Commun. Korean Math. Soc. 19 (2004), No. 4, pp. 759–764

EXPANSIVE HOMEOMORPHISMS WITH THE SHADOWING PROPERTY ON ZERO DIMENSIONAL SPACES

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ABSTRACT. Let $X = \{a\} \cup \{a_i \mid i \in \mathbb{N}\}$ be a subspace of Euclidean space E^2 such that $\lim_{i\to\infty} a_i = a$ and $a_i \neq a_j$ for $i \neq j$. Then it is well known that the space X has no expansive homeomorphisms with the shadowing property. In this paper we show that the set of all expansive homeomorphisms with the shadowing property on the space Y is dense in the space $H(Y)$ of all homeomorphisms on Y, where $Y = \{a, b\} \cup \{a_i \mid i \in \mathbb{Z}\}$ is a subspace of E^2 such that $\lim_{i\to\infty} a_i = b$ and $\lim_{i\to-\infty} a_i = a$ with the following properties; $a_i \neq a_j$ for $i \neq j$ and $a \neq b$.

1. Introduction

All spaces considered in this paper are assumed to be compact and metrizable. Let Φ be a homeomorphism from a space (X, d) onto itself. Given $\delta > 0$, a sequence $\{x_i \mid i \in \mathbb{Z}\}$ in X is called a *δ-pseudo-orbit* of Φ if

$$d(\Phi(x_i), x_{i+1}) < \delta$$

for every $i \in \mathbb{Z}$. Given $\varepsilon > 0$, a sequence $\{x_i \mid i \in \mathbb{Z}\}$ in X is said to be *ε-traced* by a point $y \in X$ if

$$d(\Phi^i(y), x_i) < \varepsilon$$

for every $i \in \mathbb{Z}$. We say that Φ has the *shadowing property*(or pseudo orbit tracing property) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of Φ can be ε-traced by a point of X. Φ is called *expansive* if there is $c > 0$ such that for every $x, y \in X$ with $x \neq y$ there is $n \in \mathbb{Z}$ for which

$$d(\Phi^n(x), \Phi^n(y)) > c.$$

Received March 12, 2004.

2000 Mathematics Subject Classification: Primary 54H20; Secondary 58F99.

Key words and phrases: expansive homeomorphism, δ- pseudo-orbit, shadowing property (pseudo orbit tracing property), Dense, Zero dimensional space.

The constant $c > 0$ is called an *expansive constant* of Φ. For a space (X, d), we denote by $H(X)$ the space of all homeomorphisms of X with the metric

$$\widetilde{d}(\Phi, \Psi) = \sup \{d(\Phi(x), \Psi(x) \mid x \in X\}$$

for $\Phi, \Psi \in H(X)$. Let

$$E(X) = \{\Phi \in H(X) \mid \Phi \text{ is expansive}\}$$

and

$$P(X) = \{\Phi \in H(X) \mid \Phi \text{ has the shadowing property}\}.$$

Given $\delta > 0$ and $a \in X$, we denote a neighborhood $U_\delta(a)$ of a by

$$U_\delta(a) = \{b \in X \mid d(a,\ b) < \delta\}.$$

Aoki [1] proved that every group automorphism at the Cantor set C has the shadowing property. Sears [5] showed that $E(C)$is dense in $H(C)$. Dateyma [2] proved that $P(C)$ is dense in $H(C)$ and Kimura [4] also proved the followings.

PROPOSITION. *Let $X = \{a\} \cup \{a_i \mid i \in \mathbb{N}\}$ be a subspace of Euclidean space E^2 such that $\lim_{i \to \infty} a_i = a$ and $a_i \neq a_j$ for $i \neq j$. Then*

a) *the set of all expansive homeomorphisms of X is dense in $H(X)$;*
b) *the set of all homeomorphisms with the shadowing property of X is dense in $H(X)$;*
c) *X has no expansive homeomorphism with the shadowing property.*

A question aries naturally as to whether a zero-dimensional countable compact space admits an expansive homeomorphism with the shadowing property. For the question, Kato and Park [3] shows a zero-dimensional countable compact space admits an expansive homeomorphism with the shadowing property.

In this paper we generalize the result of Kato and Park [3] as in the main theorem of this paper.

