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ORTHOGONAL POLYNOMIALS SATISFYING PARTIAL DIFFERENTIAL EQUATIONS BELONGING TO THE BASIC CLASS

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Abstract. We classify all partial differential equations with polynomial coefficients in x and y of the form

$$A(x)u_{xx} + 2B(x,y)u_{xy} + C(y)u_{yy} + D(x)u_x + E(y)u_y = \lambda_n u,$$

which has weak orthogonal polynomials as solutions and show that partial derivatives of all orders are orthogonal.

Also, we construct orthogonal polynomials in d-variables satisfying second order partial differential equations in d-variables.

1. Introduction

There are many characterizations of classical orthogonal polynomials ([1, 2, 7, 9, 15]). If a set $\{P_n(x)\}_{n=0}^{\infty}$ of polynomials in one variable is an orthogonal polynomial set (OPS) relative to a moment functional σ, then, for example, the following statements are all equivalent.

(i) $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a second order ordinary differential equation

$$\alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y, \tag{1.1}$$

where $\alpha(x) = ax^2 + bx + c \neq 0$, $\beta(x) = dx + e$ and $\lambda_n = an(n-1) + dn \neq 0$ for all $n \geq 1$.

(ii) σ satisfies the equation

$$(\alpha(x)\sigma)' = \beta(x)\sigma$$

for some polynomial $\alpha(x)$ of degree ≤ 2 and $\beta(x)$ of degree ≤ 1.

(iii) $\{P_n'(x)\}_{n=1}^{\infty}$ is weakly orthogonal.

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We can see that $\{P_n^{(k)}(x)\}_{n=k}^{\infty}$ $(k \geq 0)$ is an OPS (in fact, relative to a moment functional $\alpha^k(x)\sigma$) satisfying a second order ordinary differential equation

$$\alpha(x)y''(x) + \left[k\alpha'(x) + \beta(x)\right] y'(x) = \mu_n(k)y. \tag{1.2}$$

The type of the ordinary differential equation (1.2) is same as that of the ordinary differential equation (1.1). Lyskova [14] generalized this property to the polynomials in several variables. Lyskova posed and solved the problem: Determine $A_{ij}(\mathbf{x})$ and $B_i(\mathbf{x})$ so that partial derivatives of any order of polynomial solutions to the partial differential equation

$$\sum_{i,j=1}^{d} A_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i(\mathbf{x})\frac{\partial u}{\partial x_i} = \lambda_{\mathbf{n}} u, \tag{1.3}$$
$$A_{ij}(\mathbf{x}) = A_{ji}(\mathbf{x}), \mathbf{x} = (x_1, \ldots, x_d)$$

satisfy the partial differential equation of the same type as (1.3), where $\deg A_{ij}(\mathbf{x}) \leq 2$, $\deg B_i(\mathbf{x}) \leq 1$ and $\lambda_{\mathbf{n}}$ is the eigenvalue parameter depending on the multi-index $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d := \{(n_1, \ldots, n_d) | n_i \in \mathbb{N}_0, 1 \leq i \leq d\}$. Lyskova called this class of partial differential equations the *basic* class.

THEOREM 1.1. [14] *The partial differential equation* (1.3) *belongs to the basic class if and only if the coefficients in* (1.3) *have the form*

$$A_{ij}(\mathbf{x}) = a_{ij}x_i x_j + b_{ij}x_i + c_{ij}x_j + f_{ij},$$
$$B_i(\mathbf{x}) = g_i x_i + h_i,$$
$$A_{ij}(\mathbf{x}) = A_{ji}(\mathbf{x}), \ (1 \leq i, j \leq d).$$

Moreover, if u is a polynomial solution to the partial differential equation (1.3), *then* $u^{(\alpha)} = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$ $(\alpha \in \mathbb{N}_0^d, |\alpha| = \alpha_1 + \cdots + \alpha_d)$ *satisfies the partial differential equation*

$$\sum_{i,j=1}^{d} A_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d}[B_i(\mathbf{x}) + \alpha_i \frac{\partial A_{ii}}{\partial x_i}(\mathbf{x})$$
$$+2 \sum_{j=1, j\neq i}^{d} \alpha_i \frac{\partial A_{ij}}{\partial x_j}(\mathbf{x})]\frac{\partial u}{\partial x_i} = \mu_{\mathbf{n}}(\alpha)u. \tag{1.4}$$

Lyskova did not discuss the orthogonality of partial derivatives of orthogonal polynomials satisfying the partial differential equation (1.3) although he gave a method of finding the orthogonality region and the weight function for orthogonal polynomials satisfying the partial differential equation (1.3).

If $a_{ij} = a$ and $g_i = g$ for all $1 \le i, j \le d$, then the eigenvalue parameter of the differential equation (1.3) depends only the degree of polynomial solutions. In fact, we have the differential equation

$$(1.5)\quad \sum_{i,j=1}^{d} (ax_i x_j + d_{ij} x_i + e_{ij} x_j + f_{ij}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} (g x_i + h_i) \frac{\partial u}{\partial x_i} = \lambda_n u,$$

where $\lambda_n = an(n-1) + gn$. The important work about orthogonal polynomials and the partial differential equations (1.5) was done by Krall and Sheffer[6] for the case $d = 2$ in 1967. In fact, they classified, up to a complex change of variables, all weak orthogonal polynomials in two variables which satisfy the partial differential equation of the form

$$(1.6)\quad A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x)u_x + E(y)u_y = \lambda_n u,$$

where $A(x,y) = ax^2 + d_1 x + e_1 y + f_1$, $2B(x,y) = 2axy + d_2 x + e_2 y + f_2$, $C(x,y) = ay^2 + d_3 x + e_3 y + f_3$, $D(x) = gx + h_1$ and $E(y) = gy + h_2$ under the assumptions that $\lambda_n \neq \lambda_m$ for $m \neq n$ and $4f_1 f_3 - f_2^2 \neq 0$. There were discovered many weak orthogonal polynomials satisfying the partial differential equations of the form (1.6). For some weak orthogonal polynomials among them, the orthogonality was not explained. Recently, their orthogonality has been discovered by authors [8].

In this paper, we classify, up to a real change of variables, all partial differential equations (1.6) which are in the basic class, i.e., $e_1 = d_3 = 0$ on the basis of the classification by Krall and Sheffer and show that partial derivatives of any order of orthogonal polynomial solutions to the partial differential equations in the basic class are also orthogonal. Also, some examples of orthogonal polynomials in d-variables satisfying the partial differential equation (1.5) are considered under the restriction that the coefficient A_{ij} of u_{x_i, x_j} does depend on x_i and x_j only.

