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REIDEMEISTER ORBIT SETS ON THE MAPPING TORUS

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ABSTRACT. The Reidemeister orbit set plays a crucial role in the Nielsen type theory of periodic orbits, much as the Reidemeister set does in Nielsen fixed point theory. Let $f : G \to G$ be an endomorphism between the fundamental group of the mapping torus. Extending Jiang and Ferrario's works on Reidemeister sets, we obtain algebraic results such as addition formulae for Reidemeister orbit sets of f relative to Reidemeister sets on suspension groups. In particular, if f is an automorphism, an similar formula for Reidemeister orbit sets of f relative to Reidemeister sets on given groups is also proved.

0. Introduction

Nielsen fixed point theory has been extended to Nielsen type theory of periodic orbits [4, Section III.3]. In the fixed point theory, the computation of the Nielsen number often relies on our knowledge of the Reidemeister set, that is the set of Reidemeister conjugacy classes in the fundamental group. Ferrario [2] made an algebraic study of the Reidemeister set in relation to an invariant normal subgroup. He obtained addition formulae for Reidemeister numbers. Recently we extended his work on Reidemeister sets and obtained similar formulae for Reidemeister orbit numbers in [6]. Jiang [5] made a geometric study of the relation between periodic orbits and conjugacy classes in the suspension groups. Our purpose in this paper is the alternating approach on the mapping torus to obtain addition formulae for Reidemeister orbit numbers.

Given a group endomorphism $f : G \to G$, the Reidemeister set of f, denoted by $\mathcal{R}(f)$, is the set of orbits of the left action of G on G

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via $\gamma \overset{g}{\mapsto} g\gamma f(g^{-1})$, and its cardinality is the Reidemeister number $R(f)$ of f. When f is the homomorphism induced by a map $X \to X$ on the fundamental group $\pi_1(X)$, $R(f)$ is an upper bound for the Nielsen number $N(f)$, $N(f)$ is usually the minimal number of fixed points in the homotopy class of that map.

For a given integer $n > 0$, f acts on the Reidemeister set $\mathcal{R}(f^n)$ of the n-th iterate f^n. An orbit of this action is called a Reidemeister orbit, the set of all such orbits is the Reidemeister orbit set $\mathcal{RO}^{(n)}(f)$ introduced in [6]. When f is the homomorphism induced by a map $X \to X$ on $\pi_1(X)$, its cardinality $\sharp\mathcal{RO}^{(n)}(f)$ is an upper bound for the number of essential n-orbits, the latter being a lower bound for the number of n-orbits in the homotopy class.

Now let $f : G \to G$ be an endomorphism between the fundamental group of the mapping torus, and $H \subset G$ be an f-invariant normal subgroup and $\bar{G} = G/H$. The short exact sequence $1 \to H \overset{i}{\to} G \to \bar{G} \to 1$ induces an exact sequence

$$1 \to \widetilde{H} \overset{\tilde{i}}{\to} \widetilde{G}_f \to \widetilde{G}_{\bar{f}} \to 1,$$

of extended suspension groups. Under certain conditions, we have an addition formula of the form

$$\sharp\mathcal{RO}^{(n)}(f) = \sum_{j \in \mathcal{RO}^{(n)}(\bar{f})} \sharp\mathcal{R}(\varphi'_{m_j,j}),$$

where $m_j = n/\ell_j, \ell_j$ being the length of the orbit j, and $\varphi'_{m_j,j} : \widetilde{H} \to \widetilde{H}$ is an inner automorphism.

If $f : G \to G$ is an automorphism, then the extended suspension group $\widetilde{G}_f$ is an HNN extension of G relative to f (see [7]). In this case under suitable conditions, we have an addition formula of the form

$$\sharp\mathcal{RO}^{(n)}(f) = \sum_{j \in \mathcal{RO}^{(n)}(\bar{f})} \sharp\mathcal{R}(\psi'_j),$$

where $\psi'_j : H \to H$ is a twisted version of the restriction map f_H.

This paper consists of three sections. In the first section we describe some algebraic results in [2] and [6] on the Reidemeister sets and the Reidemeister orbit sets, while setting up our notation. The second section contains our results on extended suspension groups. In the last section we obtain an addition formula for an automorphism.

For the basics of Nielsen fixed point theory, the reader is referred to [1] and [4].

1. Some results on the Reidemeister sets and orbit sets

Let $f : G \to G$ be a group endomorphism. The Reidemeister set of f, denoted by $\mathcal{R}(f)$, is the set of equivalence classes for the following Reidemeister equivalence relation in G: $\gamma, \gamma' \in G$ are equivalent if and only if $\gamma' = g\gamma f(g^{-1})$ for some $g \in G$. The Reidemeister class of $\gamma \in G$ will be written $[\gamma]_f$.

