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METRICAL AND TOPOLOGICAL PRESSURE OF FLOWS WITHOUT FIXED POINTS

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ABSTRACT. We study the metrical and topological pressure for flows without fixed points on a compact metric space, and get the results as follows: (1) The metrical pressure with respect to an ergodic measure can be defined by (t, ε)-spanning sets. (2) The topological pressure is the supremum of metrical pressures with respect to all ergodic measures. (3) The properties that the topological pressure is zero, nonzero, finite or infinite respectively are invariant under weak equivalence.

1. Introduction

Let (X, d) be a compact metric space with the metric d, and $\varphi : R \times X \to X$ be a continuous flow(flow for short). Write φ_t for the homeomorphism of X defined by $\varphi_t(x) = \varphi(t, x)$. Define $\mathcal{B}$ to be the σ-algebra of Borel subsets of X. Let Φ represent φ or φ_1. The set of all Φ-invariant probability measures on $(X, \mathcal{B})$ is denoted by $M(\Phi)$. Put $E(\Phi) = \{\mu \in M(\Phi) | \mu \text{ is ergodic for } \Phi\}$. $C(X, R)$ denotes the Banach space of real-valued continuous functions on X equipped with the supremum norm. For $m \in M(\varphi)$ and $f \in C(X, R)$ we define

$$P_m(\varphi, f) = h_m(\varphi_1) + \int f dm,$$

where $h_m(\varphi_1)$ is the metrical entropy of φ_1 (with respect to m).

$P_m(\varphi, f)$ is called to be the metrical pressure of φ with respect to f and m.

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For $F, E \subset X$ we say F (t, ε)-spans E (with respect to φ), if for each $x \in E$ there is a $y \in F$ such that

$$d(\varphi_s(x), \varphi_s(y)) \leq \varepsilon, \ \forall s \in [0, t].$$

Topological pressure for a flow φ with respect to f is defined by Bowen in [1]. It is that

$$P(\varphi, f) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log P(\varphi, f, \varepsilon, t),$$

where $P(\varphi, f, \varepsilon, t) = \inf\{\sum_{x \in F} \exp \int_0^t f \circ \varphi_s(x) ds | F \ (t, \varepsilon) - \text{spans } X\}$. In particular, if $f = 0$, then $P(\varphi, 0)$ is just the topological entropy for φ.

Given φ and ψ flows on X and Y respectively, we say φ and ψ are weakly equivalent if there is a homeomorphism λ of X onto Y such that φ and $\hat{\varphi} = \{\hat{\varphi}_t = \lambda^{-1} \circ \psi_t \circ \lambda : t \in R\}$ have the same orbits ([6]). We know from [9] that, for continuous maps on compact spaces, both topological entropy and topological pressure are invariant under conjugacy. However, this does not hold for mutually weakly equivalent flows in general. In [6], Ohno proved that the properties that the topological entropy is zero, positive, finite or infinite respectively are invariant under weak equivalence for flows without fixed points (Theorem 1 in [6]). He showed also that this is not case if the flows have fixed points (Theorem 2 in [6]).

This paper is a further attempt to approach some problems of metrical and topological pressure for any flows without fixed points. In section 2, we shall give an equivalent definition of metrical pressure for an ergodic φ-invariant measure by (t, ε)-spanning sets (Theorem 1). In section 3, we shall prove that the topological pressure is the supremum of metrical pressures with respect to all ergodic φ-invariant measures (Theorem 2). In Section 4, we shall extend Theorem 1 in [6] to topological pressure (Theorem 3).

2. Metrical pressure of flows with respect to ergodic measures

In [3], Katok gave an equivalent definition of metrical entropy of a homeomorphism with respect to an ergodic invariant measure by (n, ε)-spanning sets. The following adopts the Katok's ideas.