2. Preliminaries

Through the paper, we use the multi-indices $\mathbb{N}_0^d := \{\alpha = (\alpha_1, \dots, \alpha_d) | \alpha_i \in \mathbb{N}_0, 1 \le i \le d\}$ of nonnegative integers and the lexicographic order on $\mathbb{N}_0^d$. We denote the space of all polynomials in variables $\mathbf{x} = (x_1, x_2, \dots, x_d)$ of degree less than or equal to n by Π_n^d and the space of all polynomials by Π^d. Then

$$\dim \Pi_n^d = \begin{pmatrix} n+d \\ d \end{pmatrix}$$

and

$$r_n^d := \dim\{\pi \in \Pi_n^d | \deg \pi = n\} = \begin{pmatrix} n+d-1 \\ n \end{pmatrix}.$$

A basis $\{\phi_\alpha(\mathbf{x})\}_{|\alpha|=0}^\infty$ for Π^d is called a polynomial set (PS) if $\deg \phi_\alpha = |\alpha|$ and $\Phi_n(\mathbf{x}) := \{\phi_\alpha(\mathbf{x})\}_{|\alpha|=n}$ (lexicographically ordered) is linearly independent modulo Π_{n-1}^d, where $|\alpha| = \sum_{i=0}^d \alpha_i$. We write a PS $\{\phi_\alpha(\mathbf{x})\}_{|\alpha|=n}$ by the r_n^d-dimensional column vector $\Phi_n(\mathbf{x})$, and a PS $\{\phi_\alpha(\mathbf{x})\}_{|\alpha|=0}^\infty$ by a sequence $\{\Phi_n(\mathbf{x})\}_{n=0}^\infty$ of column vectors of polynomials. A PS $\{\Phi_n(\mathbf{x})\}_{n=0}^\infty$ is monic if $\phi_\alpha(\mathbf{x}) = \mathbf{x}^\alpha$ modulo $\Pi_{|\alpha|-1}^d$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for each $\alpha \in \mathbb{N}_0^d$. The normalization of a PS $\{\Phi_n(\mathbf{x})\}_{n=0}^\infty$ is the monic PS $\{\mathbb{P}_n(\mathbf{x})\}_{n=0}^\infty$ defined by

$$\mathbb{P}_n(\mathbf{x}) = G_n^{-1}\Phi_n(\mathbf{x}),$$

where G_n is an $(n+1) \times (n+1)$ nonsingular matrix which is the leading coefficient of $\Phi_n(\mathbf{x})$ in the expression

$$\Phi_n(\mathbf{x}) = G_n \mathbf{x}^n \text{ modulo } \Pi_{n-1}^d,$$

where $\mathbf{x}^n = (x^n, x^{n-1}y, \ldots, y^n)^T$ is a vector of monomials of degree n for each $n \geq 0$.

Given any PS $\{\Phi_n(\mathbf{x})\}_{n=0}^\infty$, we can define a linear functional (moment functional) σ on Π^d, called the canonical moment functional of $\{\Phi_n(\mathbf{x})\}_{n=0}^\infty$, by the condition

$$\langle \sigma, \phi_\alpha \rangle = \begin{cases} 1 & \text{if } |\alpha| = 0, \\ 0 & \text{if } |\alpha| \geq 1. \end{cases}$$

We denote the action of a moment functional σ on any polynomial π by $\langle \sigma, \pi \rangle$. For a matrix $Q = (Q_{ij})_{i=1,j=1}^{m \quad n}$ $(Q_{ij} \in \Pi^d)$, the action of σ on Q is defined to be the matrix given by

$$\langle \sigma, Q \rangle = (\langle \sigma, Q_{ij} \rangle)_{i=1,j=1}^{m \quad n}.$$

It is very convenient to define two formal operations on a moment functional σ, i.e.,

(i) multiplication by a polynomial ϕ on σ: $\langle \phi\sigma, \pi \rangle = \langle \sigma, \phi\pi \rangle$, $\quad \pi \in \Pi^d$
(ii) partial derivatives of σ: $\langle \partial_{x_i}\sigma, \pi \rangle = -\langle \sigma, \partial_{x_i}\pi \rangle$, $\quad \pi \in \Pi^d \quad (1 \leq i \leq d)$.

Definition 2.1. Let $\{\Phi_n(\mathbf{x})\}_{n=0}^\infty$ be a PS.

(i) $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ is an orthogonal basis (OB) relative to σ if there is a nonzero moment functional σ such that $\langle \sigma, \phi_\alpha q\rangle = 0$ for any $\alpha \in \mathbb{N}_0^d$ and $q \in \Pi_{|\alpha|-1}^d$.

(ii) $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ is called a weak orthogonal polynomial set, in short WOPS, relative to σ if there is a nonzero moment functional σ such that $\langle \sigma, \phi_\alpha \phi_\beta\rangle = K_\alpha \delta_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{N}_0^d$. In particular, if $K_\alpha \neq 0$ (respectively, $K_a > 0$) for each $\alpha \in \mathbb{N}_0^d$, then $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ is called an orthogonal polynomial set ,in short, OPS (respectively, a positive-definite OPS) relative to σ.

DEFINITION 2.2. A moment functional σ is quasi-definite (respectively, positive-definite) if there is an OPS (respectively, a positive-definite OPS) relative to σ.

In the following, we give two theorems fundamental in the study of orthogonal polynomials in several variables.

THEOREM 2.1. *For a nonzero moment functional σ, the following statements are all equivalent.*

(i) *σ is quasi-definite.*

(ii) *$\Delta_n(\sigma) := \det D_n(\sigma) \neq 0$ for each $n \geq 0$, where*

$$D_n(\sigma) := (\sigma_{\alpha+\beta})_{|\alpha|=0,|\beta|=0}^{n \quad n}$$

is called the n-th Hankel matrix and $\sigma_\alpha := \langle \sigma, \mathbf{x}^\alpha\rangle$ is the α-th moment of σ for each $\alpha \in \mathbb{N}_0^d$.

(iii) *There is a unique monic OB $\{\mathbb{P}_n(\mathbf{x})\}_{n=0}^{\infty}$.*

(iv) *There is a monic OB $\{\mathbb{P}_n(\mathbf{x})\}_{n=0}^{\infty}$ such that $\langle \sigma, \mathbb{P}_n(\mathbf{x})\mathbb{P}_n^T(\mathbf{x})\rangle$ is nonsingular for each $n \geq 0$.*

THEOREM 2.2 (Favard's Theorem). [16] *Let $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ be a PS. Then the following statements are all equivalent.*

(i) *$\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ is an OB relative to a quasi-definite moment functional σ.*

(ii) *$\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ satisfies the three term recurrence relations, i.e., there are matrices $A_{ni} : r_n^d \times r_{n+1}^d$, $B_{ni} : r_n^d \times r_n^d$ and $C_{ni} : r_n^d \times r_{n-1}^d$ for all $n \geq 0$ such that*

$$x_i\Phi_n(\mathbf{x}) = A_{ni}\Phi_{n+1}(\mathbf{x}) + B_{ni}\Phi_n(\mathbf{x}) + C_{ni}\Phi_{n-1}(\mathbf{x}), \qquad (i = 1, \ldots, d)$$

and $\operatorname{rank}(C_{n1}, C_{n2}, \ldots, C_{nd}) = r_n^d$ for each $n \geq 0$. Here $M : m \times n$ means that M is an $m \times n$ matrix.

