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# THE SEMINORMED FUZZY CO-INTEGRAL

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**Abstract**

In this paper, we introduce the seminormed fuzzy co-integral as a complementary concept of seminormed fuzzy integral, and investigate its properties. Furthermore we propose an application of this new integral for decision making problems.

**Key words** : fuzzy measure, t-seminorm, t-semiconorm, seminormed fuzzy integral, seminormed fuzzy co-integral

## 1. INTRODUCTION

Sugeno [7] defined a fuzzy measure as a measure having the monotonicity instead of additive and a fuzzy integral which is an integral with respect to fuzzy measure. A generalization of the fuzzy integral was introduced by Ralescu and Adams[3]. The concept of the seminormed fuzzy integral was proposed by Suarez and Gill [5, 6]. Fuzzy integrals have been applied to insert in some decision theory, expecially decision in uncertain environment as a aggregation tool, and there were attempts to use fuzzy integral with respect to uncertainty measure, i.e., fuzzy measure, in some kind of applications[1, 2, 8].

The purpose of this paper is to introduce the seminormed fuzzy co-integral as a complementary concept of seminormed fuzzy integral, and investigate its properties. Furthermore we propose an application of this new integral for decision making problems.

## 2. THE SEMINORMED FUZZY CO-INTEGRAL

Let $X$ be a nonempty set and $\mathcal{A}$ a $\sigma$-algebra in $X$. A set function $g: \mathcal{A} \to [0, 1]$ is called a *fuzzy measure* on $\mathcal{A}$ if

(1) $g(\emptyset) = 0$, $g(X) = 1$;

(2) (monotonicity) $A, B \in \mathrm{A}$ and $A \subset B$ imply $g(A) \le g(B)$;

(3) (continuity from below) $\{A_n\} \subset \mathrm{A}$ and

$$A_1 \subset A_2 \subset \cdots \text{ imply } \lim_n g(A_n) = g(\bigcup_{n=1}^{\infty} A_n);$$

(4) (continuity from above) $\{A_n\} \subset \mathrm{A}$ and

$$A_1 \supset A_2 \supset \cdots \text{ imply } \lim_n g(A_n) = g(\bigcap_{n=1}^{\infty} A_n).$$

We call $(X, \mathcal{A}, g)$ a *fuzzy measure space* if $g$ is a fuzzy measure on $\mathcal{A}$.

Let B be the $\sigma$-algebra of Borel subsets of $[0, 1]$ and $\mathcal{A}$ a $\sigma$-algebra in $X$. A real-valued function $h: X \to [0, 1]$ is $(\mathcal{A}, \mathcal{B})$*-measurable* (i.e., measurable with respect to $\mathcal{A}$ and B) if

$$h^{-1}(B) = \{x \mid h(x) \in B\} \in \mathrm{A}$$

for any $B \in \mathcal{B}$.

We shall say measurable for $(\mathcal{A}, \mathcal{B})$-measurable if there is no confusion likely. From now on, we consider only the set

$L^0(X) = \{h: X \to [0, 1] \mid h$ is measurable with respect to $\mathcal{A}$ and $\mathcal{B}\}$.

For any given $h \in L^0(X)$ and $\alpha \in [0, 1]$, we write

$$H_\alpha = \{x \in X \mid h(x) \ge \alpha\},$$

and

$$H^\alpha = \{x \in X \mid h(x) < \alpha\}.$$

Let $A \in \mathcal{A}$, $h \in L^0(X)$. The *fuzzy integral* of $h$ on $A$ with respect to $g$ is defined by

$$\int_A h\, dg = \sup_{\alpha \in [0, 1]} [\alpha \wedge g(A \cap H_\alpha)]$$

When $A = X$, the fuzzy integral may be denoted by $\int h\, dg$. Sometimes the fuzzy integral is called Sugeno's integral in the literature.

