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HYERS-ULAM-RASSIAS STABILITY OF THE BANACH SPACE VALUED LINEAR DIFFERENTIAL EQUATIONS $y' = \lambda y$

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Abstract. The aim of this paper is to prove the stability in the sense of Hyers-Ulam-Rassias of the Banach space valued differential equation $y' = \lambda y$, where λ is a complex constant. That is, suppose f is a Banach space valued strongly differentiable function on an open interval. If f is an approximate solution of the equation $y' = \lambda y$, then there exists an exact solution of the equation near to f.

1. Introduction

In 1940, S. M. Ulam posed the following problem concerning the stability of functional equations: *"Give conditions in order for a linear mapping near an approximately linear mapping to exist"* (cf. [11, 12]). An answer has been given in the following way. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$, the set of all real numbers, for each fixed $x \in E_1$. If there exist $\theta \geq 0$ and $p \in \mathbb{R}\backslash\{1\}$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there is a unique linear mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq 2\theta\|x\|^p/|2-2^p|$ for every $x \in E_1$. In 1941, D. H. Hyers [3] obtained the result for $p = 0$. And then, Th. M. Rassias [7] generalized the above result of Hyers to the case where $0 \leq p < 1$. Moreover, he noticed in [7] that the proof also works for $p < 0$. A similar result was obtained by Z. Gajda [2] for $p > 1$. In the same paper, Gajda showed that a similar result does not hold for $p = 1$ (cf. [8]).

In connection with the stability of exponential functions, C. Alsina and R. Ger [1] remarked that the differential equation $y' = y$ has the

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Hyers-Ulam stability. More explicitly, suppose I is an open interval, $\varepsilon > 0$ and $f : I \to \mathbb{R}$ is a differentiable function such that $|f'(t) - f(t)| \le \varepsilon$ for all $t \in I$. Then there is a differentiable function $g : I \to \mathbb{R}$ such that $g' = g$ and $|f(t) - g(t)| \le 3\varepsilon$ for all $t \in I$.

The third and first authors of this paper, together with S. Miyajima [10], considered the Banach space valued differential equation $y' = \lambda y$, where λ is a complex constant. Indeed, they proved the Hyers-Ulam stability of the differential equation $y' = \lambda y$ under the condition that $\Re(\lambda) \neq 0$ (cf. Corollaries 2 and 3); Moreover, if $\Re(\lambda) = 0$ and if the diameter of I is infinite, they gave an example that the Hyers-Ulam stability does not hold. Some stability results of other differential equations are also known (cf. [5, 6]).

The aim of this paper is to prove the Hyers-Ulam-Rassias stability of the Banach space valued differential equation $y' = \lambda y$, which generalizes a result of S.-M. Jung and K. Lee [4]. In fact, they considered only real valued functions which satisfy the differential equation $y' = \lambda y$ approximately. Moreover, [10, Theorem 2.1] is obtained as an easy corollary to our main result, Theorem 1.

2. Main results

From now on, let $(X, \|\cdot\|)$ be a non-zero complex Banach space, and let $I = (a, b)$ be an open interval, where $-\infty \le a < b \le \infty$.

Recall that a function $f : I \to X$ is called strongly differentiable, if to each $t \in I$ there corresponds an $f'(t) \in X$ such that

$$\lim_{s \to 0} \left\| \frac{f(t+s) - f(t)}{s} - f'(t) \right\| = 0.$$

Suppose $f : I \to X$ is strongly differentiable and λ is a complex number. We denote by $\Re(\lambda)$ the real part of λ. Note that each of the following two statements implies the other:

(a) $f'(t) = \lambda f(t)$ for all $t \in I$;

(b) There is an $x \in X$ such that $f(t) = e^{\lambda t} x$ for all $t \in I$.

THEOREM 1. *Suppose λ is a complex number, $\epsilon : I \to [0, \infty)$ is a continuous function and $f : I \to X$ is a strongly differentiable function such that*

$$\|f'(t) - \lambda f(t)\| \le \epsilon(t) \tag{1}$$

for all $t \in I$.

(a) If $\epsilon(t)e^{-\Re(\lambda)t}$ is integrable on $(a, t_a]$ for some $t_a \in I$, then there is a unique $x_a \in X$ such that

$$\|f(t) - e^{\lambda t}x_a\| \le e^{\Re(\lambda)t}\int_a^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma$$

for all $t \in I$.

