_j^* = s_j^* = 0.$$

Condition (b) means that if both x_j^r and s_j^r converge to zero, then they do so at the same rate. In interior-point methods, this assumption does not seem to be too restrictive([12]). If $(y^*, s^*)$is nondegenerate and condition (b) holds, then the sequence of Q also satisfies condition (a), which implies that the characteristic region of ϵ-sensitivity analysis at $(\bar{x}^r, \bar{y}^r, \bar{x}^r)$converges to that of sensitivity analysis using A_B as $r \to \infty$

For example, consider the following linear programming problem ($LP1$):

$$(P1): \begin{array}{ll} Min & 4x_1 + 5x_2 \\ s.t. & 4x_1 + 3x_2 - x_3 = 12 \\ & 2x_1 + 5x_2 - x_4 = 10 \\ & 3x_1 + 4x_2 - x_5 = 11 \\ & x_j \geq 0,\ j = 1, \cdots, 5 \end{array}$$

$$(D2): \begin{array}{ll} Max & 12y_1 + 10y_2 + 11y_3 \\ s.t. & 4y_1 + 2y_2 + 3y_3 + s_1 = 4 \\ & 3y_1 + 5y_2 + 4y_3 + s_2 = 5 \\ & -y_1 + s_3 = 0 \\ & -y_2 + s_4 = 0 \\ & -y_3 + s_5 = 0 \\ & s_j \geq 0,\ j = 1, \cdots, 5 \end{array}$$

The feasible solutions and the optimal solutions of ($P1$) and ($D1$)are displayed in Figure 1. ($P1$) has a unique optimal extreme solution $\hat{x}^1$ that is degenerate,

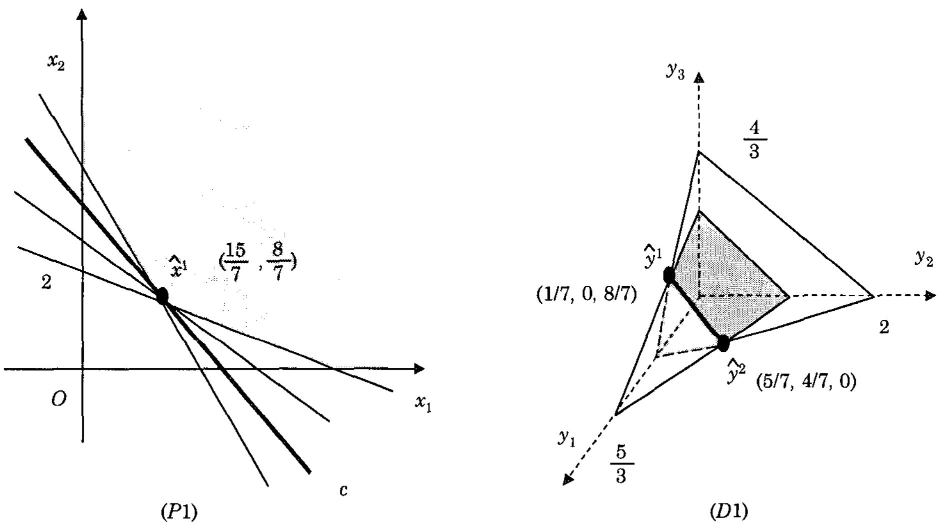

Figure 1. The optimal solutions of ($P1$) and ($D1$)

and (D1)has two optimal extreme solutions, $(\hat{y}^1,\hat{s}^1)$and $(\hat{y}^2,\hat{s}^2)$, which are both nondegenerate. Note that $\hat{x}^1 = (15/7, 8/7, 0, 0, 0)^T$, $\hat{s}^1 = (0, 0, 1/7, 0, 8/7)^T$, and $\hat{s}^2 = (0, 0, 5/7, 4/7, 0)^T$. There are two optimal bases, A_{B_1} and A_{B_2}, with basic variables that are presented in Table 1. When c_2 is perturbed, the characteristic regions of sensitivity analysis using A_{B_1} and A_{B_2}, respectively, are also given in Table 1.