A *t-seminorm* is a function $\top: [0, 1] \times [0, 1] \to [0, 1]$ which satisfies:

(1) $\top(x, 1) = \top(1, x) = x$ for each $x \in [0, 1]$,

(2) if $x_1 \le x_3$, $x_2 \le x_4$ for $x_1, x_2, x_3, x_4 \in [0, 1]$, then $\top(x_1, x_2) \le \top(x_3, x_4)$.

There exists a concept of *t-semiconorm* $\perp$ defined

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by $\perp(x, y) = 1 - \top(1 - x, 1 - y)$ for seminorm $\top$.

A t-semiconorm is a function $\perp : [0, 1]\times[0, 1] \to [0, 1]$ which satisfies:

(1) $\perp(x, 0) = \perp(0, x) = x$ for $x \in [0, 1]$,

(2) if $x_1 \le x_3$, $x_2 \le x_4$ for $x_1, x_2, x_3, x_4 \in [0, 1]$, then $\perp(x_1, x_2) \le \perp(x_3, x_4)$.

Clearly, $\top(x, 0) = \top(0, x) = 0$ and $\perp(x, 1) = \perp(1, x) = 1$.

### 2.1. Example

The following functions are *t-seminorm* $\top$ and its corresponding t-semiconorm $\perp$:

(1) $\top(x, y) = x \wedge y$, $\quad \perp(x, y) = x \vee y$

(2) $\top(x, y) = xy$, $\quad \perp(x, y) = (x + y) - xy$

(3) $\top(x, y) = 0 \vee (x + y - 1)$,

$\perp(x, y) = (x + y) \wedge 1$.

Let $A \in \mathcal{A}$, $h \in L^0(X)$. The *seminormed fuzzy integral*([5, 6]) *of* $h$ *on* $A$ *with respect to* $g$ is defined by

$$\int_A h \top g = \sup_{\alpha \in [0, 1]} \top(\alpha, g(A \cup H_\alpha)).$$

When $A = X$, the seminormed fuzzy integral is denoted by $\int h \top g$. Clearly, the seminormed fuzzy integral is the fuzzy integral for the case $\top(x, y) = x \wedge y$.

### 2.2. Definition

Let $A \in \mathcal{A}$, $h \in L^0(X)$. The *seminormed fuzzy co-integral of h on A with respect to* $g$ is defined by

$$(co)\int_A h \top g = \inf_{\alpha \in [0, 1]} \perp((1 - \alpha), g(A \cap H^\alpha)).$$

When $A = X$, the seminormed fuzzy co-integral is denoted by $(co)\int h \top g$.

### 2.3 Theorem

Let $(X, \mathcal{A}, g)$ be a fuzzy measure space, $A, B \in \mathcal{A}$, and $h, h_1, h_2 \in L^0(X)$. Then,

(1) If $g(A) = 0$, $(co)\int_A h \top g = 0$

(2) If $h_1 \le h_2$, $(co)\int_A h_1 \top g \ge (co)\int_A h_2 \top g$

(3) If $A \supset B$, $(co)\int_A h \top g \ge (co)\int_B h \top g$

(4) $(co)\int_A (h_1 \vee h_2) \top g \le (co)\int_A h_1 dg \vee (co)\int_A h_2 \top g$

(1) **Proof.** Since $g(A) = 0$,

$$\begin{aligned}(co)\int_A h \top g &= \inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), g(A \cap H^\alpha)] \\ &= \inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), 0] \\ &= \inf_{\alpha \in [0, 1]} (1 - \alpha) \\ &= 0\end{aligned}$$

(2) If $h_1 \le h_2$, then we have $g(H_1^\alpha) \ge g(H_2^\alpha)$, where $H_1^\alpha = \{x \mid h_1(x) \le \alpha\}$, $H_2^\alpha = \{x \mid h_2(x) \le \alpha\}$.