For $m \in E(\varphi), f \in C(X,R)$ and $\delta \in (0,1)$. Put

$$\begin{aligned} Q_m(\varphi, f, \varepsilon, t, \delta) &= \inf\{\sum_{x\in F} \exp \int_0^t f \circ \varphi_s(x) ds | F \text{ is a } (t,\varepsilon) \\ & \quad \text{-spanning set of a set of m-measure} \\ & \quad \geq 1-\delta \}, \\ Q_m(\varphi, f, \varepsilon, \delta) &= \limsup_{t\to+\infty} \frac{1}{t} \log Q_m(\varphi, f, \varepsilon, t, \delta), \end{aligned}$$

and
$$Q_m(\varphi, f) = \lim_{\varepsilon\to 0} Q_m(\varphi, f, \varepsilon, \delta).$$

REMARK 1.

1. Theorem 1 we shall prove shows that the limit exists and is independent of δ.
2. For any $c \in R$,

$$Q_m(\varphi, f + c) = Q_m(\varphi, f) + c.$$

THEOREM 1. *Let φ be a flow without fixed points on X. For $m \in E(\varphi)$ and $f \in C(X,R)$, we have*

$$\begin{aligned} Q_m(\varphi, f) &= h_m(\varphi_1) + \int f dm \\ &= P_m(\varphi, f). \end{aligned}$$

Let $g : X \to X$ be a continuous map, and ξ a finite measurable partition of $(X, \mathcal{B})$. Put

$$\xi^n = \xi \vee g^{-1}\xi \vee \cdots \vee g^{-(n-1)}\xi, \ \forall n \geq 1.$$

For $x \in X$, let $A_n(x)$ denote the member of the partition ξ^n to which x belongs.

LEMMA 1.[4] (Shannon-McMillan-Breiman Theorem) *For $\mu \in M(g)$ and any finite measurable partition ξ of $(X, \mathcal{B})$, there is a μ-integrable function, denoted by μ_ξ, such that*

(1) $\lim_{n\to+\infty} (-\frac{1}{n} \log \mu(A_n(x))) = \mu_\xi(x)$ *a.e.*,

(2) $\int \mu_\xi(x) d\mu = h_\mu(g, \xi)$,

where $h_\mu(g, \xi)$ is the entropy of g with respect to ξ.

LEMMA 2. *For a small constant $r > 0$, let*

$$\begin{aligned} \mathcal{A}^-_{n,r}(\xi) &= \{A_n \in \xi^n | \mu(A_n) < \exp(-n(h_\mu(g,\xi) - r))\}, \\ S^-_{n,r}(\xi) &= \bigcup_{A_n \in \mathcal{A}^-_{n,r}(\xi)} A_n, \end{aligned}$$

then for some $\delta > 0$, there is N satisfying

$$\mu(S^-_{n,r}(\xi)) > \delta, \text{ for } n > N.$$

Proof. Write $X_0 = Supp(\mu)$, then $\mu(X_0) = 1$. For $r > 0$, we set $X_r = \{y \in X_0 | \mu_\xi(y) > h_\mu(g,\xi) - \frac{r}{2}\}$. Clearly, $\mu(X_r) > 0$. Put $\mu(X_r) = 2\delta$. By Egoroff Theorem ([5]), there exists a measurable set $B \in \mathcal{B}$ such that $\mu(B) > 1 - \delta$, and the sequence $\{-\frac{1}{n} \log \mu(A_n(x))\}$ converges uniformly to $\mu_\xi(x)$ on B. Therefore, for $\frac{r}{2} > 0$ there is N such that

$$-\frac{1}{n} \log \mu(A_n(x)) > \mu_\xi(x) - \frac{r}{2},$$

for every $x \in B$ and $n > N$. Notice that $\mu(B \bigcap X_r) > \delta$, so for any $y \in B \bigcap X_r$ and $n > N$, we have

$$-\frac{1}{n} \log \mu(A_n(y)) > \mu_\xi(x) - \frac{r}{2} > h_\mu(g,\xi) - r,$$

i.e.

$$\mu(A_n(y)) < \exp(-n(h_\mu(g,\xi) - r)).$$

So $(A_n(y)) \in \mathcal{A}^-_{n,r}(\xi)$, and thus

$$\mu(S^-_{n,r}(\xi)) \geq \mu(B \bigcap X_r) > \delta.$$

Now the proof of the lemma is complete.