3. Orthogonal polynomials in two variables and partial differential equations in the basic class

We consider the following partial differential equation

$$
\begin{aligned}
&A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x)u_x + E(y)u_y \\
&= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} \\
&\quad + (ay^2 + d_3x + e_3y + f_3)u_{yy} + gxu_x + gyu_y \\
&= \lambda u.
\end{aligned}
\tag{3.1}
$$

We say that the partial differential equation (3.1) is admissible if there exists a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers such that for $\lambda = \lambda_n$ there are exactly $(n+1)$ linearly independent polynomial solution of degree n. It is easy to see that (3.1) is admissible if and only if $\lambda_n = an(n-1) + dn$ for each $n \geq 0$ and $\lambda_m \neq \lambda_n$ for $m \neq n$. From now we will assume that

$$4f_1f_3 - f_2^2 \neq 0.$$

THEOREM 3.1. *If a monic PS $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ satisfies a partial differential equation of the form (3.1), then the canonical moment functional σ of $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ satisfies the equation*

$$
\begin{aligned}
&(A\sigma)_{xx} + 2(B\sigma)_{xy} + (C\sigma)_{yy} - (D\sigma)_x - (E\sigma)_y \\
&= [(A\sigma)_x + (B\sigma)_y - D\sigma]_x + [(B\sigma)_x + (C\sigma)_y - E\sigma]_y = 0,
\end{aligned}
\tag{3.2}
$$

which is written, in terms of the moments $\sigma_{mn} = \langle \sigma, x^m y^n \rangle$ for $m,n \geq 0$, as the following;

(3.3)

$$
\begin{aligned}
A_{mn} := \; &\lambda_{m+n}\sigma_{mn} + [d_2mn + e_3n(n-1)]\sigma_{m,n-1} \\
&+[d_1m(m-1) + e_2mn]\sigma_{m-1,n} \\
&+f_1m(m-1)\sigma_{m-2,n} + f_2mn\sigma_{m-1,n-1} + f_3n(n-1)\sigma_{m,n-2} \\
&+e_1m(m-1)\sigma_{m-2,n+1} + d_3n(n-1)\sigma_{m+1,n-2} = 0.
\end{aligned}
$$

Furthermore, (3.2) has a unique solution up to a multiplication constant if (3.1) is admissible.

THEOREM 3.2. [6] *Let σ be the canonical moment functional of a monic PS $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ satisfying an admissible partial differential equation (3.1). Then the following statements are equivalent.*

(i) $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *is a WOPS relative to* σ.
(ii) $M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0$.
(iii) $M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0$.
(iv) $xM_2[\sigma] - yM_1[\sigma] = 0$.

3.1. Classification of partial differential equations in the basic class

In this section, we consider the partial differential equation in the basic class

$$\tag{3.4} \begin{aligned}&(ax^2 + d_1x + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy}\\ &+ (ay^2 + e_3y + f_3)u_{yy} + gxu_x + gyu_y = \lambda u\end{aligned}$$

and identify all differential equations having weak orthogonal polynomial solutions by using Theorem 3.2 and a real linear change of variables which preserves the orthogonality of polynomials.

The statement (iv) in Theorem 3.2 is written, in terms of the moments $\{\sigma_{mn}\}$ of σ, as the following expressions

$$\tag{3.5} \begin{aligned} D_{mn} := \quad & -[d_2(m-n) + 2e_3(n-1)]\sigma_{m,n-1}\\ & +[2d_1(m-1) - e_2(m-n)]\sigma_{m-1,n}\\ & +2f_1(m-1)\sigma_{m-2,n} - f_2(m-n)\sigma_{m-1,n-1}\\ & -2f_3(n-1)\sigma_{m,n-1} = 0.\end{aligned}$$

$D_{21} = D_{12} = D_{32} = D_{23} = 0$ give us the following necessary conditions for the partial differential equation (3.1) to have a weak OPS as solutions:

$$\tag{3.6} \begin{aligned}&2d_2f_1 + (e_2 - 2d_1)f_2 = 0,\\ &(d_2 - 2e_3)f_2 + 2e_2f_3 = 0,\end{aligned}$$

$$\tag{3.7} \begin{aligned}&(d_1 - e_2)[(d_1d_2 - 2d_2e_3 + e_2e_3)f_2 - a(f_2^2 - 4f_1f_3) = 0,\\ &(e_3 - d_2)[(d_1d_2 - 2d_2e_3 + e_2e_3)f_2 - a(f_2^2 - 4f_1f_3) = 0.\end{aligned}$$

If $d_1 - e_2 = d_2 - e_3 = 0$, then from (3.6) we have the equation for d_1 and d_2:

$$\begin{pmatrix}-f_2 & 2f_1\\ 2f_3 & -f_2\end{pmatrix}\begin{pmatrix}d_1\\ d_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

which implies that $d_1 = d_2 = e_2 = e_3 = 0$ since $4f_1f_3 - f_2^2 \neq 0$. In this case, we have a partial differential equation

$$\tag{3.8} (ax^2+f_1)u_{xx}+(2axy+f_2)u_{xy}+(ay^2+f_3)u_{yy}+gxu_x+gyu_y = \lambda_n u.$$

Note that any real linear change of variables

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (AD - BC \neq 0)$$

transforms the differential equation (3.8) into the differential equation

$$(a\xi^2 + f_1^*)u_{\xi\xi} + (2a\xi\eta + f_2^*)u_{\xi\eta} + (a\eta^2 + f_3^*)u_{\eta\eta} + g(\xi u_\xi + \eta u_\eta) = \lambda_n u,$$

where

$$f_1^* = A^2 f_1 + ABf_2 + B^2 f_3,\ f_2^* = 2ACf_1 + (AD+BC)f_2 + 2BDf_3$$

and $f_3^* = C^2 f_1 + CDf_2 + D^2 f_3$.

THEOREM 3.3. *The partial differential equation* (3.8) *is transformed, by a real linear change of variables, into the partial differential equation of the form*

$$(3.9)\quad (ax^2 + f_1^*)u_{xx} + 2axyu_{xy} + (ay^2 + f_3^*)u_{yy} + gxu_x + gyu_y = \lambda_n u$$

with $f_1^* f_3^* \neq 0$. *Furthermore, we can take (i)* $f_1^* = f_3^*$ *if* $f_2^2 - 4f_1f_3 < 0$ *and (ii)* $f_1^* = 1$, $f_3^* = -1$ *if* $f_2^2 - 4f_1f_3 > 0$.

Proof. Suppose that $f_2^2 - 4f_1f_3 < 0$. Let $u \pm iv$ be complex roots of $F(t) = f_1t^2 + f_2t + f_3$. The transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}u & 1\\ v & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

make $f_1^* = f_3^* = \frac{4f_1f_3 - f_2^2}{4f_1} \neq 0$ and $f_2^* = 0$.

Next, suppose that $f_2^2 - 4f_1f_3 > 0$ and $f_1 \neq 0$. If $f_2 \neq 0$, then the transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\frac{1}{f_2} & 1\\ -\frac{1}{f_2} & 1\end{pmatrix}\begin{pmatrix}u_1 & 1\\ u_2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

makes $f_1^* = 1$, $f_2^* = 0$ and $f_3^* = -1$, where $u_i (i = 1, 2)$ are distinct real roots of $F(t)$. If $f_2 = 0$, then we already have $f_1f_3 < 0$. By the transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}u_1 & 1\\ u_2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},$$

we have $f_1^* = f_3^* = 0$ and $f_2^* \neq 0$. For completion of proof, we follow the next argument.