Thus,

$$\begin{aligned}(co)\int_A h_1 \top g &= \inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), g(A \cap H_1^\alpha)] \\ &\ge \inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), g(A \cap H_2^\alpha)] \\ &= (co)\int_A h_2 \top g\end{aligned}$$

(3) $(co)\int_A h \top g$

$$\begin{aligned}&= \inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), g(A \cap H^\alpha)] \\ &\ge \inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), g(B \cap H^\alpha)] \\ &= (co)\int_B h \top g\end{aligned}$$

(4) Since $h_i \le h_1 \vee h_2$ for $i = 1, 2$, from (2) we get

$$(co)\int_A h_i \top g \ge (co)\int_A (h_1 \vee h_2) \top g$$

Hence we have

$$(co)\int_A (h_1 \vee h_2) \top g \le (co)\int_A h_1 dg \vee (co)\int_A h_2 \top g$$

□

### 2.4. Corollary

Let $(X, \mathcal{A}, g)$ be a fuzzy measure space, $A, B \in \mathcal{A}$, and $h, h_1, h_2 \in L^0(X)$. Then,

(1) $(co)\int_A (h_1 \wedge h_2) \top g \ge (co)\int_A h_1 \top g \wedge (co)\int_A h_2 \top g$

(2) $(co)\int_{A \cup B} h \top g \ge (co)\int_A h \top g \vee (co)\int_B h \top g$

(3) $(co)\int_{A \cap B} h \top g \le (co)\int_A h \top g \wedge (co)\int_B h \top g$

### 2.5. Remark

The co-integral can be defined completely in terms of $\top$ (not using $\perp$) as the following shows.

$$\begin{aligned}&\inf_{\alpha \in [0, 1]} \perp[(1 - \alpha), g(A \cap H^\alpha)] \\ &= 1 - \sup_{\alpha \in [0, 1]} [1 - \perp((1 - \alpha), g(A \cap H^\alpha))] \\ &= 1 - \sup_{\alpha \in [0, 1]} \top[\alpha, 1 - g(A \cap H^\alpha)]\end{aligned}$$

where the last equality holds because

$$1 - \perp(1 - x, 1 - y) = \top(x, y).$$

### 2.6. Theorem

(Monotone Convergence Theorem) Let $(X, \mathcal{A}, g)$ be a fuzzy measure space and let $\top$ be a continuous $t$-seminorm. Suppose that the relations

(1) $h_1(x) \le h_2(x) \le \cdots$
( $h_1(x) \ge h_2(x) \ge \cdots$ )

(2) $h(x) = \lim_{n\to\infty} h_n(x)$

hold at every point $x$ in X. Then

$$\lim_n (co) \int_A h_n \top g = (co) \int_A h \top g.$$

**Proof**. We may assume that $A = X$ without any loss of generality. Let

$$H_n^\alpha = \{x \in X \mid h_n(x) < \alpha\}, \quad n = 1, 2, \cdots$$
$$H^\alpha = \{x \in X \mid h(x) < \alpha\}.$$

Then it is easy to see that

$$\bigcap_{n=1}^{\infty} H_n^\alpha = H^\alpha \; ( \bigcup_{n=1}^{\infty} H_n^\alpha = H^\alpha)$$

Hence by continuity of $t$-semiconorm $\perp$ and continuity from above(below) of $g$, we have

$$\begin{aligned}
\lim_{n\to\infty} (co) \int h_n \top g &= \lim_{n\to\infty} \inf_{\alpha\in[0,1]} \perp(1-\alpha,\; g(H_n^\alpha)) \\
&= \inf_{\alpha\in[0,1]} \perp\left(1-\alpha,\; \lim_{n\to\infty} g(H_n^\alpha)\right) \\
&= \inf_{\alpha\in[0,1]} \perp\left(1-\alpha,\; g(\bigcap_{n=1}^{\alpha} H_n^\alpha)\right) \\
\Big( &= \inf_{\alpha\in[0,1]} \perp\left(1-\alpha,\; g(\bigcup_{n=1}^{\alpha} H_n^\alpha)\right)\Big) \\
&= \inf_{\alpha\in[0,1]} \perp(1-\alpha,\; g(H^\alpha)) \\
&= (co) \int h \top g
\end{aligned}$$

□

## 2. AN APPLICATION

Let us apply seminormed fuzzy integrals and co-integrals to a evaluation problem. First, we consider the following Proposition.

