J. Korean Math. Soc. **41** (2004), No. 6, pp. 1023–1034

LIMIT THEOREMS FOR PARTIAL SUM PROCESSES OF A GAUSSIAN SEQUENCE

YONG-KAB CHOI, KYO-SHIN HWANG,
HEE-JIN MOON, TAE-SUNG KIM AND JONG-IL BAEK

ABSTRACT. In this paper we establish limsup and liminf theorems for the increments of partial sum processes of a dependent stationary Gaussian sequence.

1. Introduction and results

Let $\{X_j;\ j = 1, 2, \dots\}$ be a sequence of independent identically distributed (i.i.d) random variables and let $\mathbf{S}_0 = 0$ and $\mathbf{S}_n = \sum_{j=1}^{n} X_j$. For an integer sequence $\{a_n;\ n = 1, 2, \dots\}$ with $1 \le a_n \le n$, put

$$U_n = \max_{1 \le k \le n - a_n} (\mathbf{S}_{k+a_n} - \mathbf{S}_k).$$

Csörgő and Révész [6] obtained the following strong limit law

$$\lim_{n\to\infty} \frac{U_n}{b_n} = 1 \qquad \text{a.s.} \tag{1.1}$$

under some conditions of $\{X_j\}$ and $\{a_n\}$, where $\{b_n;\ n = 1, 2, \dots\}$ is some sequence of constants. For further various results on this limit law (1.1) about the sequence of i.i.d. random variables, we refer to ([5], [7], [8], [9], [11], [19], [20], [21]).

On the other hand, Lin ([15], [16], [18]) established large increment results for a sequence of independent or mixing dependent random variables. Theoretically and practically, strong dependent sequences are

Received September 3, 2003.
2000 Mathematics Subject Classification: 60F10, 60F15, 60G15.
Key words and phrases: Gaussian sequence, large deviation probability, regularly varying function.
This work was supported by KRF Grant (2002-042-D00008).

important and interesting. Usually one considers the case of Gaussian sequences.

Horvàth and Shao [10] studied extreme value limit distributions for the maximum of partial sums of a stationary Gaussian sequence with long-range dependence.

Recently, Csáki and Gonchigdanzan [4] investigated almost sure central limit theorems for the maximum of dependent stationary Gaussian sequences.

In this paper we are interested in the strong limit law types as in (1.1) about partial sum processes of a dependent stationary Gaussian sequence. Let $\{\xi_j; j = 1, 2, \dots\}$ be a centered stationary Gaussian sequence with $E\xi_1^2 = 1$ and $\rho_n = E(\xi_1 \xi_{1+n})$, $n \geq 1$. Put $S_0 = 0$, $S_n = \sum_{j=1}^n \xi_j$ and $\sigma(n) = \sqrt{ES_n^2}$. Assume that $\sigma(n)$ can be extended to a continuous function $\sigma(t)$ of $t > 0$ which is nondecreasing and regularly varying with exponent α for some $0 < \alpha < 1$. Suppose that $\{a_n; n \geq 1\}$ is a sequence of positive integers such that

(i) $1 \leq a_n \leq n$.

Denote $\beta_n = \{2(\log(n/a_n) + \log\log n)\}^{1/2}$ for $n > e$.

Recently, Choi et al. [3] proved the following Theorems A and B.

THEOREM A. *Suppose that the sequence $\{a_n;\ n \geq 1\}$ satisfies conditions* (i) *and*

(ii) $\limsup_{n\to\infty} a_n/n =: \mu < 1$,

(iii) *there exist $0 < \mu_2 \leq \mu_1 \leq 1$ such that, for any $m < n$, we have $\mu_1 a_m \leq a_n$ and $\mu_2 a_m/m \geq a_n/n$.*

Assume that, for $n \geq 1$, either

(iv) $\rho_n \leq 0$

or

(v) $|\rho_n| \leq \sigma^2(n)/n^2$.

Then we have

$$(1.2)\qquad \begin{aligned} &\limsup_{n\to\infty} \sup_{0\leq i\leq n} \sup_{1\leq j\leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} = 1 \qquad a.s., \\ &\limsup_{n\to\infty} \frac{|S_{n+a_n} - S_n|}{\sigma(a_n)\beta_n} = 1 \qquad a.s. \end{aligned}$$

Next, consider the case of a limit result.