2. The main theorem

THEOREM 2.1. *Let $Y = \{a, b\} \cup \{a_i \mid i \in \mathbb{Z}\}$ be a subspace of Euclidean space E^2 such that $\lim_{i \to \infty} a_i = b$ and $\lim_{i \to -\infty} a_i = a$, where $a_i \neq a_j$ for $i \neq j$* and *$a \neq b$. Then the set of all expansive homeomorphisms with the shadowing property on Y is dense in $H(Y)$.*

PROOF. Without loss of generality, we assume that for each $i \in \mathbb{Z}$,

$$d(a_{i+1},\ a_i) < d(a_i,\ a_{i-1}) \text{ if } i \geq 0,$$

and

$$d(a_{i-1},\ a_i) < d(a_i,\ a_{i+1}) \ \text{ if } i < 0.$$

Let $\Psi \in H(Y)$ and $\varepsilon > 0$. We construct a homeomorphism Φ satisfying

$$\Phi \in E(Y) \cap P(Y) \text{ and } \widetilde{d}(\Phi, \Psi) < \varepsilon.$$

To do this, we consider the following two cases:
Case 1. $\Psi(a) = a$ and $\Psi(b) = b$.
Case 2. $\Psi(a) = b$ and $\Psi(b) = a$.

Case 1: For $\varepsilon > 0$, we take $n \in \mathbb{N}$ such that for $|i| \geq n,\ i \in \mathbb{Z}$,

$$d(a_i,\ a) < \varepsilon \ \text{ or } \ d(a_i,\ b) < \varepsilon.$$

we define the sets $C_1,\ C_2,\ C_3$ by

$$C_1 = \{a_i \mid i \leq -n\},\ C_2 = \{a_i \mid -n < i < n\}$$

and

$$C_3 = \{a_i \mid n \leq i\}.$$

Since Ψ is a homeomorphism, there is a $q \in \mathbb{N}$ such that

$$\Psi(a_i) \in C_1 \text{ for all } i < -q \text{ and } \Psi(a_i) \in C_3 \text{ for all } i > q.$$

Put $k = \max\{q, n\}$. For every $i \in \mathbb{Z}$, we take $l,\ l' \in \mathbb{Z}$ such that

$$l = \max\{i \mid \Psi(a_i) = a_j,\ |j| < k\}$$

and

$$l' = \min\{i \mid \Psi(a_i) = a_j,\ |j| < k\}.$$

we define the sets $B,\ B_1,\ B_2$ by

$$B = \{a_i \mid l' \leq i \leq l\} \setminus \{a_i \mid \Psi(a_i) = a_j, |j| < k\}$$

and

$$B_1 = \{a_i \mid \Psi(a_i) \in C_1, a_i \in B\} \text{ and } B_2 = \{a_i \mid \Psi(a_i) \in C_3, a_i \in B\}.$$

Then we know that $B = B_1 \cup B_2$.

Define a mapping Φ from Y onto itself as follows:

i) $\Phi(a) = a$ and $\Phi(b) = b$;

ii) for any $a_i \in Y$,

$$\Phi(a_i) = \begin{cases} \Psi(a_i), & \text{if } i \in \{i \mid \Psi(a_i) = a_j,\ |j| < k\} \\ a_{i+1}, & \text{if } i \geq l+1 \text{ or if } i \leq l'-2; \end{cases}$$

iii) $\Phi(a_{l'-1})$ is a point in the set
$\{a_i \mid i = l', \cdots, l\} \setminus \{a_i \mid |i| < k\}$;

iv) $\Phi(a_i) \in \{a_i \mid l' \leq i \leq -k\} \setminus \Phi(a_{l'-1})$ if $a_i \in B_1$,
$\Phi(a_i) \in \{a_i \mid k \leq i \leq l+1\} \setminus \Phi(a_{l'-1})$ if $a_i \in B_2$ and
$\Phi(a_i) \neq \Phi(a_j), a_i \neq a_j \in B_s, s = 1, 2$.

Then Φ is bijective and $\Phi \in H(Y)$. By the construction of Φ, it is clear that

$$\widetilde{d}(\Phi, \Psi) < \varepsilon.$$

Now we show that Φ is expansive. Put

$$c = \min\{d(a_{l+1},\ a_{l+2}),\ d(a_{l'-1},\ a_{l'-2})\}.$$

Then we have

$$U_c(a_{l'-1}) = a_{l'-1} \text{ and } U_c(a_{l+1}) = a_{l+1}.$$

Let x and y be two points in $Y \setminus \{a,\ b\}$ with $x \neq y$. If

$$x \in \{a_i \mid l' - 1 \leq i \leq l + 1\},$$

then we get $d(x,\ y) > c$. Let x be a point in the set $x \in \{a_i \mid i > l+1\} \cup \{a_i \mid i < l' - 1\}$. Then we can chose $k,\ k' \in \mathbb{Z}$ such that