Table 1 The characteristic regions of two optimal bases

optimal basis	index set of basic variables	optimal solution	characteristic region
A_{B^1}	$B^1 = \{1, 2, 4\}$	$(\hat{x}^1, \hat{y}^1, \hat{s}^1)$	$-2 \le \theta \le 1/3$
A_{B^2}	$B^2 = \{1, 2, 5\}$	$(\hat{x}^1, \hat{y}^2, \hat{s}^2)$	$-2 \le \theta \le 5$

Consider an ϵ-optimal interior solution $(\bar{x}^1, \bar{y}^1, \bar{s}^1)$:

$$\bar{x}^1 = (15/7 + \delta, 8/7 + \delta, 7\delta, 7\delta, 7\delta)^T,$$
$$\bar{y}^1 = (1/7 - \delta, \delta, 8/7 - \delta)^T,$$
$$\bar{s}^1 = (5\delta, 2\delta, 1/7 - \delta, \delta, 8/7 - \delta)^T,$$

where δ is a small positive number. As δ converges to 0, the ϵ-optimal interior solution $(\bar{x}^1, \bar{y}^1, \bar{s}^1)$converges to the optimal solution $(\hat{x}^1, \hat{y}^1, \hat{s}^1)$, and condition (b) is satisfied. Therefore, the characteristic region of ϵ-optimal solution at $(\bar{x}^1, \bar{y}^1, \bar{s}^1)$ converges to that of sensitivity analysis using the optimal basis A_{B^1}. Consider another ϵ-optimal interior solution $(\bar{x}^1, \bar{y}^2, \bar{s}^2)$:

$$\bar{x}^1 = (15/7 + \delta, 8/7 + \delta, 7\delta, 7\delta, 7\delta)^T,$$
$$\bar{y}^2 = (5/7 - \delta, 4/7 - \delta, \delta)^T,$$
$$\bar{s}^2 = (3\delta, 4\delta, 5/7 - \delta, 4/7 - \delta, \delta)^T.$$

As δ converges to 0, the ϵ-optimal interior solution $(\bar{x}^1, \bar{y}^2, \bar{s}^2)$ converges to the optimal solution $(\hat{x}^1, \hat{y}^2, \hat{s}^2)$, and condition (b) is satisfied. Therefore, the characteristic region of ϵ-optimal solution at $(\bar{x}^1, \bar{y}^2, \bar{s}^2)$converges to that of sensitivity analysis using the optimal basis A_{B^2}.

4. The relationship between ϵ-sensitivity analysis and sensitivity analysis using an optimal basis under primal and dual degeneracy

In this section, we investigate the limiting behavior of ϵ-sensitivity analysis at an ϵ-optimal interior solution that converges to an optimal extreme solution that is both primal and dual degenerate. Consider an example of linear programming problem, (*LP*2), which has a primal and dual degenerate optimal extreme solution.

$$(P2): \begin{array}{ll} Min & 2x_1 + 5x_2 + x_3 \\ s.t. & 2x_1 + 5x_2 + x_3 \qquad = 15 \\ & 4x_1 + 3x_2 \qquad - x_4 = 9 \\ & x_j \geq 0, j = 1, \cdots, 4 \end{array}$$

$$(D2): \begin{array}{ll} Max & 15y_1 + 9y_2 \\ s.t. & 2y_1 + 4y_2 + s_1 \qquad = 2 \\ & 5y_1 + 3y_2 \qquad + s_2 \qquad = 5 \\ & y_1 \qquad + s_3 \qquad = 1 \\ & - y_2 \qquad + s_4 = 0 \\ & s_j \geq 0, j = 1, \cdots, 4 \end{array}$$

Figure 2 shows the feasible solutions and the optimal solutions of (*P*2) and (*D*2). (*P*2) has three optimal extreme solutions, $\hat{x}^1, \hat{x}^2, \hat{x}^3$, and (*D*2) has a unique optimal extreme solution, $(\hat{y}^1, \hat{s}^1)$.

$$\hat{x}^1 = (15/2, 0, 0, 21)^T, \hat{x}^2 = (0, 3, 0, 0)^T, \hat{x}^3 = (9/4, 0, 21/2, 0)^T,$$
$$\hat{y}^1 = (1, 0)^T, \hat{s}^1 = (0, 0, 0, 0)^T$$

Moreover, (*LP*2) has five optimal bases, with basic variables that are listed in Table 2. The optimal solution $(\hat{x}^2, \hat{y}^1, \hat{s}^1)$ has three associated optimal bases, A_{B^1}, A_{B^4} and A_{B^5}. The characteristic region of A_{B^1} is equal to that of A_{B^5}, but not equal to that of A_{B^4}. However, ϵ-sensitivity analysis depends only on an ϵ-optimal interior solution. This property of ϵ-sensitivity analysis raises the following question: If an ϵ-optimal interior solution $(\bar{x}, \bar{y}, \bar{s})$ converges to $(\hat{x}^2, \hat{y}^1, \hat{s}^1)$, with which of the three optimal bases should the characteristic region of ϵ-sensitivity analysis be compared? For example, consider an ϵ-optimal interior solution $(\bar{x}, \bar{y}, \bar{s})$ to (*LP*2):

$$\bar{x} = (2\delta, 3 - \delta, \delta, 5\delta)^T,$$
$$\bar{y} = (1 - 3\delta, \delta)^T,$$
$$\bar{s} = (2\delta, 12\delta, 3\delta, \delta)^T$$

where δ is a small positive number. As δ converges to zero, $(\bar{x}, \bar{y}, \bar{s})$ converges to $(\hat{x}^2, \hat{y}^1, \hat{s}^1)$, and the characteristic region of ϵ-sensitivity analysis at $(\bar{x}, \bar{y}, \bar{s})$ converges to those of A_{B^1} and A_{B^5}, but not to that of A_{B^4}.

