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A SUMMATION FORMULA OF ${}_6F_5(1)$

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ABSTRACT. The authors aim at obtaining an interesting result for a special summation formula for ${}_6F_5(1)$, by comparing two generalized Watson's theorems on the sum of a ${}_3F_2$ obtained earlier by Mitra and Lavoie *et.al.*

The aim of this note is, by comparing special cases of two known summation formulas for ${}_3F_2(1)$ with the aid of another known summation formula for ${}_5F_4$, to derive the following interesting summation formula for ${}_6F_5(1)$:

$$\begin{aligned}&{}_6F_5\left(\begin{matrix}\frac{1}{2}c+\frac{3}{4}, & c, & 1, & \frac{3}{2}, & \frac{1}{2}a, & \frac{1}{2}b\\ \frac{1}{2}c-\frac{1}{4}, & \frac{1}{2}, & c-1, & c-\frac{1}{2}a+\frac{1}{2}, & c-\frac{1}{2}b+\frac{1}{2}\end{matrix}\;\middle|\; 1\right)\\ &=\frac{\alpha\,(2c-a-1)\,(2c-b-1)}{(2c-a-b-3)\,(2c-a-b-1)\,(2c-1)\,(c-1)}\\ &\qquad(\Re(2c-a-b)>3),\end{aligned} \tag{1}$$

where, for convenience,

$$\alpha=2\,c^2-(a+b+5)\,c+(a+1)\,(b+1)+2.$$

In 1943, Mitra [3] generalized the classical Watson's theorem on the sum of a ${}_3F_2(1)$ [1, p. 16]:

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}a, & b, & c\\ \frac{1}{2}(a+b+1), & 2c\end{matrix}\;\middle|\; 1\right)\\ &=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\\ &\qquad(\Re(2c-a-b)>-1)\end{aligned} \tag{2}$$

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in the following form:

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & \delta \end{matrix} \,\middle|\, 1\right) \\
&= \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)} \\
&\cdot\left\{\frac{\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\mathcal{A}(a,\,b,\,c,\,\delta)\right. \\
&\left.+\frac{2c-\delta}{\delta}\,\frac{\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}\mathcal{B}(a,\,b,\,c,\,\delta)\right\} \\
&\qquad (\Re(2\delta-2c-a-b)>-1),
\end{aligned} \tag{3}
$$

where, for convenience,

$$
\mathcal{A}(a,\,b,\,c,\,\delta) := {}_7F_6\left(\begin{matrix} c-\frac{1}{2},\ \frac{1}{2}c+\frac{3}{4},\ c,\ c-\frac{1}{2}\delta,\ c-\frac{1}{2}\delta+\frac{1}{2},\ \frac{1}{2}a,\ \frac{1}{2}b \\ \frac{1}{2}c-\frac{1}{4},\ \frac{1}{2},\ \frac{1}{2}\delta+\frac{1}{2},\ \frac{1}{2}\delta,\ c-\frac{1}{2}a+\frac{1}{2},\ c-\frac{1}{2}b+\frac{1}{2} \end{matrix} \,\middle|\, 1\right)
$$

and

$$
\mathcal{B}(a,\,b,\,c,\,\delta) := {}_7F_6\left(\begin{matrix} c+\frac{1}{2},\ \frac{1}{2}c+\frac{5}{4},\ c,\ c-\frac{1}{2}\delta+\frac{1}{2},\ c-\frac{1}{2}\delta+1,\ \frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}c+\frac{1}{4},\ \frac{3}{2},\ \frac{1}{2}\delta+1,\ \frac{1}{2}\delta+\frac{1}{2},\ c-\frac{1}{2}a+1,\ c-\frac{1}{2}b+1 \end{matrix} \,\middle|\, 1\right).
$$

The special case $\delta = 2c$ of (3) yields the Watson's theorem (1).

In 1992, by using a different method from that of Mitra in [3], Lavoie, Grondin and Rathie [2] have obtained twenty five summation formulas closely related to the classical Watson's theorem:

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix} \,\middle|\, 1\right) \quad (i,\,j=0,\,\pm1,\,\pm2),
$$

the case $(i, j) = (0, -2)$ of which is as follows:

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c-2 \end{matrix} \,\middle|\, 1\right) \\
&= \frac{2^{a+b-2}\,\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{3}{2}\right)}{2(c-1)\,\Gamma\left(\frac{1}{2}\right)\Gamma(a)\,\Gamma(b)} \\
&\cdot\left[\frac{\{(c-a-1)(c-b-1)+(c-1)(c-2)\}\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)}\right. \\
&\left.+\frac{4\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a-1\right)\Gamma\left(c-\frac{1}{2}b-1\right)}\right] \quad (\Re(2c-a-b)>3)
\end{aligned} \tag{4}
$$

shall be used here.

On the other hand, the special case $\delta = 2c - 2$ of (3) is written as follows:

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c-2 \end{matrix}\,\middle|\, 1\right) \\
&= \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)} \\
&\cdot\left[\frac{\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\right. \\
&\cdot {}_7F_6\left(\begin{matrix} c-\frac{1}{2}, & \frac{1}{2}c+\frac{3}{4}, & c, & 1, & \frac{3}{2}, & \frac{1}{2}a, & \frac{1}{2}b \\ \frac{1}{2}c-\frac{1}{4}, & \frac{1}{2}, & c-\frac{1}{2}, & c-1, & c-\frac{1}{2}a+\frac{1}{2}, & c-\frac{1}{2}b+\frac{1}{2} \end{matrix}\,\middle|\, 1\right) \\
&+\frac{1}{c-1}\,\frac{\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)} \\
&\left.\cdot {}_5F_4\left(\begin{matrix} c+\frac{1}{2}, & \frac{1}{2}c+\frac{5}{4}, & 2, & \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}c+\frac{1}{4}, & c-\frac{1}{2}, & c-\frac{1}{2}a+1, & c-\frac{1}{2}b+1 \end{matrix}\,\middle|\, 1\right)\right],
\end{aligned}
\tag{5}
$$

provided $\Re(2c-a-b) > 3$.

If we apply the known summation formula for the ${}_5F_4$ due to Bailey [1, p. 27] to the right-hand side of (5), comparing its resulting identity and (4), after a little simplification, we arrive at the desired result (1).

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