Finally, if $f_2^2 - 4f_1f_3 > 0$ and $f_1 = 0$, then $f_2 \neq 0$ and the transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\frac{1}{f_2} & 1\\ -\frac{1}{f_2} & 1\end{pmatrix}\begin{pmatrix}f_3 & -f_2\\ f_2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

makes $f_1^* = 1$, $f_2^* = 0$ and $f_3^* = -1$. Thus theorem is proved. □

THEOREM 3.4. *If* $(d_1 - e_2)^2 + (d_2 - e_3)^2 \neq 0$ *and the partial differential equation* (3.4) *has a weak OPS as solutions, then it is transformed into* *one of the following partial differential equations:*

(3.10)

$$(d_1x + f_1)u_{xx} + (e_3y + f_3)u_{yy} + gxu_x + gyu_y = gnu,$$

(3.11)

$$(x^2 - x)u_{xx} + 2xyu_{xy} + (y^2 - y)u_{yy} + [(\alpha+\beta+\gamma+3)x - (\alpha+1)]u_x + [(\alpha+\beta+\gamma+3)y - (\beta+1)]u_y = \lambda_n u,$$

(3.12)

$$x^2u_{xx} + 2xyu_{xy} + y(y-1)u_{yy} + g(x-1)u_x + g(y-\gamma)u_y = \lambda_n,$$

(3.13)

$$f_1u_{xx} + f_2u_{xy} + f_3u_{yy} + gxu_x + gyu_y = gnu,$$

(3.14)

$$xu_{xx} + 2yu_{xy} + yu_{yy} + (gx + h_1)u_x + (gy + h_2)u_y = gnu,$$

(3.15)

$$xu_{xx} + 2yu_{xy} + (gx + h_1)u_x + (gy + h_2)u_y = gnu.$$

Proof. We have three cases from (3.6) and (3.7):

Case 1: $2d_2f_1 + (e_2 - 2d_1)f_2 = 0$, $(d_2 - 2e_3)f_2 + 2e_2f_3 = 0$, $a = f_2 = 0$.

It is easy to see that $d_2 = e_2 = 0$ since $f_1f_3 \neq 0$. Then we obtain (3.10).

Case 2: $2d_2f_1 + (e_2 - 2d_1)f_2 = 0$, $(d_2 - 2e_3)f_2 + 2e_2f_3 = 0$ and $(d_1d_2 - 2d_1e_3 + e_2e_3)f_2 = a(f_2^2 - 4f_1f_3)(\neq 0)$.

In this case, we can show that

$$d_2e_2 = 2af_2(\neq 0)$$

and (3.4) can be written as

$$\tfrac{1}{4a}(2ax + e_2)(2ax + 2d_1 - e_2)u_{xx} + \tfrac{1}{2a}(2ax + e_2)(2ay + d_2)u_{xy} + \tfrac{1}{4a}(2ay + d_2)(2ay + 2e_3 - d_2)u_{yy} + gxu_x + gyu_y = \lambda_n u.$$

If $(d_1 - e_2)(d_2 - e_3) \neq 0$, then we have the partial differential equation (3.11) through the transformation

$$\xi = \frac{a}{e_2 - d_1}x + \frac{d_2}{2(e_2 - d_1)}, \quad \eta = \frac{a}{e_2 - d_3}y + \frac{d_2}{2(d_2 - e_3)}.$$

Similarly, if $d_1 - e_2 = 0$ but $d_2 - e_3 \neq 0$, then we have the partial differential equation (3.12) through the transformation

$$\xi = \frac{2a}{e_2}x + 1, \quad \eta = \frac{a}{e_2 - d_3}y + \frac{d_2}{2(d_2 - e_3)}.$$

Case 3: $2d_2f_1+(e_2-2d_1)f_2=0$, $(d_2-2e_3)f_2+2e_2f_3=0$, $a=0$ and $d_1d_2-2d_1e_3+e_2e_3=0$.

If $f_2=0$, then we are reduced to Case 1 with $d_1e_3=0$.

If $f_2\neq 0$, then we have $d_2e_2(f_2^2-4f_1f_3)=0$, which implies that

$$d_2e_2=0.$$

If $d_2=e_2=0$, then $d_1=e_3=0$ and we obtain (3.13). Thus we may assume that

$$d_2=0, \quad e_2\neq 0,$$

which implies that

$$2d_1=e_2(\neq 0),\ e_2f_3=e_3f_2.$$

If $e_3\neq 0$, then the differential equation (3.1) becomes

$$(d_1x+f_1)u_{xx}+2d_1(y+\frac{f_3}{e_3})u_{xy}+e_3(y+\frac{f_3}{e_3})u_{yy}+gxu_x+gyu_y=gnu,$$

which is transformed into the partial differential equation (3.14) by the change of variables

$$\xi=\frac{2d_1x+f_1}{d_1^2}, \quad \eta=\frac{e_3y+f_3}{e_3^2}.$$

If $e_3=0$, then $f_3=0$ and we have the partial differential equation

$$(d_1x+f_1)u_{xx}+(2d_1y+f_2)u_{xy}+gxu_x+gyu_y=gnu,$$

which is transformed into the partial differential equation (3.15) by the linear change of variables

$$\xi=\frac{2d_1x+f_1}{d_1^2}, \quad \eta=2d_1y+f_2.$$

□

In the following, we identify all partial differential equations which can be obtained from Theorem 3.3 and Theorem 3.4 by a suitable real linear change of variables. According to the form of polynomial solutions, we divide them into three groups. We refer [10] for the classical orthogonal polynomials such as the (twisted) Jacobi, Laguerre, (twisted) Hermite and Bessel polynomials.

I. OPS's related to the (twisted) Jacobi polynomials (see [6, 8]).

If $a\neq 0$ in (3.9), we may assume that $a=1$. By the real change of variables $x=\sqrt{|f_1|}\xi$ and $y=\sqrt{|f_3|}\eta$, the differential equation (3.9) is transformed into the differential equation

$$(\xi^2+\operatorname{sgn} f_1)u_{\xi\xi}+2\xi\eta u_{\xi\eta}+(\eta^2+\operatorname{sgn} f_3)u_{\eta\eta}+g\xi u_\xi+g\eta u_\eta=\lambda_n u,$$

where $\operatorname{sgn} x = 1$ if $x > 0$ and $\operatorname{sgn} x = -1$ if $x < 0$. Thus we have the differential equations

$$(3.16) \qquad (x^2 - 1)u_{xx} + 2xyu_{xy} + (y^2 - 1)u_{yy} + gxu_x + gyu_y = \lambda_n u;$$

$$(3.17) \qquad (x^2 + 1)u_{xx} + 2xyu_{xy} + (y^2 + 1)u_{yy} + gxu_x + gyu_y = \lambda_n u;$$

$$(3.18) \qquad (x^2 + 1)u_{xx} + 2xyu_{xy} + (y^2 - 1)u_{yy} + gxu_x + gyu_y = \lambda_n u.$$

The polynomial solutions to the differential equation (3.16) are called circle polynomials which are expressed by Gegenbauer polynomials. For $g > 1$, they are orthogonal with respect to the weight function $w(x, y) = (1 - x^2 - y^2)^{\frac{g-3}{2}}$ in the unit disk. (3.17) and (3.18) could not be obtained in classification by Krall and Sheffer because they used a complex change of variables. Now we have the explicit expression of their polynomial solutions in terms of the twisted Jacobi polynomials. Another differential equation related to the Jacobi polynomials is

(3.19)
$$(x^2 - x)u_{xx} + 2xyu_{xy} + (y^2 - y)u_{yy} + [(\alpha + \beta + \gamma + 3)x - (\alpha + 1)]u_x$$
$$+[(\alpha + \beta + \gamma + 3)y - (\beta + 1)]u_y = \lambda_n u,$$

which has polynomial solutions which are orthogonal with respect to the weight function $w(x, y) = x^\alpha y^\beta (1-x-y)^\gamma$ in the triangle $\{(x, y) | x, y, 1-x-y > 0\}$. We can give a explicit form of polynomials by Jacobi polynomials.