THEOREM B. *Suppose that the condition* (i) *and one of* (iv) *and* (v) *are satisfied. Further suppose that*

(vi) $$\lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = \infty.$$

Then we have

$$(1.3)\qquad \begin{aligned} &\lim_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} = 1 \qquad \text{a.s.,} \\ &\lim_{n\to\infty} \sup_{0\le i\le n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\beta_n} = 1 \qquad \text{a.s.} \end{aligned}$$

Note that the condition (v) implies that, for $n \geq 1$,

$$-n^{2\alpha-2}L(n) \leq \rho_n \leq n^{2\alpha-2}L(n),$$

where $L(n)$ is a slowly varying function.

For the Wiener process $\{W(t),\ 0 \leq t < \infty\}$ with independent increments, Book and Shore [1] proved that liminf results are different from limsup results if the following condition

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty$$

of Theorem 1.2.1 in [6] is replaced by

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = r, \qquad 0 \leq r < \infty.$$

On this point of view, the main objective of this paper is to show that liminf results are different from the results (1.2) and (1.3) for dependent Gaussian sequences if the condition (vi) is replaced by

$$\lim_{n\to\infty} \frac{\log(n/a_n)}{\log_\theta \log n} = r, \qquad 0 \leq r < \infty,$$

where $\theta = 1 + \varepsilon$ for $\varepsilon > 0$ small enough.

The main results are as follows:

THEOREM 1.1. *If the condition* (i) *and*

$$\text{(vii)} \qquad \lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = r, \qquad 0 \le r \le \infty$$

are satisfied, then we have

$$(1.4) \qquad \liminf_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} \le \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.}$$

The following theorem is straightforward from Theorem 1.1.

THEOREM 1.2. *If the condition* (i) *and*

$$\text{(vii)}' \qquad \lim_{n\to\infty} \frac{\log(n/a_n)}{\log_\theta \log n} = r, \qquad 0 \le r \le \infty$$

are satisfied, then we have

$$(1.5) \qquad \liminf_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta'_n} \le \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.,}$$

where $\beta'_n = \{2(\log(n/a_n) + \log_\theta \log n)\}^{1/2}$ *for* $n > e$.

THEOREM 1.3. *Suppose that conditions* (i), (vii)′ *and one of* (iv) *and* (v) *are satisfied. Then we have*

$$(1.6) \qquad \liminf_{n\to\infty} \sup_{0\le i\le n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\beta'_n} \ge \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.}$$

Combining Theorems 1.2 and 1.3, we obtain the following liminf result.

COROLLARY 1.1. *Under the assumptions of Theorem 1.3, we have*

$$(1.7) \qquad \begin{aligned} &\liminf_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta'_n} = \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.,} \\ &\liminf_{n\to\infty} \sup_{0\le i\le n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\beta'_n} = \left(\frac{r}{1+r}\right)^{1/2} \quad \text{a.s.} \end{aligned}$$

Note that if $r = \infty$ in (vii)′, then (1.3) follows from (1.6) and Theorem 1.1 in Choi et al. [3]; if $0 \le r < \infty$ in (vii)′, then (1.7) differs from (1.2) under conditions (ii), (iii) and (vii)′.

2. Proofs of main theorems

The following Lemmas 2.1 and 2.2 are used for the proof of Theorem 1.1, and Lemma 2.1 is an analogue of Lemma 2.2 in [3] (See also Lemma 2.2 in [17]).

LEMMA 2.1. *For any $\varepsilon > 0$, there exists a positive constant c_ε such that*

$$P\Big\{\sup_{0\le i\le n}\sup_{1\le j\le a_n}\frac{|S_{i+j}-S_i|}{\sigma(a_n)}\ge u\Big\}\le c_\varepsilon\frac{n}{a_n}\exp\Big(-\frac{u^2}{2+\varepsilon}\Big)$$

for all $u > 1$.

The next Lemma 2.2 is obvious.