Similar to the proof of Lemma 2, we can get the following

LEMMA 3. *For a small constant $r > 0$, let*

$$\begin{aligned} \mathcal{A}^+_{n,r}(\xi) &= \{A_n \in \xi^n | \mu(A_n) > \exp(-n(h_\mu(g,\xi) + r))\}, \\ S^+_{n,r}(\xi) &= \bigcup_{A_n \in \mathcal{A}^+_{n,r}(\xi)} A_n, \end{aligned}$$

then there exist some $\delta > 0$ and N satisfying

$$\mu(S^+_{n,r}(\xi)) > 2\delta, \text{ for } n > N.$$

Proof of Theorem 1, we give the proof in two steps.

Step 1. $Q_m(\varphi, f) \leq h_m(\varphi_1) + \int f dm.$

For any $\sigma < 0$, take $\varepsilon > 0$ such that if $d(x,y) < \varepsilon$ then

$$(1) \qquad d(\varphi_s(x), \varphi_s(y)) < \sigma, \ \ \forall s \in [0,1],$$

for all $x, y \in X$.

Since $m \in E(\varphi)$, it is easy to check $m \in M(\varphi_1)$ and is non-atomic. Take a finite partition ξ of X such that $diam(\xi) < \frac{\varepsilon}{2}$. By Lemma 3, for $r > 0$ there are $\eta > 0$ and N such that

$$(2) \qquad \mu(S^+_{n,r}(\xi)) > 2\eta, \ \text{for } n > N.$$

It follows from $m \in E(\varphi)$ that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t f\circ\varphi_s(x)ds = \int fdm \ \text{ a.e..}$$

Let $g_n(x) = \frac{1}{n}\int_0^n f\circ\varphi_s(x)ds$, then $g_n \to \int fdm$ a.e.. By Egoroff Theorem, for $\eta > 0$ there is a measurable set B such that $m(B) > 1-\eta$, and $\{g_n\}$ converges uniformly to $\int fdm$ on B. Obviously $\frac{1}{t}\int_0^t f\circ\varphi_s(x)ds$ also converges uniformly to $\int fdm$ on B, and $m(E_n := B\bigcap S^+_{n,r}(\xi)) > \eta := 1-\delta, \forall n \geq N$.

Let

$$Q_m(\varphi, f, \sigma, \delta) = \lim_{j\to+\infty} \frac{1}{t_j}\log Q_m(\varphi, f, \sigma, t_j, \delta).$$

We can put $\{t_j\}$ satisfying

$$(3) \qquad |\frac{1}{t}\int_0^t f\circ\varphi_s(x)ds - \int fdm| < \frac{1}{j}, \ \forall x \in B, \ t > t_j.$$

Write $n_j = t_j + s_j, 0 \leq s_j < 1$. Let F_{n_j} be an (n_j, ε)-spanning set with the smallest cardinality of $E_{n_j} := B\bigcap S^+_{n_j,r}(\xi)$ with respect to φ_1, then F_{n_j} is a (t_j, σ)-spanning set of E_{n_j} for φ. For any $A_{n_j} \in \xi^{n_j}$, we have $diam(A_{n_j}) < \varepsilon$. So there is at most one element of F_{n_j} in A_{n_j}, and therefore

$$(4) \qquad Card(F_{n_j}) \leq \exp(n_j(h_m(\varphi_1, \xi) + r)).$$

By (3) and (4), we have

$$Q_m(\varphi, f, \sigma, t_j, \delta) \leq \exp(n_j(h_m(\varphi_1, \xi) + r) + n_j(\int fdm + \frac{1}{j})).$$

Hence

$$\begin{aligned} Q_m(\varphi, f, \sigma, \delta) &\leq h_m(\varphi_1, \xi) + r + \int fdm \\ &\leq h_m(\varphi_1) + r + \int fdm. \end{aligned}$$

Let $r \to 0$ and then $\sigma \to 0$, we get

$$Q_m(\varphi, f) \le h_m(\varphi_1) + \int f dm.$$

Step 2. Put

$$\begin{aligned} q_m(\varphi, f, \varepsilon, \delta) &= \liminf_{t \to +\infty} \frac{1}{t} \log Q_m(\varphi, f, \varepsilon, t, \delta), \\ \text{and} \qquad q_m(\varphi, f) &= \lim_{\varepsilon \to 0} q_m(\varphi, f, \varepsilon, \delta). \end{aligned}$$

We shall prove $q_m(\varphi, f) \ge h_m(\varphi_1) + \int f dm$.