II. Product of the classical orthogonal polynomials in one variable

We may assume that $g = 1$ in the differential equation (3.10). If $d_1 \neq 0$ and $e_3 = 0$ in the differential equation (3.10), the real change of variables $d_1 x + f_1 = -d_1^2 \xi$ and $y = \sqrt{|f_3|}\eta$ gives us the differential equation

(3.20)
$$-\xi u_{\xi\xi} + \operatorname{sgn} f_3 u_{\eta\eta} + (\xi - \alpha - 1)u_\xi + \eta u_\eta = \lambda_n u \qquad (\beta + 1 = -f_1/d_1^2).$$

If $d_1 = e_3 = 0$ (or $a = 0$ in (3.9)), then we have, thorough the real change of variables $x = \sqrt{|f_1|}\xi$ and $y = \sqrt{|f_3|}\eta$, the differential equation

$$(3.21) \qquad \operatorname{sgn} f_1 u_{\xi\xi} + \operatorname{sgn} f_3 u_{\eta\eta} + \xi u_\xi + \eta u_\eta = \lambda_n u.$$

We note that (3.20) with $\operatorname{sgn} f_3 = 1$ and (3.21) with $\operatorname{sgn} f_1 = 1$ or $\operatorname{sgn} f_3 = 1$ are omitted in Krall and Sheffer's paper. If $d_1^2 + e_3^2 \neq 0$, then, by the real change of variables $d_1 x + f_1 = -d_1^2 \xi$ and $e_3 y + f_3 = -e_3^2 \eta$, we can obtain the differential equation

$$-\xi u_{\xi\xi} - \eta u_{\eta\eta} + (\xi - \alpha - 1)u_\xi + (\eta - \beta - 1)u_\eta = \lambda_n u.$$

Summarizing these, we have the set of differential equations

$$
\begin{aligned}
&u_{xx} - u_{yy} + xu_x + yu_y = \lambda_n u,\\
&u_{xx} - u_{yy} - xu_x - yu_y = \lambda_n u,\\
&u_{xx} + u_{yy} + xu_x + yu_y = \lambda_n u,\\
&u_{xx} + u_{yy} - xu_x - yu_y = \lambda_n u,\\
&xu_{xx} + yu_{yy} + (\alpha + 1 - x)u_x + (\beta + 1 - y)u_y = \lambda_n u,\\
&u_{xx} + yu_{yy} - xu_x + (\beta + 1 - y)u_y = \lambda_n u,\\
&-u_{xx} + yu_{yy} - xu_x + (\beta + 1 - y)u_y = \lambda_n u,\\
&xu_{xx} + u_{yy} + (\alpha + 1 - x)u_x - yu_y = \lambda_n u,\\
&xu_{xx} - u_{yy} + (\alpha + 1 - x)u_x - yu_y = \lambda_n u,
\end{aligned}
$$

whose polynomial solutions are characterized by the following theorem.

THEOREM 3.5. [4] *Let* $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *be a monic PS satisfying the differential equation of the form*

$$A(x)u_{xx} + C(y)u_{yy} + D(x)u_x + E(y)u_y = \lambda_n u.$$

Then

(a) $P_{n0}(x,y) = P_{n0}(x)$, $P_{0n}(x,y) = P_{0n}(y)$, *and* $P_{mn}(x,y) = P_{m0}(x)P_{0n}(y)$, *m and* $n \geq 0$;
(b) $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *is a WOPS;*
(c) *For the canonical moment functional* σ *of* $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$, *the following statements are all equivalent:*
  (i) σ *is quasi-definite (respectively, positive-definite);*
  (ii) $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *is an OPS(respectively, a positive-definite OPS);*
  (iii) $\{P_{n0}(x)\}_{n=0}^{\infty}$ *and* $\{P_{0n}(y)\}_{n=0}^{\infty}$ *are classical OPS's (respectively, positive-definite classical OPS's).*

III. OPS's related to the Bessel polynomials (see [6, 8])

$$x^2u_{xx} + 2xyu_{xy} + y(y-1)u_{yy} + g(x-1)u_x + g(y-\gamma)u_y = \lambda_n; \tag{3.22}$$

$$xu_{xx} + 2yu_{xy} + yu_{yy} + (gx + \alpha + 2)u_x + (gy + \alpha - \beta + 1)u_y = gnu; \tag{3.23}$$

$$xu_{xx} + 2yu_{xy} + (gx + \alpha + 2)u_x + (gy + \beta)u_y = gnu. \tag{3.24}$$

We see, by Theorem 3.7 in the next subsection, that the differential equation (3.22) is closely related to the Bessel polynomials. Apparently it seems that the differential equations (3.23) and (3.24), which were omitted from the classification by Krall and Sheffer since they used a complex change of variables, does not involve the Bessel polynomials. A further work shows that we can express polynomial solutions in terms of Bessel polynomials. See [11].

3.2. Orthogonality of partial derivatives

Recently, we obtained some results on the partial derivatives of orthogonal polynomials in two variables [12].

THEOREM 3.6. [12] *Let* $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ *be an OPS relative to a moment functional* σ *and a monic PS* $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *be the normalization of* $\{\Phi_n(x,y)\}_{n=0}^{\infty}$. *If* $\partial_x P_{n0}=0$ *(respectively,* $\partial_y P_{0n}=0$*) for each* $n \geq 1$, *then the following statements are equivalent.*

(i) σ *satisfies the functional equation*

$$(\alpha(x,y)\sigma)_x = \beta(x,y)\sigma, \ (\textit{respectively, } (\bar{\alpha}(x,y)\sigma)_y = \bar{\beta}(x,y)\sigma\ ), \tag{3.25}$$

where $\alpha(x,y)$ *(respectively,* $\bar{\alpha}(x,y)$*) is a polynomial of degree* ≤ 2 *and* $\beta(x,y)$ *(respectively,* $\bar{\beta}(x,y)$*) is a polynomial of degree* ≤ 1.

(ii) $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *satisfy a system of partial differential equation*

$$\begin{aligned} &\alpha(x,y)\partial_x^2\overrightarrow{U}_n + \beta(x,y)\partial_x\overrightarrow{U}_n \\ &= \Lambda_n\overrightarrow{U}_n, \ (\textit{respectively, } \bar{\alpha}(x,y)\partial_y^2\overrightarrow{U}_n + \bar{\beta}(x,y)\partial_y\overrightarrow{U}_n = \Gamma_n\overrightarrow{U}_n), \end{aligned} \tag{3.26}$$

where $\alpha(x,y)$ *(respectively,* $\bar{\alpha}(x,y)$*) is a polynomial of degree* ≤ 2, $\beta(x,y)$ *(respectively,* $\bar{\beta}(x,y)$*) is a polynomial of degree* ≤ 1, Λ_n *(respectively,* Γ_n*) is a constant matrix of order* $(n+1)\times(n+1)$ *for* $n \geq 0$ *and* $\overrightarrow{U}_n = (U_{n0}, U_{n-1,1}, \ldots, U_{0n})^T$ *is a vector of polynomials* $n \geq 0$.

(iii) $\{\mathbb{P}_n^x(x,y)\}_{n=0}^{\infty}$ *(respectively,* $\{\mathbb{P}_n^y(x,y)\}_{n=0}^{\infty}$*) is a monic OB, where* $\mathbb{P}_n^x(x,y) = \{\frac{\partial_x P_{n+1-k,k}}{n+1-k}\}_{k=0}^{n}$ *and* $\mathbb{P}_n^y(x,y) = \{\frac{\partial_y P_{n-k,k+1}}{k+1}\}_{k=0}^{n}$.