LEMMA 2.2. *Let $\{\xi, \xi_n; n \ge 1\}$ be a sequence of random variables. If*

$$P\{\xi_n > \xi\}\to 0 \quad as \quad n\to\infty,$$

then there is a subsequence $\{\xi_{n_k}\}$ such that

$$\limsup_{k\to\infty}\xi_{n_k}\le\xi \qquad a.s.$$

So we have

$$\liminf_{n\to\infty}\ \xi_n\le\xi \qquad \text{a.s.}$$

Proof of Theorem 1.1. First, suppose that $0 < r \le \infty$. From (vii), there exists $\gamma > 0$ such that $n/a_n \ge (\log n)^\gamma$ for sufficiently large n. Thus by Lemma 2.1 we have, for any $\varepsilon > 0$,

$$\begin{aligned}&P\Big\{\sup_{0\le i\le n}\sup_{1\le j\le a_n}\frac{|S_{i+j}-S_i|}{\sigma(a_n)\{2\log(n/a_n)\}^{1/2}}>\sqrt{1+\varepsilon}\Big\}\\&\le c_\varepsilon\frac{n}{a_n}\exp\Big(-\frac{2+2\varepsilon}{2+\varepsilon}\log\frac{n}{a_n}\Big)\\&\le c_\varepsilon(\log n)^{-\gamma\varepsilon/(2+\varepsilon)}\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

It follows from Lemma 2.2 that

$$\liminf_{n\to\infty}\sup_{0\le i\le n}\sup_{1\le j\le a_n}\frac{|S_{i+j}-S_i|}{\sigma(a_n)\{2\log(n/a_n)\}^{1/2}}\le 1 \qquad \text{a.s.}$$

Hence by (vii) we obtain

$$
\begin{aligned}
&\liminf_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j}-S_i|}{\sigma(a_n)\beta_n} \\
&= \liminf_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j}-S_i|}{\sigma(a_n)\{2\log(n/a_n)\}^{1/2}} \\
(2.1)\qquad &\times \Big(\frac{2\log(n/a_n)}{2(\log(n/a_n)+\log\log n)}\Big)^{1/2} \\
&\le \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}
\end{aligned}
$$

On the other hand, consider the case $r = 0$. It follows from (vii) that for any small $\varepsilon > 0$ we have

$$\frac{n}{a_n} < (\log n)^{\varepsilon/(2+\varepsilon)}$$

for n sufficiently large. Applying Lemma 2.1 again, we get

$$
\begin{aligned}
&P\Big\{\sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j}-S_i|}{\sigma(a_n)\beta_n} > \sqrt{\varepsilon}\Big\} \\
&\le c_\varepsilon(\log n)^{-\varepsilon/(2+\varepsilon)} \to 0 \quad \text{as} \quad n\to\infty
\end{aligned}
$$

and hence Lemma 2.2 gives

$$(2.2)\qquad \liminf_{n\to\infty} \sup_{0\le i\le n} \sup_{1\le j\le a_n} \frac{|S_{i+j}-S_i|}{\sigma(a_n)\beta_n} \le 0 \qquad \text{a.s.}$$

Combining (2.1) with (2.2) completes the proof of Theorem 1.1. □

The following Lemmas 2.3-2.5 are essential to prove Theorem 1.3.

LEMMA 2.3. (cf. Corollary 1.2.2 in [14]) *Let $\xi = (\xi_{ij})$ and $\eta = (\eta_{ij})$, $1 \le i \le n$, $1 \le j \le m$, be centered Gaussian random vectors such that*

$$
\begin{aligned}
&E(\xi_{ij}^2) = E(\eta_{ij}^2) \quad \text{for all} \quad i, j, \\
&E(\xi_{ij}\xi_{ik}) \le E(\eta_{ij}\eta_{ik}) \quad \text{for all} \quad i, j, k, \\
&E(\xi_{ij}\xi_{lk}) \ge E(\eta_{ij}\eta_{lk}) \quad \text{for all} \quad i \ne l, \ j \text{ and } k.
\end{aligned}
$$