Let $\xi = \{A_1, A_2, \dots, A_k, A_{k+1}\}$ be a finite partition of $(X, \mathcal{B})$ such that

1. $A_1, A_2, \dots, A_k$ are pairwise disjoint and closed,
2. $A_{k+1} = X \backslash \bigcup_{i=1}^{k} A_i$.

The inequality is a consequence of the following claim.

Claim. For any $r > 0$ and any partition ξ satisfying the properties 1.–2. above, it follows that

$$r + q_m(\varphi, f) \ge h_m(\varphi_1, \xi) + \int f dm. \tag{5}$$

In order to prove the claim, we choose a positive integer L so that $\frac{1}{L} \log 6 < r$, and let

$$\xi^n = \xi \vee \varphi_L^{-1} \xi \vee \cdots \vee \varphi_L^{-(n-1)} \xi.$$

Take a small constant $a > 0$, by Lemma 2, there are $\delta > 0$ and N such that

$$m(S_{n,a}^{-}(\xi)) > 4\delta, \text{ for } n > N. \tag{6}$$

Since

$$\liminf_{t \to +\infty} \frac{1}{t} \log Q_m(\varphi, f, \varepsilon, t, \delta) = \liminf_{n \to +\infty} \frac{1}{nL} \log Q_m(\varphi, f, \varepsilon, nL, \delta), \tag{7}$$

it is enough to prove the claim for $t = nL$.

For any $\sigma > 0$, take $0 < \varepsilon < \sigma$ such that if $d(x, y) < \varepsilon$ then

$$|f \circ \varphi_s(x) - f \circ \varphi_s(y)| < \sigma, \tag{8}$$

for any $x, y \in X$ and any $s \in [0, 1]$.

Let F'_{nL} be an (nL, ε)-spanning set of E of m-measure greater than or equal to $1 - \delta$ with respect to φ, then it is an (n, ε)-spanning set of E for φ_L.

For $\delta > 0$ and a set B given by Egoroff Theorem, i.e. $m(B) > 1-\delta$ and $\int_0^t f \circ \varphi_s(x)ds$ converges uniformly to $\int f dm$ $(t \to +\infty)$ on B, let $E_{nL} := B \bigcap E$, then

$$m(E_{nL}) > 1 - 2\delta. \tag{9}$$

Let $F_{nL} \subset F'_{nL}$ be an (nL, ε)-spanning set with the smallest cardinality of E_{nL} for φ. Let $N(1-\delta, \varphi, \varepsilon, nL)$ be the smallest cardinality of any (nL, ε)-spanning set of a set of m-measure greater than or equal to $1-\delta$ with respect to φ. Clearly

$$N(1-\delta, \varphi, \varepsilon, nL) \geq N(1-2\delta, \varphi, \varepsilon, nL). \tag{10}$$

It follows from the proof of Theorem 5 in [7] that

$$\begin{aligned} 6^n \mathrm{Card}(F'_{nL}) &\geq 6^n \mathrm{Card}(F_{nL}) \\ &\geq 6^n N(1-2\delta, \varphi, \varepsilon, nL) \\ &\geq 2\delta \exp(n(h_m(\varphi_L, \xi) - a)). \end{aligned} \tag{11}$$

Choose $x_{nL} \in F_{nL}$ satisfying

$$\int_0^{nL} f \circ \varphi_s(x_{nL})ds = \min\{\int_0^{nL} f \circ \varphi_s(x)ds | x \in F_{nL}\}, \tag{12}$$

and notice that the cardinality of $F_{nL} \subset F'_{nL}$ is the smallest, so there is $y_{nL} \in E_{nL}$ such that

$$d(\varphi_s(x_{nL}), \varphi_s(y_{nL})) < \varepsilon, \ \ \forall s \in [0, nL]. \tag{13}$$