### 2.1 Proposition

Let $(X, \mathcal{A}, g)$ be a fuzzy measure space, and $h \in L^0([0, 1])$.

(1) If $h$ is a constant function, then

$$\int h \top g + (co) \int h \top g = 1$$

(2) $X \equiv [0, 1]$, $\mathcal{A} \equiv$ {Lebesgue measurable sets}, and $g \equiv$ the Lebesque measure, then

$$\int h \top g + (co) \int h \top g = 1$$

**Proof**. Let $h(x) = c, \; 0 \le c \le 1$.

(1) $\int h \top g$

$$\begin{aligned}
&= \sup_{\alpha\in[0,c]} \top[\alpha,\; g(X)] \vee \sup_{\alpha\in(c,1]} \top[\alpha,\; g(\emptyset)] \\
&= \sup_{\alpha\in[0,c]} \top[\alpha,\; 1] \vee \sup_{\alpha\in(c,1]} \top[\alpha,\; 0] \\
&= c
\end{aligned}$$

$(co) \int h \top g$

$$\begin{aligned}
&= \inf_{\alpha\in[0,c]} \perp[1-\alpha,\; g(\emptyset)] \wedge \inf_{\alpha\in(c,1]} \perp[1-\alpha,\; g(X)] \\
&= \inf_{\alpha\in[0,c]} \perp[1-\alpha,\; 0] \wedge \inf_{\alpha\in(c,1]} \perp[1-\alpha,\; 1] \\
&= 1 - c
\end{aligned}$$

(2) Since $g$ is the Lebesgue measure,

$$g(H_\alpha) = 1 - g(H^\alpha).$$

Hence

$$\begin{aligned}
(co) \int h \top g &= \inf_{\alpha\in[0,1]} \perp[(1-\alpha),\; g(H^\alpha)] \\
&= \inf_{\alpha\in[0,1]} \perp[(1-\alpha),\; 1 - g(H_\alpha)] \\
&= \inf_{\alpha\in[0,1]} [1 - \top(\alpha,\; g(H_\alpha))] \\
&= 1 - \sup_{\alpha\in[0,1]} \top[\alpha,\; g(H_\alpha)] \\
&= 1 - \int h \top g
\end{aligned}$$

□

The second statement does not hold for general fuzzy measures so that

$$\int h \top g \text{ and } 1 - (co) \int h \top g$$

are different in general. In spirit, $1 - (co) \int h \top g$ is another "fuzzy integral". For some cases, it may be better to use this fuzzy co-integral (together with the fuzzy integral) for a decision making problem for a better judgement.

For example, consider the problem of evaluating a Chinese dish. Assume that the quality factors we consider are taste, smell, and appearance(including, e.g., the color, shape and general arrangement of the dish).

We denote these factors by $T$, $S$, and $A$ respectively; hence, we can set $X = \{T, S, A\}$. Assume further that following set function $g$ is employed as an importance measure (a fuzzy measure) :

$$g(\emptyset) = 0,$$

$g(\{T\}) = 0.7$, $\quad g(\{S\}) = 0.1$, $\quad g(\{A\}) = 0$
$g(\{T, S\}) = 0.9$ $\quad g(\{T, A\}) = 0.8$ $\quad g(\{S, A\}) = 0.3$
$g(\{T, S, A\}) = 1$

Assume that 5 dishes were presented to the Chinese food contest. An expert is invited as an adjudicator to judge each quality factor of those dishes and he scores the quality factors as follows :

Table 1. The scores of dishes

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
|---|---|---|---|---|---|
| $h(T)$ | 0.9 | 0.9 | 0.75 | 0.8 | 0.5 |
| $h(S)$ | 0.8 | 0.85 | 0.8 | 0.8 | 0.9 |
| $h(A)$ | 0.9 | 0.9 | 0.6 | 0.8 | 0.9 |

The seminormed fuzzy integrals have been used for the synthetic evaluation of fuzzy objects [7, 10].