Then, for all real numbers λ_{ij},

$$P\Big\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(\eta_{ij} > \lambda_{ij})\Big\} \le P\Big\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(\xi_{ij} > \lambda_{ij})\Big\}.$$

LEMMA 2.4. ([12], [13]) *Let ξ_j $(j = 1, 2, \ldots, n)$ be standardized normal random variables with* $\mathrm{Cov}(\xi_i, \xi_j) = \Lambda_{ij}$ *such that* $\delta = \max_{i\neq j} |\Lambda_{ij}| < 1$. *Then for any real number u and integers $1 \le l_1 < l_2 < \cdots < l_k \le n$ with $k \le n$, we have*

$$(2.3)\quad P\Big\{ \max_{1\le j\le k} \xi_{l_j} \le u \Big\} \le (\Phi(u))^k + c \sum_{1\le i<j\le k} |\rho_{ij}| \exp\Big(- \frac{u^2}{1+|\rho_{ij}|} \Big),$$

where $\rho_{ij} = \Lambda_{l_i l_j}$ and $c = c(\delta)$ is a constant independent of n and u, and $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\, dy$.

Under the stationary condition on ρ_{ij}, we can estimate an upper bound for the second term of the right hand side of (2.3) as follows:

LEMMA 2.5. ([2]) *Let ξ_j, δ, k and ρ_{ij} be as in Lemma 2.4. Assume that for some $\nu > 0$*

$$|\rho_{ij}| < |i-j|^{-\nu} \quad \text{for all} \quad i \ne j.$$

Put $u = \{(2-\eta)\log k\}^{1/2}$, where $0 < \eta < (1-\delta)\nu/(1+\nu+\delta)$. Then we have

$$\sum := \sum_{1\le i<j\le k} |\rho_{ij}| \exp\Big(- \frac{u^2}{1+|\rho_{ij}|} \Big) \le c\, k^{-\delta_0},$$

where $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$ and c is a constant independent of n and u.

Proof of Theorem 1.3. (1.6) is obvious when $r = 0$. In what follows, we assume that $0 < r \le \infty$. For $\theta > 1$, let

$$A_{k,l} = \{n : \theta^{k-1} \le n \le \theta^k, \theta^{l-1} \le a_n \le \theta^l\},$$

where $k = 1, 2, \ldots$; $l = 1, 2, \ldots$. The condition (vii)$'$ implies that, for sufficiently large k, there exists $\gamma > 0$ such that

$$1 \le l \le k + 1 - \gamma \log((k-1)\log\theta)/(\log\theta)^2 =: K$$

and there exists $M > 0$ such that

$$\theta(k,l) := \big[\theta^{k-l-1}/M\big] > 1.$$

Noting that

$$\lim_{n\to\infty} \frac{\sqrt{2\log(n/a_n)}}{\beta_n'} = \begin{cases} (r/(1+r))^{1/2} & \text{if } 0 < r < \infty, \\ 1 & \text{if } r = \infty \end{cases}$$

by (vii)′, then (1.6) is proved if we show that

$$\liminf_{n\to\infty} \sup_{0\le i\le n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\sqrt{2\log(n/a_n)}} \ge 1 \qquad \text{a.s.} \tag{2.4}$$

By the regular variation of $\sigma(\cdot)$, we have

$$\sigma(\theta^{l-1}) \ge (\theta-1)^{-\alpha}\sigma(\theta^l - \theta^{l-1})$$

for some $0 < \alpha < 1$. Thus

$$\begin{aligned} &\liminf_{n\to\infty} \sup_{0\le i\le n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\sqrt{2\log(n/a_n)}} \\ &\ge \liminf_{k\to\infty} \inf_{1\le l\le K} \inf_{n\in A_{k,l}} \sup_{0\le i\le n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\sqrt{2\log(n/a_n)}} \\ &\ge \liminf_{k\to\infty} \inf_{1\le l\le K} \sup_{0\le i\le \theta^{k-1}} \frac{|S_{i+\theta^l} - S_i|}{\sigma(\theta^l)\sqrt{2\log\theta^{k-l}}} \\ &\quad - \limsup_{k\to\infty} \inf_{1\le l\le K} \sup_{0\le i\le\theta^k} \sup_{\theta^{l-1}\le j\le\theta^l} \frac{(\theta-1)^\alpha|S_{i+j} - S_{i+\theta^l}|}{\sigma(\theta^l-\theta^{l-1})\sqrt{2\log\theta^{k-l}}} \\ &=: J_1 - J_2. \end{aligned} \tag{2.5}$$