(b) If $\epsilon(t)e^{-\Re(\lambda)t}$ is integrable on $[t_b, b)$ for some $t_b \in I$, then there is a unique $x_b \in X$ such that

$$\|f(t) - e^{\lambda t}x_b\| \le e^{\Re(\lambda)t}\int_t^b \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma$$

for all $t \in I$.

As noted above, given any $x \in X$, $e^{\lambda t}x$ is a solution of the X-valued differential equation $y' = \lambda y$. That is, Theorem 1 says that if f is an approximate solution of the equation $y' = \lambda y$, then there is an exact solution of the equation near to f.

Proof. Let X^* be the dual space of X. We associate to each $\varphi \in X^*$ the function $f_\varphi : I \to \mathbb{C}$ defined by

$$f_\varphi(t) = \varphi(f(t))$$

for all $t \in I$. For any chosen $\varphi \in X^*$, it holds by the continuity of φ that $(f_\varphi)'(t) = \varphi(f'(t))$ for all $t \in I$. Hence, it follows from (1) that

$$\begin{aligned} |(f_\varphi)'(t) - \lambda f_\varphi(t)| &= |\varphi(f'(t)) - \varphi(\lambda f(t))| \\ &\le \|\varphi\|\,\|f'(t) - \lambda f(t)\| \\ &\le \|\varphi\|\,\epsilon(t) \end{aligned}$$

for all $t \in I$. Since $\epsilon(t)e^{-\Re(\lambda)t}$ is continuous on I, in view of the last inequality, we get

$$\begin{aligned} \left|\int_s^t \{e^{-\lambda\sigma}f_\varphi(\sigma)\}'d\sigma\right| &= \left|\int_s^t \{(f_\varphi)'(\sigma) - \lambda f_\varphi(\sigma)\}e^{-\lambda\sigma}d\sigma\right| \\ &\le \|\varphi\|\left|\int_s^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma\right| \end{aligned} \tag{2}$$

for all $s, t \in I$. Although $\{e^{-\lambda\sigma}f_\varphi(\sigma)\}'$ need not be continuous, (2) implies that

$$\int_s^t \{e^{-\lambda\sigma}f_\varphi(\sigma)\}'d\sigma = e^{-\lambda t}f_\varphi(t) - e^{-\lambda s}f_\varphi(s) \tag{3}$$

for all $s, t \in I$ (cf. [9, Theorem 7.21]). It follows from (2) and (3) that

$$\begin{aligned} |\varphi(e^{-\lambda t}f(t) - e^{-\lambda s}f(s))| &= |e^{-\lambda t}f_\varphi(t) - e^{-\lambda s}f_\varphi(s)| \\ &\leq \|\varphi\| \left| \int_s^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma \right| \end{aligned} \tag{4}$$

for all $s, t \in I$.

Recall that

$$\|x\| = \sup\{|\psi(x)| : \psi \in X^*, \|\psi\| = 1\}$$

for each $x \in X$ (cf. [9, Remarks 5.21]). Since $\varphi \in X^*$ was arbitrary, it follows from (4) that

$$\begin{aligned} &\|e^{-\lambda t}f(t) - e^{-\lambda s}f(s)\| \\ &\quad = \sup\{|\psi(e^{-\lambda t}f(t) - e^{-\lambda s}f(s))| : \psi \in X^*, \|\psi\| = 1\} \\ &\quad \leq \sup\left\{ \|\psi\| \left| \int_s^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma \right| : \psi \in X^*, \|\psi\| = 1 \right\} \\ &\quad = \left| \int_s^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma \right| \end{aligned} \tag{5}$$

for all $s, t \in I$.