Although we know, by theorem 3.6, that partial derivatives of a certain OPS form an OB , it is difficult to find such an OPS explicitly if it does not arise as solutions of a partial differential equation. Note that an OPS in theorem 3.6 do not need to satisfy a partial differential equation.

If the differential equation (3.1) is admissible, its unique monic PS of solutions has the following interesting property.

THEOREM 3.7. [4] *Let* $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *be a monic PS satisfying the partial differential equation* (3.1). *If* $A_y = 0$ *(respectively,* $C_x = 0$*), then we have*

(i) $P_{n0}(x,y) = P_{n0}(x)$ (*respectively,* $P_{0n}(x,y) = P_{0n}(y)$), $n \geq 0$;

(ii) $\{P_{n0}(x,y)\}_{n=0}^{\infty}$ (*respectively,* $\{P_{0n}(x,y)\}_{n=0}^{\infty}$) *satisfy the ordinary differential equation*

$$A(x)P''_{n0}(x) + D(x)P'_{n0}(x) = \lambda_n P_{n0}(x)$$
$$(\text{respectively, } C(y)P''_{0n}(y) + E(y)P'_{0n}(y) = \lambda_n P_{0n}(y));$$

(iii) $\{P_{n0}(x)\}_{n=0}^{\infty}$ (*respectively,* $\{P_{0n}(y)\}_{n=0}^{\infty}$) *is a classical OPS if and only if*

$$A(-u_n/s_{2n}) \neq 0 \qquad (\text{respectively, } C(-v_n/s_{2n}) \neq 0),$$

where $s_n = an + g$, $u_n = d_1 n$ *and* $v_n = e_3 n$ *for each* $n \geq 0$.

EXAMPLE 3.1. [3][6] Let a monic PS $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ satisfy the partial differential equation

$$3yu_{xx} + 2u_{xy} - xu_x - yu_y + nu = 0. \tag{3.27}$$

Then $P_{0n}(x,y) = y^n$ by Theorem 3.7 since $C(x,y) = 0$.We can check that $\partial_x \mathbb{P}_{n+1}(x,y)$ spans a vector space of polynomials of degree n and satisfies the differential equation (3.27) but $\partial_y \mathbb{P}_{n+1}(x,y)$ does not. For example, we consider the differential equation (3.27) with $n = 2$. The monic polynomial solutions of degree 2 are given by

$$P_{2,0}(x,y) = x^2 - 6y, \quad P_{1,1}(x,y) = xy - 1, \quad p_{0,2}(x,y) = y^2.$$

Observe that $\partial_y \mathbb{P}_2(x,y) = \{-6, x, 2y\}$ does not satisfy the differential equation (3.27) with $n = 1$ and does not span a vector space of polynomials with degree 1 while $\partial_x \mathbb{P}_2(x,y) = \{2x, y, 0\}$ does. Note that $A_y \neq 0$.

Now we define a monic PS $\{\mathbb{P}_n^{(k,\ell)}(x,y)\}_{n=0}^{\infty}$ from a monic PS $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ as the following: for $k, \ell \geq 0$

$$\begin{aligned} P_{n-j,j}^{(k+1,\ell)}(x,y) &= \tfrac{1}{n-j+1}\tfrac{\partial}{\partial x}P_{n-j+1,j}^{(k,\ell)}(x,y) && (0 \leq j \leq n),\\ P_{n-j,j}^{(k,\ell+1)}(x,y) &= \tfrac{1}{j}\tfrac{\partial}{\partial y}P_{n-j+1,j}^{(k,\ell)}(x,y) && (1 \leq j \leq n+1),\\ P_{n-j,j}^{(0,0)}(x,y) &= P_{n-j,j}(x,y), && (0 \leq j \leq n). \end{aligned} \tag{3.28}$$

THEOREM 3.8. *Let* $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *be the normalization of an OPS* $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ (*relative to* σ) *satisfying the partial differential equation* (3.1).

(i) *If* $C_x = 0$ *and there exists a polynomial* $f(x,y)$ *such*

$$Af_x + Bf_y = A_x f, \quad Bf_x + Cf_y = 2B_x f, \tag{3.29}$$

then $\{\mathbb{P}_n^{(i,0)}(x,y)\}_{n=0}^{\infty}$ *is an OB relative to* $f^i(x,y)\sigma$ *and satisfies the differential equation*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + (D + iA_x)u_x + (E + 2iB_x)u_y = \mu_n(i)u.$$

(ii) *If* $A_y = 0$ *and there is a polynomial* $g(x,y)$ *such that*

$$Ag_x + Bg_y = 2B_y g, \quad Bg_x + Cg_y = C_y g, \tag{3.30}$$

then $\{\mathbb{P}_n^{(0,j)}(x,y)\}_{n=0}^{\infty}$ *is an OB relative to* $g^j(x,y)\sigma$ *and satisfies the differential equation*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + (D + 2jB_y)u_x + (E + jC_y)u_y = \mu_n(j)u.$$

Proof. (*i*) : If a monic polynomial $u = P_{n-k,k}(x,y)(0 \le k \le n)$ satisfies the partial differential equation

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(y)u_{yy} + D(x)u_x + E(y)u_y = \lambda_n u, \tag{3.31}$$

then we can show that, by differentiating (3.31) i-times with respect to x, $u^{(i,0)} = \partial_x^i u$ satisfies the partial differential equation

$$\begin{aligned} &Au_{xx} + 2Bu_{xy} + Cu_{yy} + (D + iA_x)u_x + (E + 2B_x)u_y \\ &= \mu_n(i)u, \quad \mu_n(i) = \lambda_n + 2ai. \end{aligned} \tag{3.32}$$

On the other hand, $\tau = f^i(x,y)\sigma$ satisfies the moment equation for (3.32)

$$\begin{aligned} (A\tau)_x + (B\tau)_y &= (D + iA_x)\tau, \\ (B\tau)_x + (C\tau)_y &= (E + 2B_x)\tau, \end{aligned}$$

which implies that $\{\mathbb{P}_n^{(i,0)}(x,y)\}_{n=0}^{\infty}$ is an OB relative to $f^i(x,y)\sigma$.

(*ii*) : We omit the proof because it is proved in a quite similar way as (*i*). □

In order that we can discuss the orthogonality of partial derivatives of all order of a PS, we need to require that $\partial_x^i \partial_y^j \mathbb{P}_{n+i+j}(x,y)$ spans a vector space consisting of polynomials with degree n only, where $\{\mathbb{P}_n(\mathbf{x})\}_{n=0}^{\infty}$ is the normalization of a PS $\{\Phi_n(x,y)\}_{n=0}^{\infty}$. Theorem 3.7 tells that this is true if $A_y = C_x = 0$. The next theorem shows that a monic PS $\{\mathbb{P}_n^{(k,\ell)}(x,y)\}_{n=0}^{\infty}$ defined by (3.28) is an OB and satisfies a certain partial differential equation if an OB $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ satisfies a partial differential equation (3.4) in the basic class.