By (8) and (11)-(13), we get

$$\begin{aligned} &6^n \sum_{x \in F'_{nL}} \exp \int_0^{nL} f \circ \varphi_s(x)ds \\ \geq\ &2\delta \exp(n(h_m(\varphi_L, \xi) - a) + \int_0^{nL} f \circ \varphi_s(y_{nL})ds - nL\sigma). \end{aligned}$$

Take $\{n_j\}$ such that

$$|\frac{1}{n_j} \int_0^{n_j L} f \circ \varphi_s(x)ds - \int f dm| < \frac{1}{j}, \ \ \forall x \in B.$$

and

$$q_m(\varphi, f, \varepsilon, \delta) = \liminf_{j \to +\infty} \frac{1}{n_j L} \log Q_m(\varphi, f, \varepsilon, n_j L, \delta).$$

Therefore

$$
\begin{aligned}
6^n \sum_{x\in F'_{n_jL}} \exp \int_0^{n_jL} f\circ\varphi_s(x)ds &\geq 2\delta\exp(n_j(h_m(\varphi_L,\xi)-a \\
&\quad +L\int fdm-\frac{L}{j}-L\sigma)),
\end{aligned}
$$

and then

$$
\begin{aligned}
6^n Q_m(\varphi,f,\varepsilon,n_jL,\delta) &\geq 2\delta\exp(n_j(h_m(\varphi_L,\xi)-a+L\int fdm \\
&\quad -\frac{L}{j}-L\sigma)).
\end{aligned}
$$

Thus

$$\frac{1}{L}\log 6+q_m(\varphi,f,\varepsilon,\delta)\geq\frac{1}{L}(h_m(\varphi_L,\xi)-a)+\int fdm-\sigma.$$

Let $\sigma\to 0$ (so $\varepsilon\to 0$), we get

$$q_m(\varphi,f)+r\geq\frac{1}{L}(h_m(\varphi_L,\xi)-a)+\int fdm.$$

Notice that ξ and a are arbitrary and $h_m(\varphi_L)=Lh_m(\varphi_1)$, therefore

$$q_m(\varphi,f)+r\geq h_m(\varphi_1)+\int fdm.$$

Now the proof of the theorem is complete. □

Corollary 1. *For any $m\in E(\varphi)$,$\tau\in R\backslash\{0\}$ and $f\in C(X,R)$,*

$$P_m(\varphi,f)=\frac{1}{|\tau|}h_m(\varphi_\tau)+\int fdm.$$

3. The variational principle of topological pressures

For a given flow φ on X and $f\in C(X,R)$, we have known from [1] and the variational principle of topological pressure of continuous maps [9] that

$$
\begin{aligned}
P(\varphi,f) &= P(\varphi_1,f_1), \\
&= \sup\{h_m(\varphi_1)+\textstyle\int fdm|m\in M(\varphi_1)\}, \\
&= \sup\{h_m(\varphi_1)+\textstyle\int fdm|m\in E(\varphi_1)\}, \\
&= \sup\{h_m(\varphi_1)+\textstyle\int fdm|m\in M(\varphi)\},
\end{aligned}
\tag{14}
$$

where $f_1(x) = \int_0^1 f \circ \varphi_s(x)ds$, and $P(\varphi_1, f_1)$ is the topological pressure of φ_1 with respect to f_1.

Since there exist significant non-parallel gradients between φ and φ_1, for instance, a φ_1-invariant measure is not φ-invariant in general, and an ergodic measure for φ is not necessary ergodic for φ_1, it is reasonable to give the following.