(a) Assume that there exists a $t_a \in I$ such that $\epsilon(t)e^{-\Re(\lambda)t}$ is integrable on $(a, t_a]$. Since $\epsilon(t)e^{-\Re(\lambda)t}$ is continuous on I, our hypothesis indeed implies that $\epsilon(t)e^{-\Re(\lambda)t}$ is integrable on $(a, t_0]$ for any $t_0 \in I$. By (5), $\{e^{-\lambda s}f(s)\}_{s\in I}$ is a Cauchy net, i.e., $e^{-\lambda s}f(s)$ converges to an element, say $x_a \in X$, as $s \to a^+$. It thus follows from (5) again that

$$\begin{aligned} &\|f(t) - e^{\lambda t}x_a\| \\ &\quad \leq \|f(t) - e^{\lambda(t-s)}f(s)\| + \|e^{\lambda(t-s)}f(s) - e^{\lambda t}x_a\| \\ &\quad \leq e^{\Re(\lambda)t} \left| \int_s^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma \right| + e^{\Re(\lambda)t}\|e^{-\lambda s}f(s) - x_a\| \end{aligned} \tag{6}$$

for all $s, t \in I$. Since $e^{-\lambda s}f(s) \to x_a$ as $s \to a^+$, we have

$$\|f(t) - e^{\lambda t}x_a\| \leq e^{\Re(\lambda)t} \int_a^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma \tag{7}$$

for all $t \in I$.

If, in addition, $x \in X$ also satisfies

$$\|f(t) - e^{\lambda t}x\| \leq e^{\Re(\lambda)t} \int_a^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma$$

for all $t \in I$, then we have

$$\begin{aligned}\|x_a - x\| &\leq e^{-\Re(\lambda)t}\|e^{\lambda t}x_a - f(t)\| + e^{-\Re(\lambda)t}\|f(t) - e^{\lambda t}x\| \\ &\leq 2\int_a^t \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma \\ &\to 0\end{aligned}$$

as $t \to a^+$, and hence $x = x_a$, which proves the uniqueness of x_a.

(b) Now, let us assume that $\epsilon(t)e^{-\Re(\lambda)t}$ is integrable on $[t_b, b)$ for some $t_b \in I$. As we mentioned in (a), $\epsilon(t)e^{-\Re(\lambda)t}$ is integrable on $[t_0, b)$ for any $t_0 \in I$. Following the first part of (a), we can show by (5) that $e^{-\lambda s}f(s)$ converges to a point, say $x_b \in X$, as $s \to b^-$. Similarly to (6) and (7), we get

$$\|f(t) - e^{\lambda t}x_b\| \leq e^{\Re(\lambda)t}\int_t^b \epsilon(\sigma)e^{-\Re(\lambda)\sigma}d\sigma$$

for each $t \in I$. By a similar way given in (a), we may easily verify the uniqueness of x_b. □

REMARK 1. Note that if λ is a real number, then the above proof works for the real Banach space case, and hence a result similar to Theorem 1 holds. Moreover, if λ is a real number, the following corollaries and remarks are also true for real Banach space case.

In the rest of this paper, we define

$$m = \inf\{e^{-\Re(\lambda)t} : t \in I\} \text{ and } M = \sup\{e^{-\Re(\lambda)t} : t \in I\}$$

for a given complex number λ. It seems to be interesting that we compare the following three corollaries with [10, Theorem 2.1].

COROLLARY 2. *Let $f : I \to X$ be a strongly differentiable function that satisfies the inequality*

$$\|f'(t) - \lambda f(t)\| \leq \varepsilon \tag{8}$$

for all $t \in I$ and for some $\varepsilon > 0$. If $\Re(\lambda) \neq 0$ and $m = 0$, there exists a unique $x_0 \in X$ such that

$$\sup_{t\in I}\|f(t) - e^{\lambda t}x_0\| < \infty.$$

Moreover, for the above $x_0 \in X$, the following estimate

$$\|f(t) - e^{\lambda t}x_0\| \leq \frac{\varepsilon}{|\Re(\lambda)|} \tag{9}$$

holds for all $t \in I$.

Corollary 2 states that if $\Re(\lambda) \neq 0$ and $m = 0$, then there exists one and only one solution $e^{\lambda t}x_0$ of the equation $y' = \lambda y$ such that "the distance" between $f(t)$ and $e^{\lambda t}x_0$ is finite; Moreover, the distance is at most $\varepsilon/|\Re(\lambda)|$. Therefore,

$$\sup_{t\in I} \|f(t) - e^{\lambda t}x\| = \infty$$

for all $x \in X\backslash\{x_0\}$.