$$f^k(x) = a_{l+1} \text{ or } f^{k'}(x) = a_{l'-1}.$$

Hence Φ is an expansive homeomorphism. Now we are going to show that Φ has the shadowing property. Let $\varepsilon_0 > 0$ and let $\varepsilon_1 = \min\{\epsilon_0,\ \varepsilon\}$. Take $P,\ P' \in \mathbb{Z}$ with $P' < P$ satisfying the following properties:

$$\text{if } \ i \leq P' \text{ then } \ d(a,\ a_i) < \varepsilon_1,$$

and

$$\text{if } \ i \geq P \ \text{ then } \ d(b,\ a_i) < \varepsilon_1.$$

Put

$$\delta = \min\{d(a_{p'-1},\ a_{p'-2}),\ d(a_{p+1},\ a_{p+2})\}.$$

To show that Φ has the shadowing property, it is sufficient that every δ-pseudo orbit of Φ can be ε_1-traced by a point of Y. Let $\zeta = \{y_i \mid i \in \mathbb{Z}\}$ be a δ-pseudo orbit of Φ. we define the sets $A_1,\ A_2,\ A_3$ by

$$A_1 = \{a_i \mid i < P'\},\ A_2 = \{a_i \mid P' \leq i \leq P\}$$

and

$$A_3 = \{a_i \mid i > P\}.$$

Then we have the following three possibilities:

1) $\zeta \cap A_i \neq \phi$, for each $i = 1, 2, 3$;
2) $\zeta \subset A_1$;
3) $\zeta \subset A_3$.

1): If the point y_0 of the δ-pseudo orbit $\zeta = \{y_i \mid i \in \mathbb{Z}\}$ is in A_2, then ζ is ε_1-traced by the point y_0. But if $y_0 \in A_1$ or $y_0 \in A_3$, then we take a point p in the set $\{y_i \mid i \in \mathbb{Z}\} \cap A_2$ say $p = y_r$. Consider the sequence $\{y_i \mid i \in \mathbb{Z}\}$ and we denote it by $\{y_{i-r} \mid i \in \mathbb{Z}\}$.

Then the the sequence $\{y_{i-r} \mid i \in \mathbb{Z}\}$ is ε_1 -traced by the point y_{r-r}.

2): If $\zeta \subset A_1$, then it is clear that the δ-pseudo orbit $\zeta = \{y_i \in A_1 \mid i \in \mathbb{Z}\}$ is ε_1-traced by the point a.

3): If $\zeta \subset A_3$, then it is also obvious that the δ-pseudo orbit $\zeta = \{y_i \in A_3 \mid i \in \mathbb{Z}\}$ is ε_1-traced by the point b.

Case 2: Let $\varepsilon > 0$ and $l,\ l'$ as in Case 1. Define a mapping Φ from Y onto itself as follows:

i) $\Phi(a) = b$ and $\Phi(b) = a$;
ii) for any $a_i \in Y$, $\Phi(a_i) = \Psi(a_i)$, if $i \in \{i \mid \Psi(a_i) = a_j,\ |j| < k\}$;
iii) $\Phi(a_{l'-1})$ is a point in the set $\{a_i \mid i = l', \cdots, l\} \setminus \{a_i \mid |i| < k\}$;
iv) $\Phi(a_i) \in \{a_i \mid l' \leq i \leq -k\} \setminus \phi(a_{l'-1})$ if $a_i \in B_1$
$\Phi(a_i) \in \{a_i \mid k \leq i \leq l+1\} \setminus \Phi(a_{l'-1})$ if $a_i \in B_2$ and
$\Phi(a_i) \neq \Phi(a_j), a_i \neq a_j \in B_s, s = 1, 2$;
v) $\Phi(a_{l+2t-1}) = a_{l'-2t}, t = 1, 2, \cdots,$
$\Phi(a_{l+2t}) = a_{l'-2t+1}, t = 1, 2, \cdots,$
$\Phi(a_{l'-2t}) = a_{l+2t+1}, t = 1, 2, \cdots,$
$\Phi(a_{l'-2t-1}) = a_{l+2t}, t = 1, 2, \cdots.$

By the same techniques as in the proof of Case 1, we know that Φ is bijective, $\Phi(a) = b$ and $\Phi(b) = a$. Hence $\Phi \in H(Y)$. By the construction of Φ, we have

$$\widetilde{d}(\Phi, \Psi) < \varepsilon \text{ and } \Phi \in E(Y) \cap P(Y).$$

This completes the proof of our main theorem. □

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