First, we will show that, for any small $\varepsilon > 0$,

$$J_2 \le \varepsilon \qquad \text{a.s.} \tag{2.6}$$

We claim that, for some $R > 2$,

$$\limsup_{k\to\infty} \sup_{1\le l\le K} \sup_{0\le i\le\theta^k} \sup_{\theta^{l-1}\le j\le\theta^l} \frac{|S_{i+j} - S_{i+\theta^l}|}{\sigma(\theta^l-\theta^{l-1})\sqrt{2\log\theta^{k-l}}} \le R \qquad \text{a.s.} \tag{2.7}$$

By the same way as the proof of Lemma 2.1, we can obtain

$$P\Big\{ \sup_{0\le i\le\theta^k} \sup_{\theta^{l-1}\le j\le\theta^l} \frac{|S_{i+j} - S_{i+\theta^l}|}{\sigma(\theta^l-\theta^{l-1})} > u \Big\} \le c_\varepsilon \theta^{k-l} e^{-u^2/(2+\varepsilon)}$$

for all $u > 1$. Thus,

$$P\Big\{\sup_{1\le l\le K}\sup_{0\le i\le\theta^k}\sup_{\theta^{l-1}\le j\le\theta^l}\frac{|S_{i+j}-S_{i+\theta^l}|}{\sigma(\theta^l-\theta^{l-1})} > R\sqrt{2\log\theta^{k-l}}\Big\}$$
$$\le c_\varepsilon\sum_{l=1}^{K}\theta^{k-l}\exp\Big(-\frac{8}{2+\varepsilon}\log\theta^{k-l}\Big)$$
$$\le c_\varepsilon\sum_{l=1}^{K}\theta^{-2(k-l)}\le c\,k^{-\gamma/\log\theta}.$$

Since $\gamma/\log\theta > 1$, the Borel-Cantelli lemma implies (2.7), and thus (2.6) follows if $\theta\to 1$.

Next, consider J_1. For $0\le m\le\theta(k,l)$, let

$$S(m) = S_{m\theta^l M+\theta^l} - S_{m\theta^l M}.$$

It follows from (vii)′ that, for any $0<\varepsilon<1$,

$$\begin{aligned}P\Big\{\sup_{0\le i\le\theta^{k-1}}\frac{S_{i+\theta^l}-S_i}{\sigma(\theta^l)\sqrt{2\log\theta^{k-l}}}\le\sqrt{1-\varepsilon}\Big\}\\ \le P\Big\{\max_{0\le m\le\theta(k,l)}\frac{S(m)}{\sigma(\theta^l)}\le\sqrt{1-\varepsilon}\sqrt{2\log\theta(k,l)}\Big\}.\end{aligned}\tag{2.8}$$

Assume that (iv) holds. By Lemma 2.3, we have

$$\begin{aligned}P\Big\{\max_{0\le m\le\theta(k,l)}\frac{S(m)}{\sigma(\theta^l)}\le\{(2-2\varepsilon)\log\theta(k,l)\}^{1/2}\Big\}\\ \le\big(\Phi(\{(2-2\varepsilon)\log\theta(k,l)\}^{1/2})\big)^{\theta(k,l)}\le\exp\big(-c(\theta^{k-l})^{\varepsilon}\big),\end{aligned}\tag{2.9}$$

where c is a positive constant. Hence by (2.8) and (2.9), we have

$$P\Big\{\inf_{1\le l\le K}\sup_{0\le i\le\theta^{k-1}}\frac{S_{i+\theta^l}-S_i}{\sigma(\theta^l)\sqrt{2\log\theta^{k-l}}}\le\sqrt{1-\varepsilon}\Big\}$$
$$\le\sum_{l=1}^{K}\exp\big(-c(\theta^{k-l})^{\varepsilon}\big)\le\exp\big(-c\,k^{\varepsilon\gamma/\log\theta}\big)$$