Proof. If $\Re(\lambda) < 0$, $\varepsilon e^{-\Re(\lambda)t}$ is integrable on $(a, t_a]$ for any $t_a \in I$. According to (a) of Theorem 1, there exists an $x_a \in X$ such that

$$\|f(t) - e^{\lambda t}x_a\| \leq \varepsilon e^{\Re(\lambda)t} \int_a^t e^{-\Re(\lambda)\sigma} d\sigma = \frac{\varepsilon}{|\Re(\lambda)|}$$

for all $t \in I$. (We notice that $m = \lim_{\sigma\to a^+} e^{-\Re(\lambda)\sigma} = 0$).

Similarly, if $\Re(\lambda) > 0$, $\varepsilon e^{-\Re(\lambda)t}$ is integrable on $[t_b, b)$ for any $t_b \in I$. In view of (b) of Theorem 1, there is an $x_b \in X$ such that

$$\|f(t) - e^{\lambda t}x_b\| \leq \varepsilon e^{\Re(\lambda)t} \int_t^b e^{-\Re(\lambda)\sigma} d\sigma = \frac{\varepsilon}{|\Re(\lambda)|}$$

for all $t \in I$. (We remark that $m = \lim_{\sigma\to b^-} e^{-\Re(\lambda)\sigma} = 0$ when $\Re(\lambda) > 0$).

If we set

$$x_0 = \begin{cases} x_a & \text{for } \Re(\lambda) < 0, \\ x_b & \text{for } \Re(\lambda) > 0, \end{cases}$$

this x_0 satisfies the inequality (9) for all $t \in I$.

Finally, we show the uniqueness of $x_0 \in X$. So, suppose $x_1 \in X$ satisfies $\|f(t) - e^{\lambda t}x_1\| \leq K$ for all $t \in I$ and for some $0 < K < \infty$. It follows from (9) that

$$\begin{aligned} \|x_0 - x_1\| &\leq e^{-\Re(\lambda)t}\|e^{\lambda t}x_0 - f(t)\| + e^{-\Re(\lambda)t}\|f(t) - e^{\lambda t}x_1\| \\ &\leq e^{-\Re(\lambda)t}\left(\frac{\varepsilon}{|\Re(\lambda)|} + K\right) \end{aligned}$$

for all $t \in I$. Since $m = \inf\{e^{-\Re(\lambda)t} : t \in I\} = 0$, we thus obtain $x_0 = x_1$, proving the uniqueness. □

REMARK 2. In Corollary 2, we proved the uniqueness of an $x_0 \in X$, under the hypothesis $m = 0$, for which the inequality (9) is true for all $t \in I$. On the other hand, such a uniqueness need not hold if $m > 0$ (cf. [10, Remark 2.2]). Indeed, suppose $f : I \to X$ is a strongly differentiable function such that $f'(t) = \lambda f(t)$ for all $t \in I$ and for a given complex number λ with $\Re(\lambda) \neq 0$. Then $f(t)$ is of the form $f(t) = e^{\lambda t} x_0$ for all $t \in I$ and for some $x_0 \in X$. If $m > 0$, then

$$\|x_0 - x\| \leq \frac{m\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right) \tag{10}$$

implies that

$$\|f(t) - e^{\lambda t}x\| = e^{\Re(\lambda)t}\|x_0 - x\| \leq \frac{\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right)$$

for all $t \in I$, since $m = \inf\{e^{-\Re(\lambda)t} : t \in I\}$. Thus, the inequality (9) is true for all $x \in X$ satisfying (10).

In Remark 2, we gave a function $f : I \to X$ such that there are infinitely many $x \in X$ which satisfies (9). In the following corollary, we moreover show that if $m > 0$, then to each function $f : I \to X$ with (8) there correspond infinitely many $x \in X$ such that the inequality (9) is true.

Corollary 3. *Assume that a strongly differentiable function $f : I \to X$ satisfies the inequality (8) for all $t \in I$ and for some $\varepsilon > 0$. If $\Re(\lambda) \neq 0$ and $m > 0$, then there are infinitely many $x_0 \in X$ for which the inequality*

$$\|f(t) - e^{\lambda t}x_0\| \leq \frac{\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right) \tag{11}$$

holds for all $t \in I$. More explicitly, if S is the set of all $x_0 \in X$ satisfying (11), then the cardinal number of S is at least $\mathfrak{c}$, where $\mathfrak{c}$ denotes that of the continuum.