THEOREM 3.9. *Suppose that an OPS* $\{\Phi_n(x,y)\}_{n=0}^{\infty}$ *relative to* σ *satisfies the partial differential equation* (3.4) *in the basic class. Let* $\{\mathbb{P}_n(x,y)\}_{n=0}^{\infty}$ *be the normalization of* $\{\Phi_n(x,y)\}_{n=0}^{\infty}$. *If a polynomial*

$f(x,y)$ satisfies the equations (3.29), and a polynomial $g(x,y)$ the equations (3.30), then $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ satisfy the partial differential equation
(3.33)
$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + (D + iA_x + 2jB_y)u_x + (E + 2iB_x + jC_y)u_y = \mu_n(i;j)u$$
and are orthogonal relative to the moment functional
$$\sigma^{(i,j)} = f^i(x,y)g^j(x,y)\sigma.$$

Proof. We can easily prove that $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ satisfy the partial differential equation (3.33) by differentiating (3.4) i-times with respect to x and j-times with respect to y. Let $f(x,y)$ and $g(x,y)$ be polynomials satisfying the equations (3.29) and (3.30). We claim that $\sigma^{(i,j)} = f^i(x,y)g^j(x,y)\sigma$ satisfies the moment equations for the partial differential equation (3.33)

$$\text{(3.34)} \qquad \begin{cases} (A\tau)_x + (B\tau)_y = (D + iA_x + 2jB_y)\tau, \\ (B\tau)_x + (C\tau)_y = (E + 2iB_x + jC_y)\tau. \end{cases}$$

It is quite easy to prove our claim since
$$\begin{aligned}
&(A\sigma^{(i,j)})_x + (B\sigma^{(i,j)})_y \\
&= [f^i g^j (A\sigma)_x + (B\sigma)_y] + if^{i-1}g^j [Af_x + Bf_y]\sigma + jf^i g^{j-1}[Ag_x + Bg_y]\sigma \\
&= (D + iA_x + 2jB_y)f^i g^j \sigma \\
&= (D + iA_x + 2jB_y)\sigma^{(i,j)}.
\end{aligned}$$

Similarly, $\sigma^{(i,j)} = f^i(x,y)g^j(x,y)\sigma$ satisfies
$$(B\sigma^{(i,j)})_x + (C\sigma^{(i,j)})_y = (E + 2iB_x + jC_y)\sigma^{(i,j)}.$$

Thus if τ is the canonical moment functional of $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$, then $\sigma^{(i,j)} = \tau$ by the uniqueness of solution to (3.34) by Theorem 3.1. □

To complete the discussion on the orthogonality of partial derivatives, we should be able to show that for partial differential equations in the basic we identified in the previous subsection, the equations (3.29) and (3.30) have nonzero polynomial solutions. Several important partial differential equations are considered in next examples. Other partial differential equations in the basic class are left to the reader.

EXAMPLE 3.2. Consider the partial differential equation of the form
$$(ax^2 + f_1)u_{xx} + 2axyu_{xy} + (ay^2 + f_3)u_{yy} + gxu_x + gyu_y = \lambda u.$$

We have

$$f(x,y) = g(x,y) = f_1f_3 + af_3x^2 + af_1y^2$$

and $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional

$$\sigma^{(i,j)} = (f_1f_3 + af_3x^2 + af_1y^2)^{i+j}\sigma.$$

EXAMPLE 3.3. Consider the partial differential equation

$$(d_1x + f_1)u_{xx} + (e_3y + f_3)u_{yy} + gx + u_x + gyu_y = \lambda u.$$

In this case, we have

$$f(x,y) = d_1x + f_1, \quad g(x,y) = e_3y + f_3$$

and $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional

$$\sigma^{(i,j)} = (d_1x + f_1)^i(e_3y + f_3)^j\sigma.$$

EXAMPLE 3.4. Consider the partial differential equation

$$(x^2 - x)u_{xx} + 2xyu_{xy} + (y^2 - y)u_{yy} + [(\alpha+\beta+\gamma+3)x - (\alpha+1)]u_x$$
$$+[(\alpha+\beta+\gamma+3)y - (\beta+1)]u_y = \lambda_n u.$$

Solving the equations (3.29) and (3.30), we have

$$f(x,y) = x(1-x-y), \quad g(x,y) = y(1-x-y)$$

and $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional

$$\sigma^{(i,j)} = x^iy^j(1-x-y)^{i+j}\sigma.$$

EXAMPLE 3.5. Consider the partial differential equation

$$x^2u_{xx} + 2xyu_{xy} + (y^2 - y)u_{yy} + (gx - g)u_x + (gy - g\gamma)u_y = \lambda_n y$$

The equations (3.29) and (3.30) have polynomials $f(x,y) = x^2$ and $g(x,y) = xy$ as solutions, respectively. Thus $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional

$$\sigma^{(i,j)} = x^{2i}y^j\sigma.$$

EXAMPLE 3.6. Consider the partial differential equation

$$xu_{xx} + 2yu_{xy} + yu_{yy} + (gx + \alpha + \beta + 2)u_x + (gy + \beta + 1)u_y = \lambda u.$$

Solving the moment equations, we have $\sigma = e^{gx}(x-y)^{\alpha}y^{\beta}$. We see that

$$f(x,y) = x - y, \quad g(x,y) = y(x-y)$$

and $\{\mathbb{P}_n^{(i,j)}(x,y)\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional

$$\sigma^{(i,j)} = (x-y)^{i+j}y^j\sigma.$$

EXAMPLE 3.7. Consider the partial differential equation

$$xu_{xx} + 2yu_{xy} + (gx + \alpha + 2)u_x + (gy + \beta)u_y = \lambda u.$$

We have

$$f(x, y) = y, \quad g(x, y) = y^2.$$

and $\{\mathbb{P}_n^{(i,j)}(x, y)\}_{n=0}^{\infty}$ is orthogonal relative to a moment functional $\sigma^{(i,j)} = y^{i+2j}\sigma$.

4. Orthogonal polynomials in d-variables and second order partial differential equations in the basic class

In this section, we investigate the relation between orthogonal polynomials in d-variables and second order partial differential equations of the form

$$L[u] = \sum_{i,j=1}^{d} A_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i(\mathbf{x}) \frac{\partial u}{\partial x_i} = \lambda u, \tag{4.1}$$

where $\mathbf{x} = (x_1, \ldots, x_d)$ and $A_{ij}(\mathbf{x}) = A_{ji}(\mathbf{x})$ for all $1 \le i, j \le d$ and λ is the eigenvalue parameter. We say that the partial differential equation (4.1) is admissible if there is a sequence $\{\lambda_n | n \ge 0\}$ of real numbers such that for $\lambda = \lambda_n$ there are precisely $\binom{n+d-1}{n}$ linearly independent polynomial solutions of degree n.

It is not difficult to see that if the partial differential equation (4.1) is admissible if and only if (i) it has the form

$$\begin{aligned} A_{ij}(\mathbf{x}) &= ax_i x_i + \textstyle\sum_{k=1}^{d} d_k^{ij} x_k + f_{ij}, \\ B_i(\mathbf{x}) &= gx_i + h_i, \\ \lambda_n &= an(n-1) + gn \end{aligned}$$

and (ii) $\lambda_m \ne \lambda_n$ for $m \ne n$. The partial differential equation (4.1) has a unique monic PS as solutions if and only if it is admissible.

THEOREM 4.1. *Let σ be the canonical moment functional of a PS $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$. If $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ satisfies the partial differential equation (4.1), then σ satisfies the equation*

$$L^*[\sigma] = \sum_{i,j=1}^{d} (A_{ij}(\mathbf{x})\sigma)_{x_i, x_j} - \sum_{i=1}^{d} (B_i(\mathbf{x})\sigma)_{x_i} = 0, \tag{4.2}$$

where $L^*[\cdot]$ *is the Lagrangian adjoint of* $L[\cdot]$ *defined by*

$$L^*[u] = \sum_{i,j=1}^{d} (A_{ij}(\mathbf{x})u)_{x_i,x_j} - \sum_{i=1}^{d} (B_i(\mathbf{x})u)_{x_i}.$$

Furthermore, (4.2) *has a unique solution up to a multiplication constant if the partial differential equation* (4.1) *is admissible.*

THEOREM 4.2. *Let* $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ *be an OPS relative to a moment functional* σ. *Then the followings are equivalent.*

(i) $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ *satisfies the partial differential equation* (4.1).