for all large k. It follows from the Borel-Cantelli lemma that

$$J_1\ge 1\qquad\text{a.s.}\tag{2.10}$$

Consider the case when (v) holds. In this case, we can estimate an upper bound of the right hand side of (2.8) by Lemmas 2.4 and 2.5. Define

$$r(m,m') = Cov\left(\frac{S(m)}{\sigma(\theta^l)}, \frac{S(m')}{\sigma(\theta^l)}\right), \quad m > m' = 0,1,\dots,\theta(k,l)$$

and let $q = m - m' \geq 1$. Then by (v) we have

$$\begin{aligned}
&|r(m,m')| \\
&= \frac{1}{\sigma^2(\theta^l)}\Big|E\Big\{\xi_{m\theta^l M+1}\xi_{m'\theta^l M+1} + \cdots + \xi_{m\theta^l M+1}\xi_{m'\theta^l M+\theta^l} \\
&\quad + \cdots + \xi_{m\theta^l M+\theta^l}\xi_{m'\theta^l M+1} + \cdots + \xi_{m\theta^l M+\theta^l}\xi_{m'\theta^l M+\theta^l}\Big\}\Big| \\
&= \frac{1}{\sigma^2(\theta^l)}\big|\rho_{q\theta^l M} + \cdots + \rho_{q\theta^l M+1-\theta^l} + \cdots + \rho_{q\theta^l M+\theta^l-1} + \cdots + \rho_{q\theta^l M}\big| \\
&\leq \frac{(\theta^l)^2}{\sigma^2(\theta^l)}\big|\rho_{q\theta^l M+1-\theta^l}\big| \leq \frac{\theta^{2l}}{\sigma^2(\theta^l)}\frac{\sigma^2(q\theta^l M+1-\theta^l)}{(q\theta^l M+1-\theta^l)^2} \\
&\leq \frac{1}{(qM-1)^2}\frac{\sigma^2((qM-1)\theta^l)}{\sigma^2(\theta^l)} \leq c(qM-1)^{2\alpha-2} < q^{-\nu},
\end{aligned}$$

where $\nu = 1 - \alpha > 0$. Applying Lemmas 2.4 and 2.5 for

$$\begin{aligned}
&\xi_{lj} = \frac{S(m)}{\sigma(\theta^l)}, \quad m = 0,1,\dots,\theta(k,l), \\
&u = \big\{(2-\eta)\log\theta(k,l)\big\}^{1/2}, \quad \eta = 2\varepsilon, \\
&|\rho_{ij}| = |r(m,m')| < |m-m'|^{-\nu}, \quad m \neq m',
\end{aligned}$$

then the right hand side of (2.8) is less than or equal to

$$\big(\Phi(u)\big)^{\theta(k,l)} + c\big(\theta(k,l)\big)^{-\delta_0}.$$

Thus we have

$$\begin{aligned}
&P\Big\{\sup_{0\leq i\leq\theta^{k-1}}\frac{S_{i+\theta^l}-S_i}{\sigma(\theta^l)\sqrt{2\log\theta^{k-l}}} \leq \sqrt{1-\varepsilon}\Big\} \\
&\leq \exp\big(-c(\theta^{k-l})^{\varepsilon}\big) + c(\theta^{k-l})^{-\delta_0} \leq c\,(\theta^{k-l})^{-\delta_0}
\end{aligned}$$

for all large k. Considering J_1 in (2.5), we have

$$P\Big\{\inf_{1\le l\le K}\sup_{0\le i\le \theta^{k}-1}\frac{S_{i+\theta^l}-S_i}{\sigma(\theta^l)\sqrt{2\log\theta^{k-l}}}\le\sqrt{1-\varepsilon}\Big\}$$
$$\le\sum_{l=1}^{K}c\,(\theta^{k-l})^{-\delta_0}\le c\,k^{-\gamma\delta_0/\log\theta}.$$

Thus the Borel-Cantelli lemma gives (2.10). From (2.5), (2.6) and (2.10), we obtain (2.4). This completes the proof of Theorem 1.3. □

ACKNOWLEDGEMENT. The authors wish to thank the referee for careful reading and helpful suggestion on this paper.