Proof. Following the first part of the proof of Corollary 2 and considering the proof of Theorem 1, we conclude that if we define

$$x_0 = \begin{cases} \lim_{s \to a^+} e^{-\lambda s} f(s) & \text{for } \Re(\lambda) < 0, \\ \lim_{s \to b^-} e^{-\lambda s} f(s) & \text{for } \Re(\lambda) > 0, \end{cases}$$

then the inequality (11) is true for all $t \in I$.

Since

$$m = \begin{cases} \lim_{s \to a^+} e^{-\Re(\lambda)s} & \text{for } \Re(\lambda) < 0, \\ \lim_{s \to b^-} e^{-\Re(\lambda)s} & \text{for } \Re(\lambda) > 0, \end{cases}$$

we may define

$$J = \{e^{-\Re(\lambda)t} : m < e^{-\Re(\lambda)s} \leq e^{-\Re(\lambda)t} < M \text{ implies } e^{-\lambda s} f(s) = x_0\}.$$

Note that the possibility $J = \emptyset$ is not excluded. By Remark 2, we consider only the case where $J \subsetneq (m, M)$.

Now, assume that either $J = \emptyset$ or $J \neq \emptyset$ with $\sup J < M$. We define

$$\alpha = \begin{cases} \sup J & \text{for } J \neq \emptyset, \\ m & \text{for } J = \emptyset. \end{cases}$$

By the definitions of J and α, we see that $e^{-\lambda s} f(s) \to x_0$ as $e^{-\Re(\lambda)s} \to \alpha$, and hence there is a $0 < \delta_0 < \alpha(1 - m/M)$ such that

$$\text{(12)} \quad \alpha < e^{-\Re(\lambda)s} < \alpha + \delta_0 \text{ implies } \|x_0 - e^{-\lambda s} f(s)\| \leq \frac{m\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right).$$

By the definition of J, there is an $s_0 \in I$ such that

$$\text{(13)} \qquad \alpha < e^{-\Re(\lambda)s_0} < \alpha + \delta_0$$

and $x_0 \neq e^{-\lambda s_0} f(s_0)$. Put $r_0 = \|x_0 - e^{-\lambda s_0} f(s_0)\| > 0$. Since the function $s \mapsto \|x_0 - e^{-\lambda s} f(s)\|$ is continuous, it follows from the intermediate value theorem that there corresponds to each $r \in (0, r_0)$ an $s_r \in I$ such that

$$\text{(14)} \qquad \alpha < e^{-\Re(\lambda)s_r} < e^{-\Re(\lambda)s_0}$$

and

$$\text{(15)} \qquad \|x_0 - e^{-\lambda s_r} f(s_r)\| = r.$$

Now, put $x_r = e^{-\lambda s_r} f(s_r)$ for each $r \in (0, r_0)$. Then by (15), $x_{r_1} \neq x_{r_2}$ whenever $r_1, r_2 \in (0, r_0)$ and $r_1 \neq r_2$.

We shall show that the inequality (11) holds for each element of $\{x_r : r \in (0, r_0)\}$. It follows from (12), (13) and (14) that

$$\text{(16)} \qquad \|x_0 - e^{-\lambda s_r} f(s_r)\| \leq \frac{m\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right)$$

for all $r \in (0, r_0)$. Pick $t \in I$ arbitrarily. There are now two possibilities: either $e^{-\Re(\lambda)t} \leq \alpha$ or $\alpha < e^{-\Re(\lambda)t}$.

In the former case, $e^{-\lambda t} f(t) = x_0$ by the definition of J, and hence (16) gives

$$\begin{aligned} \|f(t) - e^{\lambda t} x_r\| &= e^{\Re(\lambda)t} \|x_0 - e^{-\lambda s_r} f(s_r)\| \\ &\leq \frac{\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right) \end{aligned}$$

for all $r \in (0, r_0)$. (We notice that $e^{\Re(\lambda)t} m \leq 1$).