THEOREM 2. *Let φ be a flow without fixed points on X. For any $f \in C(X, R)$, we have*

$$P(\varphi, f) = \sup\{h_m(\varphi_1) + \int f dm | m \in E(\varphi)\}.$$

Proof. It follows from (14) and $E(\varphi) \subset M(\varphi) \subset M(\varphi_1)$ that

$$\sup\{h_m(\varphi_1) + \int f dm | m \in E(\varphi)\} \leq P(\varphi, f).$$

To prove the converse inequality, it is enough to prove

$$\begin{aligned} & \sup\{h_m(\varphi_1) + \int f dm | m \in E(\varphi)\} \\ \geq \ & \sup\{h_m(\varphi_1) + \int f dm | m \in E(\varphi_1)\}. \end{aligned}$$

For each $\mu \in E(\varphi_1)$ and $t \in R$, we define

(15) $$\mu_t(B) = \mu(\varphi_t(B)), \text{ for } B \in \mathcal{B};$$

(16) $$m(B) = \int_0^1 \mu_t(B)dt, \text{ for } B \in \mathcal{B}.$$

It is easily seen that $\mu_t \in E(\varphi_1)$ and $m \in E(\varphi)$.

Let Supp(μ_t)(resp. Supp(m)) denote the support set of μ_t(resp. m). By (15) and $\varphi_t : X \to X$ is a homeomorphism, one can easily show

(17) $$\varphi_t(\text{Supp}(\mu_t)) = \text{Supp}(\mu),$$

and

$$\text{Supp}(m) = \overline{\bigcup_{t \in [0,1]} \text{Supp}(\mu_t)}.$$

For $\delta \in (0, 1)$ and $E_m \subset \text{Supp}(m)$ whose m-measure is greater than or equal to $1 - \delta$, suppose F_m is a (t, ε)-spanning set of E_m for φ. Put $t = n + s$, where n is a positive integer and $s \in [0, 1)$. It is easily seen from the equality (16) that there is $t_0 \in [0, 1]$ such that $\mu_{t_0}(E_m) \geq 1 - \delta$. Put $E(t_0) = E_m \bigcap \text{Supp}(\mu_{t_0})$, then $\mu_{t_0}(E(t_0)) \geq 1 - \delta$. We can see from the equality (17) that $\varphi_{t_0}(E(t_0)) \subset \text{Supp}(\mu)$ and $\mu(\varphi_{t_0}(E(t_0))) = \mu_{t_0}(E(t_0)) \geq 1-\delta$. Write $E_0 = \varphi_{t_0}(E(t_0))$.

Since F_m is a (t,ε)-spanning set of E_m with respect to φ, $F_0 := \varphi_{t_0}(F_m)$ is an $(n-1,\varepsilon)$-spanning set of E_0 for φ_1. By Remark 1, we can assume that $f \geq 0$. Therefore

$$\begin{aligned}\sum_{x\in F_m} \exp \int_0^t f\circ\varphi_s(x)ds &\geq \sum_{x\in F_0} \exp \int_0^{n-1} f\circ\varphi_s(x)ds \\ &= \sum_{x\in F_m} \exp \sum_{i=0}^{n-2} f_1\circ\varphi_1^i(x) \\ &\geq P_\mu^*(\varphi_1, f_1, \varepsilon, n-1),\end{aligned}$$

where $P_\mu^*(\varphi_1, f_1, \varepsilon, n-1) = \inf\{\sum_{x\in F} \exp \sum_{i=0}^{n-1} f_1\circ\varphi_1^i(x) | F$ is a (t,ε)-spanning set of a set of μ-measure $\geq 1-\delta$ with respect to $\varphi_1\}$ (see [2]).

Since E_m is arbitrary, we have

$$Q_m(\varphi, f, \varepsilon, t, \delta) \geq P_\mu^*(\varphi_1, f_1, \varepsilon, n-1).$$

It follows from Theorem 2.1 in [2] and Theorem 1 that

$$h_m(\varphi_1) + \int f dm \geq h_\mu(\varphi_1) + \int f d\mu.$$

This proves

$$\sup\{P_m(\varphi_1, f) | m \in E(\varphi)\} \geq \sup\{P_m(\varphi_1, f) | m \in E(\varphi_1)\}.$$

Now the proof of the theorem is complete. □

4. Topological pressures of mutually weakly equivalent flows

THEOREM 3. *Let φ and ψ be flows on compact metric spaces X and Y respectively. If φ and ψ are weakly equivalent and they have no fixed points, then we have*

$$P(\varphi, f) = C_{\varphi\psi} P(\psi, f\circ\lambda^{-1}), \text{ for } f \in C(X,R),$$

where $C_{\varphi\psi}$ is a constant, and λ is the homeomorphism in the definition of weak equivalence.