From the above example, we can conclude that under primal and dual degeneracy of an optimal extreme solution (x^*, y^*, s^*), the characteristic region of ϵ-sensitivity analysis at an ϵ-optimal interior solution, which converges to (x^*, y^*, s^*), may not converges to that of sensitivity analysis using an optimal basis associated with (x^*, y^*, s^*)

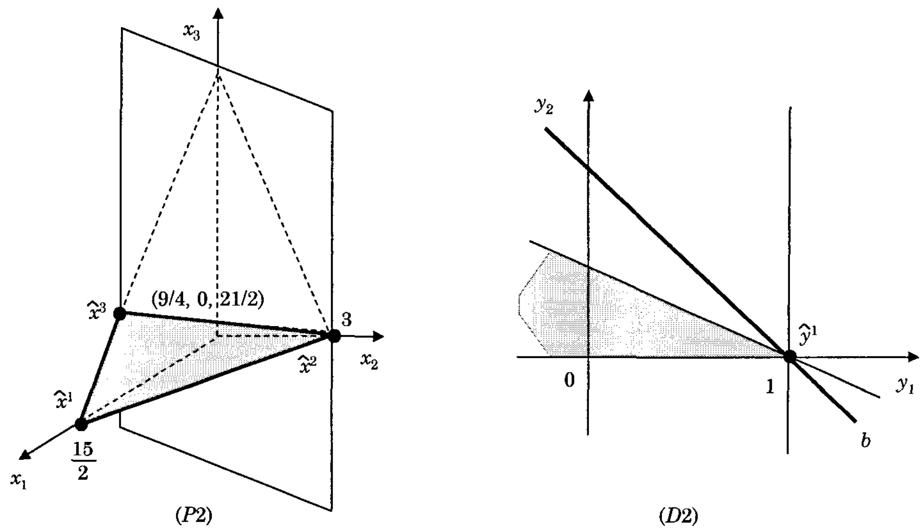

Figure 2. The optimal solutions of $(P2)$ and $(D2)$

Table 2. The characteristic regions of five optimal bases

optimal basis	index set of basic variables	optimal solution	characteristic region
A_{B^1}	$B^1 = \{1, 2\}$	$(\hat{x}^2, \hat{y}^1, \hat{s}^1)$	$\theta \le 0$
A_{B^2}	$B^2 = \{1, 3\}$	$(\hat{x}^3, \hat{y}^1, \hat{s}^1)$	$\theta \ge 0$
A_{B^3}	$B^3 = \{1, 4\}$	$(\hat{x}^1, \hat{y}^1, \hat{s}^1)$	$\theta \ge 0$
A_{B^4}	$B^4 = \{2, 3\}$	$(\hat{x}^2, \hat{y}^1, \hat{s}^1)$	$0 \le \theta \le 0$
A_{B^5}	$B^5 = \{2, 4\}$	$(\hat{x}^2, \hat{y}^1, \hat{s}^1)$	$\theta \le 0$

5. CONCLUSION

In this paper, we examined some properties of ϵ-sensitivity analysis. First, we showed that the value $\bar{x}^T F e_k$ converges to zero as an ϵ-optimal interior solution converges to any optimal solution, from which the simpler formula of ϵ-sensitivity analysis was derived. Because of this property, we can regard the value of $\bar{x}^T F e_k$ as zero when an ϵ-optimal solution is sufficiently close to an optimal solution.

To examine the limiting behavior of ϵ-sensitivity analysis when an ϵ-optimal solution is sufficiently close to an optimal extreme solution, we compared the characteristic region of ϵ-sensitivity analysis with that of sensitivity analysis using an optimal basis. Under primal nondegeneracy or dual nondegeneracy of an optimal extreme solution, the characteristic region of ϵ-sensitivity analysis converges to that of sensitivity analysis using the optimal basis associated with the optimal extreme solution. However, if an optimal extreme solution is primal and dual degenerate, it has more than one associated optimal basis, and each optimal basis may have a different characteristic region. In this case, the characteristic region of ϵ-sensitivity analysis may not converge to that of sensitivity analysis using an optimal basis associated with the optimal extreme solution.

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