In the latter case, since $0 < \delta_0 < \alpha(1 - m/M)$, (13) and (14) give

$$\text{(17)} \qquad \alpha < e^{-\Re(\lambda)s_r} < \alpha\left(2 - \frac{m}{M}\right)$$

for all $r \in (0, r_0)$. If we divide the terms in (17) by $e^{-\Re(\lambda)t}$, then

$$\frac{\alpha}{e^{-\Re(\lambda)t}} < \frac{e^{-\Re(\lambda)s_r}}{e^{-\Re(\lambda)t}} < \frac{\alpha}{e^{-\Re(\lambda)t}}\left(2 - \frac{m}{M}\right),$$

and hence

$$\frac{m}{M} < \frac{e^{-\Re(\lambda)s_r}}{e^{-\Re(\lambda)t}} < 2 - \frac{m}{M},$$

since $m \le \alpha < e^{-\Re(\lambda)t} < M$. Thus, we have

$$\left|\frac{e^{-\Re(\lambda)s_r}}{e^{-\Re(\lambda)t}} - 1\right| \le 1 - \frac{m}{M} \tag{18}$$

for all $r \in (0, r_0)$. Note that, by (5), the following inequality

$$\|e^{-\lambda t}f(t) - e^{-\lambda s_r}f(s_r)\| \le \frac{\varepsilon}{|\Re(\lambda)|}|e^{-\Re(\lambda)t} - e^{-\Re(\lambda)s_r}| \tag{19}$$

holds for every $r \in (0, r_0)$. It follows from (18) and (19) that

$$\begin{aligned} \|f(t) - e^{\lambda t}x_r\| &\le e^{\Re(\lambda)t}\frac{\varepsilon}{|\Re(\lambda)|}|e^{-\Re(\lambda)t} - e^{-\Re(\lambda)s_r}| \\ &\le \frac{\varepsilon}{|\Re(\lambda)|}\left(1 - \frac{m}{M}\right) \end{aligned}$$

for all $r \in (0, r_0)$. Thus, each element of $\{x_r : r \in (0, r_0)\}$ satisfies the inequality (11) for all $t \in I$. Recall that, by (15), the cardinal number of the set $\{x_r : r \in (0, r_0)\}$ is that of $(0, r_0)$, and hence $\mathfrak{c}$. This completes the proof. □

COROLLARY 4. *Assume that a strongly differentiable function $f : I \to X$ satisfies the inequality* (8) *for all $t \in I$ and for some $\varepsilon > 0$. If $\Re(\lambda) = 0$ and the diameter $\delta(I)$ of I is finite, then there exist unique $x_a \in X$ and $x_b \in X$ such that*

$$\|f(t) - e^{\lambda t}x_a\| \le \varepsilon(t - a) \text{ and } \|f(t) - e^{\lambda t}x_b\| \le \varepsilon(b - t)$$

for all $t \in I$, respectively.

Proof. Since $\delta(I)$ is finite, ε is integrable on (a, b). Note that $\Re(\lambda) = 0$. According to (a) and (b) of Theorem 1, there exist unique $x_a \in X$ and $x_b \in X$ such that

$$\|f(t) - e^{\lambda t}x_a\| \le \varepsilon(t - a) \text{ and } \|f(t) - e^{\lambda t}x_b\| \le \varepsilon(b - t)$$

for all $t \in I$, respectively. □

REMARK 3. Suppose $I = (a, \infty)$ and $\epsilon(t)$ is a nonnegative polynomial on I with real coefficients. Then $\epsilon(t)e^{-t}$ is integrable on (t_0, ∞) for any

$t_0 \in I$. The second author and K. Lee gave an explicit formula of the function $e^t \int_t^\infty \epsilon(\sigma)e^{-\sigma}d\sigma$ (see [4, Theorem 4]).

References

[1] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373–380.

[2] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[4] S.-M. Jung and K. Lee, *Hyers-Ulam-Rassias stability of linear differential equations*, to appear.

[5] T. Miura, S. Miyajima and S.-E. Takahasi, *Hyers-Ulam stability of linear differential operator with constant coefficients*, Math. Nachr. **258** (2003), 90–96.

[6] ———, *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286** (2003), 136–146.

[7] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[8] Th. M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[9] W. Rudin, *Real and Complex Analysis* (3rd Edition), McGraw-Hill, 1987.

[10] S.-E. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc. **39** (2002), 309–315.

[11] S. M. Ulam, *Problems in Modern Mathematics*, Chap. VI, Science Editions, Wiley, New York, 1964.

[12] ———, *Sets, Numbers and Universes* Selected Works, Part III, MIT Press, Cambridge, MA, 1974.

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