Proof. Let $\hat\varphi = \{\hat\varphi_t = \lambda^{-1}\circ\psi_t\circ\lambda : t \in R\}$. It is trivial from the definition of weak equivalence that

$$P(\hat\varphi, f) = P(\psi, f\circ\lambda^{-1}), \text{ for } f \in C(X,R).$$

It is enough to prove that there exists a constant $C_{\varphi\psi}$ such that

$$P(\varphi, f) = C_{\varphi\psi} P(\hat{\varphi}, f), \text{ for } f \in C(X, R).$$

By Theorem 2, we have

$$P(\varphi, f) = \sup\{h_m(\varphi_1) + \int f dm | m \in E(\varphi)\},$$

and

$$P(\hat{\varphi}, f) = \sup\{h_{\hat{m}}(\hat{\varphi}_1) + \int f d\hat{m} | \hat{m} \in E(\hat{\varphi})\}.$$

In addition, we get from Definition 2.2 in [8] and Proposition 1 in [6] that there exist continuous functions $\theta : X \times R \to R$ and $\hat{\theta} : Y \times R \to R$ such that for each $m \in E(\varphi)$,

(18)

$$\int f d\hat{m} = \frac{1}{\int \theta(1,x) dm} \int (\int_0^{\theta(1,x)} f \circ \hat{\varphi}_s(x) ds) dm, \text{ for } f \in C(X, R),$$

defines an $\hat{m} \in E(\hat{\varphi})$, the map $\Lambda : E(\varphi) \to E(\hat{\varphi}), m \mapsto \hat{m}$ is bijective, and the inverse map $\Lambda^{-1} : E(\hat{\varphi}) \to E(\varphi), \hat{m} \mapsto m$ is defined by

(19)

$$\int f dm = \frac{1}{\int \hat{\theta}(1,x) d\hat{m}} \int (\int_0^{\hat{\theta}(1,x)} f \circ \varphi_s(x) ds) dm, \text{ for } f \in C(X, R).$$

It follows from [6] that

$$h_{\hat{m}}(\hat{\varphi}_1) = \frac{1}{\int \theta(1,x) dm} h_m(\varphi_1).$$

Therefore

$$\frac{1}{A} h_m(\varphi_1) \le h_{\hat{m}}(\hat{\varphi}_1) \le \frac{1}{a} h_m(\varphi_1), \tag{20}$$

where $A = \max\{\theta(1,x) | x \in X\}, a = \min\{\theta(1,x) | x \in X\}$. It is easily checked by (18) and (19) that

$$C_1 \int f dm \le \int f d\hat{m} \le C_2 \int f dm \tag{21}$$

where $C_1 = \frac{b}{B}, B = \max\{\hat{\theta}(1,y) | y \in Y\}$, $b = \min\{\hat{\theta}(1,y) | y \in Y\}$, and $C_2 = \frac{a}{A}$.

Combining (20) with (21), we have that there exist constants D_1 and D_2 satisfying

$$\begin{aligned} D_1(h_m(\varphi_1) + \int f dm) &\leq h_{\hat{m}}(\hat{\varphi}_1) + \int f d\hat{m} \\ &\leq D_2(h_m(\varphi_1) + \int f dm). \end{aligned}$$

It follows that

$$D_1 P(\varphi, f) \leq P(\hat{\varphi}, f) \leq D_2 P(\varphi, f),$$

and so there is a constant $C_{\varphi\psi}$ such that

$$P(\varphi, f) = C_{\varphi\psi} P(\hat{\varphi}, f), \quad \text{for } f \in C(X, R).$$

Now the proof of the theorem is complete. □

COROLLARY 2. [6] *If* $f \equiv 0$ *then* $h(\varphi) = C_{\varphi\psi} h(\psi)$.

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