References

[1] S. A. Book and T. R. Shore, *On large intervals in the Csörgő-Révész theorem on increments of a Wiener process*, Z. Wahrsch. Verw. Gebiete **46** (1978), 1–11.

[2] Y. K. Choi, *Erdös-Rényi-type laws applied to Gaussian processes*, J. Math. Kyoto Univ. **31(1)** (1991), 191–217.

[3] Y. K. Choi, Z. Y. Lin, W. S. Wang, K. S. Hwang and T. S. Kim, *Asymptotic behaviors for partial sum processes of a Gaussian sequence*, Acta Math. Hungar. **103(1-2)** (2004), 43–54.

[4] E. Csáki and K. Gonchigdanzan, *Almost sure limit theorems for the maximum of stationary Gaussian sequences*, Statist. Probab. Lett. **58** (2002), 195–203.

[5] M. Csörgő, *A glimpse of Pál Erdös's impact on probability and statistics*, Canad. J. Statist. **30(4)** (2002).

[6] M. Csörgő and P. Révész, *Strong Approximations in Probability and Statistics*, Academic Press, New York, 1981.

[7] M. Csörgő and J. Steinebach, *Improved Erdös-Rényi and strong approximation laws for increments of partial sums*, Ann. Probab. **9(6)** (1981), 988–996.

[8] P. Deheuvels and J. Steinebach, *Exact convergence rates in strong approximation laws for large increments of partial sums*, Probab. Theory Related Fields **76** (1987), 369–393.

[9] A. N. Frolov, *On one-sided strong laws for large increments of sums*, Statist. Probab. Lett. **37** (1998), 155–165.

[10] L. Horvàth and Q. M. Shao, *Darling-Erdös-type theorems for sums of Gaussian variables with long-range dependence*, Stochastic Process. Appl. **63** (1996), 117–137.

[11] H. Lanzinger and U. Stadtmüller, *Maxima of increments of partial sums for certain subexponential distributions*, Stochastic Process. Appl. **86** (2000), 307–322.

[12] M. R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, New York, 1983.

[13] W. V. Li and Q. M. Shao, *A normal comparison inequality and its applications*, Probab. Theory Related Fields **122** (2002), 494–508.

[14] Z. Y. Lin, *Path properties of Gaussian processes*, Zhejiang University Press, 2001.

[15] ______, *On Csörgő-Révész's increments of sums of non-i.i.d. random variables*, Scientia Sinica (Series A) **30** (1987), no. 9, 921–931.

[16] ______, *On increments of sums of independent non-identically distributed random variables*, Scientia Sinica (Series A) **31** (1988), no. 8, 927–937.

[17] Z. Y. Lin, Y. K. Choi and K. S. Hwang, *Some limit theorems on the increments of a multi-parameter fractional Brownian motion*, Stochastic Anal. Appl. **19** (2001), no. 4, 499–517.

[18] Z. Y. Lin and Q. M. Shao, *On the limiting behaviors of increments of sums of random variables without moment conditions*, Chinese Ann. Math. **14B** (1993), no. 3, 307–318.

[19] J. Steinebach, *On the increments of partial sum processes with multidimensional indices*, Z. Wahrsch. Verw. Gebiete **63** (1983), 59–70.

[20] ______, *Improved Erdös-Rényi and strong approximation laws for increments of renewal processes*, Ann. Probab. **14** (1986), no. 2, 547–559.

[21] W. S. Wang, *Self-normalized lag increments of partial sums*, Statist. Probab. Lett. **58** (2002), 41–51.

Yong-Kab Choi, Kyo-Shin Hwang, and Hee-Jin Moon
Department of Mathematics, College of Natural Science
Gyeongsang National University
Chinju 660-701, Korea
E-mail: math-ykc@hanmail.net

Tae-Sung Kim and Jong-Il Baek
Department of Mathematics and Institute of Basic Science
Wonkwang University,
Iksan 570-749, Korea
E-mail: starkim@wonkwang.ac.kr