(ii) $\sigma L[\cdot]$ *is formally symmetric on polynomials in the sense that*

$$\langle L[p]\sigma, q\rangle = \langle L]q]\sigma, p\rangle \qquad \text{for all } p, q \in \Pi^d. \tag{4.3}$$

(iii) σ *satisfies the following equations*

$$M_i[\sigma] = \sum\nolimits_{j=1}^{d} (A_{ij}(\mathbf{x})\sigma)_{x_i} - B_i(\mathbf{x})\sigma = 0 \text{ for } i = 1, \ldots, d. \tag{4.4}$$

We call $M_i[\sigma] = 0(1 \le i \le d)$ the moment equations for the partial differential equation (4.1).

THEOREM 4.3. *Suppose that the admissible partial differential equation* (4.1) *is in the basic class. Let* $\{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}$ *be an OPS relative to* σ. *Then for each* $\alpha \in \mathbb{N}_0^d$, $\{\mathbb{P}_n^{(\alpha)}(\mathbf{x})\}_{n=0}^{\infty}$ *is an OB relative to a moment functional*

$$\sigma^{(\alpha)} = f_1^{\alpha_1}(\mathbf{x}) \cdots f_d^{\alpha_d}(\mathbf{x})\sigma,$$

if there are polynomials $f_i(\mathbf{x})$ $(i = 1, \ldots, d)$ *such that*

$$\mathbf{A}(\nabla f_i)^T = \mathbf{B}_i, \qquad (i = 1, \ldots, d), \tag{4.5}$$

where $\mathbf{A} = (A_{ij}(\mathbf{x}))_{i,j=1}^{d}, \mathbf{B}_i(\mathbf{x})$ $(j = 1, \ldots, d)$ *is a column vector with components*

$$(\mathbf{B}_i(\mathbf{x}))_j = (2 - \delta_{ij})\frac{\partial A_{ij}}{\partial x_i}, \qquad (j = 1, \ldots, d)$$

and $\{\mathbb{P}_n^{(\alpha)}(\mathbf{x})\}_{n=0}^{\infty}$ *is a PS defined by*

$$\begin{cases} P_{\beta-\mathbf{e}_i}^{(\alpha+\mathbf{e}_i)}(\mathbf{x}) = \frac{\partial}{\partial x_i} P_{\beta}^{(\alpha)}(\mathbf{x}), & |\beta| = n+1, \beta_i \neq 0, \\ P_{\beta}^{(\mathbf{0})}(\mathbf{x}) = P_{\beta}(\mathbf{x}) & : \text{normalization of } \{\Phi_n(\mathbf{x})\}_{n=0}^{\infty}. \end{cases}$$

Now we give examples of orthogonal polynomials in d-variables satisfying a partial differential equation of the form (4.1) which is in the basic class.

EXAMPLE 4.1. [13] Consider the partial differential equation

$$\sum_{i=1}^{d}(x_i^2 - x_i)\frac{\partial^2 u}{\partial x_i^2} + \sum_{i,j=1,i\neq j}^{d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\left(\left(\sum_{j=1}^{d}\gamma_j + \gamma + d + 1\right)x_i - (\gamma_i + 1)\right)\frac{\partial u}{\partial x_i} = \lambda_n u. \tag{4.6}$$

It is obvious that the partial differential equation (4.6) is in the basic class. By Theorem 4.2, the partial differential equation (4.6) has orthogonal polynomial solutions (which will be called the d-dimensional simplex polynomials) since the weight function

$$w(x_1, \ldots, x_d) = x_1^{\gamma_1} \cdots x_d^{\gamma_d}(1 - x_1 - \cdots - x_d)^{\gamma} \qquad (\gamma_i > -1, \gamma > -1)$$

satisfies the moment equation (4.4) in the classical sense on the simplex

$$\{(x_1, \ldots, x_d) | x_i \geq 0, \textstyle\sum_{i=1}^{d} x_i \leq 1\}.$$

Furthermore, (4.6) has an OPS as solutions if

$$\gamma_i + 1, \gamma + 1 \neq 0, -1, \ldots \ (1 \leq i \leq d),$$
$$\gamma + \Sigma_{i=k}^{d}\gamma_i + d - k + 2 \neq 0, -1, \ldots \ (1 \leq k \leq d).$$

Also we see that $\{\mathbb{P}_n^{(\alpha)}(\mathbf{x})\}_{n=0}^{\infty}$ are orthogonal relative to the moment functional

$$\sigma^{(\alpha)} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}(1 - x_1 - \cdots - x_d)^{|\alpha|}\sigma.$$

since $f_i(\mathbf{x}) = x_i(1 - x_1 - \cdots - x_d)$ $(i = 1, \ldots, d)$ satisfies the equations (4.5).

EXAMPLE 4.2. [13] Consider the partial differential equation

$$\sum_{i=1}^{d}(x_i^2 - \varepsilon_i)\frac{\partial^2 u}{\partial x_i^2} + \sum_{i,j=1,i\neq j}^{d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} g x_i \frac{\partial u}{\partial x_i} = \lambda_n u \quad (\varepsilon_i = \pm 1). \tag{4.7}$$

It is obvious that the partial differential equation (4.7) is in the basic class. In the case that $\varepsilon_i = 1$ $(1 \leq i \leq d)$, the weight function

$$w(x_1, \ldots, x_d) = \left(1 - x_1^2 - \cdots - x_d^2\right)^{\frac{g-d-1}{2}} \qquad (g > d - 1)$$

satisfies the moment equation (4.4) in the classical sense on the d-dimensional sphere. Thus the partial differential equation (4.7) has orthogonal polynomial solutions (which will be called the d-dimensional sphere polynomials) by Theorem 4.2. Solving the equations (4.5) for f_i $(i = 1, \ldots, d)$, then we have

$$f_i(\mathbf{x}) = 1 - x_1^2 - \cdots - x_d^2 \quad (i = 1, \ldots, d).$$

We see that $\{\mathbb{P}_n^{(\alpha)}(\mathbf{x})\}_{n=0}^{\infty}$ are orthogonal relative to the moment functional

$$\sigma^{(\alpha)} = (1 - x_1^2 - \cdots - x_d^2)^{|\alpha|}\sigma.$$

Generally, if either $\varepsilon_i = -1$ for some i, or $g \neq d-1, d-2, d-3, \ldots$, then the partial differential equation (4.7) has an OPS (not necessarily a positive-definite OPS) as solutions. It is not hard to see that $\{\mathbb{P}_n^{(\alpha)}(\mathbf{x})\}_{n=0}^{\infty}$ are orthogonal relative to the moment functional

$$\sigma^{(\alpha)} = \left(1 - \sum_{i=1}^{d} \varepsilon_i x_i^2\right)^{|\alpha|} \sigma$$

since $f_i(\mathbf{x}) = 1 - \sum_{i=1}^{d} \varepsilon_i x_i^2$ $(i = 1, \ldots, d)$ satisfies the equations (4.5).

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