Let us take $\top(x, y) = x \wedge y$ as in [10]. The synthetic evaluation of the quality of $D_1$ is then calculated as follows :

$$\begin{aligned}
&\int h \top g \\
&= \sup_{\alpha \in [0,\, 1]} (\alpha \wedge g(H_\alpha)) \\
&= [\sup_{\alpha \in [0,\, 0.8]} (\alpha \wedge g(X))] \\
&\qquad \vee [\sup_{\alpha \in (0.8,\, 0.9]} (\alpha \wedge g(\{T, A\}))] \\
&\qquad \vee [\sup_{\alpha \in (0.9,\, 1]} (\alpha \wedge g(\emptyset))] \\
&= [\sup_{\alpha \in [0,\, 0.8]} (\alpha \wedge 1)] \\
&\qquad \vee [\sup_{\alpha \in (0.8,\, 0.9]} (\alpha \wedge 0.8)] \\
&\qquad \vee [\sup_{\alpha \in (0.9,\, 1]} (\alpha \wedge 0)] \\
&= 0.8 \vee 0.8 \vee 0 \\
&= 0.8
\end{aligned}$$

One can use

$$1 - (co)\int h \top g \Big( \quad \text{or} \quad \frac{1}{2}\Big[\int h \top g + (1 - (co)\int h \top g)\Big]\Big)$$

for the synthetic evaluation of the quality of $D_1$. Since

$$\begin{aligned}
&(co)\int h \top g \\
&= \inf_{\alpha \in [0,\, 1]} ((1 - \alpha) \vee g(H^\alpha)) \\
&= [\inf_{\alpha \in [0,\, 0.8]} ((1 - \alpha) \vee g(\emptyset))] \\
&\qquad \wedge [\inf_{\alpha \in (0.8,\, 0.9]} ((1 - \alpha) \vee g(\{S\}))] \\
&\qquad \wedge [\inf_{\alpha \in (0.9,\, 1]} ((1 - \alpha) \vee g(X))] \\
&= [\inf_{\alpha \in [0,\, 0.8]} ((1 - \alpha) \vee 0)] \\
&\qquad \wedge [\inf_{\alpha \in (0.8,\, 0.9]} ((1 - \alpha) \vee 0.1)] \\
&\qquad \wedge [\inf_{\alpha \in (0.9,\, 1]} ((1 - \alpha) \vee 1)] \\
&= 0.2 \wedge 0.1 \wedge 1 \\
&= 0.1
\end{aligned}$$

the synthetic evaluation of the quality of $D_1$ is

$$1 - (co)\int h \top g = 0.9$$

$$\text{or} \quad \frac{1}{2}\Big[\int h \top g + \Big(1 - (co)\int h \top g\Big)\Big] = 0.85$$

The synthetic evaluations of five dishes are given in the following table;

Table 4. The synthetic evaluations of dishes

| Dish | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
|---|---|---|---|---|---|
| $\int h \top g$ | 0.8 | 0.85 | 0.75 | 0.8 | 0.5 |
| $1 - (co)\int h \top g$ | 0.9 | 0.9 | 0.75 | 0.8 | 0.5 |
| $\frac{1}{2}\Big[\int h \top g + \Big(1 - (co)\int h \top g\Big)\Big]$ | 0.85 | 0.875 | 0.75 | 0.8 | 0.5 |

Thus $\frac{1}{2}\Big[\int h \top g + \Big(1 - (co)\int h \top g\Big)\Big]$ is the best reasonable synthetic evaluation for our example. But, one could choose one from these three, depending the context of the problem.

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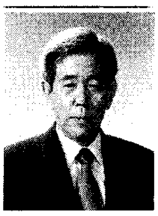

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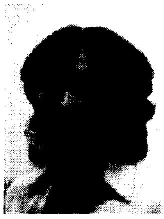

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