If $H \subset G$ is an f-invariant normal subgroup, then the short exact sequence

$$1 \to H \xrightarrow{i} G \xrightarrow{q} \bar{G} \to 1,$$

where $\bar{G} = G/H$, and $i : H \to G$ and $q : G \to \bar{G}$ are the inclusion and quotient homomorphisms, induces an exact sequence (in the category of pointed sets)

$$(\mathcal{R}(f_H), [1]_{f_H}) \xrightarrow{i_*} (\mathcal{R}(f), [1]_f) \xrightarrow{q_*} (\mathcal{R}(\bar{f}), [1]_{\bar{f}}) \to 1$$

of Reidemeister sets, where $\mathcal{R}(f_H)$ is the Reidemeister set of the restriction map $f_H : H \to H$, and $\mathcal{R}(\bar{f})$ is the Reidemeister set of the induced map $\bar{f} : \bar{G} \to \bar{G}$.

The function i_* is not injective in general. In a paper of Davide Ferrario [2], an f-invariant normal subgroup $T_f(K)$ in H is identified so that under certain conditions, the image $i_*\mathcal{R}(f_H)$ is in one-to-one correspondence with the Reidemeister set $\mathcal{R}(\widehat{f_H})$ of the induced map $\widehat{f_H} : \bar{H} \to \bar{H}$, where $\bar{H} = H/T_f(K)$.

Definition 1.1 [2]. Suppose $K \subset G$ is an f-invariant subgroup and $KH = q^{-1}\mathrm{Fix}(\bar{f})$. Such a K exists, for example the subgroup $q^{-1}\mathrm{Fix}(\bar{f})$ itself. Let $[K, H]$ denote the subgroup of G generated by all $khk^{-1}h^{-1}$ such that $k \in K$ and $h \in H$. Let K^G denote the smallest normal subgroup of G containing K. Define

$$O_f K := \{kf(k^{-1}) \mid k \in K\}.$$

Define

$$T_f(K) := [K^G, H] \cup O_f K$$

to be the smallest subgroup of G containing both $[K^G, H]$ and $O_f K$.

Proposition 1.2 [2]. *The subgroup $T_f(K)$ is normal in H, f-invariant and the equality $T_f(K) = \{\alpha k f(k^{-1}) \mid \alpha \in [K^G, H], k \in K\}$ holds true.*

LEMMA 1.3 [2]. *For any f-invariant subgroup K of G such that*

$$KH = q^{-1}\mathrm{Fix}(\bar{f})$$

there exists a surjection

$$A : q_*^{-1}([1]_{\bar{f}}) = i_*\mathcal{R}(f_H) \to \mathcal{R}(\widehat{f_H} : \bar{H} \to \bar{H}),$$

where $\bar{H} = H/T_f(K)$, and $\widehat{f_H} : \bar{H} \to \bar{H}$ is induced by $f_H : H \to H$. A is defined by $A([h]_f) := [p(h)]_{\widehat{f_H}})$ for all $h \in H$, where p is the projection $p : H \to \bar{H}$.

Moreover, A is injective whenever

$$\mathcal{R}(f) = \mathcal{R}\left(\hat{f} : G/[K^G, H] \to G/[K^G, H]\right),$$

where $\hat{f}$ is induced by $f : G \to G$.

COROLLARY 1.4 [2]. *If $\mathrm{Fix}(\bar{f}) = \{1\}$ then*

$$i_* : \mathcal{R}(f_H) \to q_*^{-1}([1]_{\bar{f}}) \subset \mathcal{R}(f)$$

is a bijection.

How do we deal with $q_*^{-1}([\bar{\alpha}]_{\bar{f}}) \subset \mathcal{R}(f)$ for an arbitrary $\alpha \in G$?

LEMMA 1.5 [2]. *For any $\alpha \in G$, let φ_α denote the endomorphism of G defined by $\varphi_\alpha(g) := \alpha f(g)\alpha^{-1}$ for all $g \in G$, and let $\varphi_{\alpha H} : H \to H$ and $\bar{\varphi}_{\bar{\alpha}} : \bar{G} \to \bar{G}$ be its restriction and projection respectively. Then there is a canonical bijection of the Reidemeister sets of φ_α and f, denoted by $\alpha_* : \mathcal{R}(\varphi_\alpha) \to \mathcal{R}(f)$, given by $\alpha_*([g]_{\varphi_\alpha}) = [g\alpha]_f$.*

Moreover, we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccccc}
(\mathcal{R}(\varphi_{\alpha H}), [1]_{\varphi_{\alpha H}}) & \xrightarrow{i_*} & (\mathcal{R}(\varphi_\alpha), [1]_{\varphi_\alpha}) & \xrightarrow{q_*} & (\mathcal{R}(\bar{\varphi}_{\bar{\alpha}}), [1]_{\bar{\varphi}_{\bar{\alpha}}}) & \longrightarrow & 1 \\
 & & \Big\downarrow{\alpha_*} & & \Big\downarrow{\bar{\alpha}_*} & & \\
 & & (\mathcal{R}(f), [\alpha]_f) & \xrightarrow{q_*} & (\mathcal{R}(\bar{f}), [\bar{\alpha}]_{\bar{f}}) & \longrightarrow & 1.
\end{array}$$

From [6], we introduce our's results on the Reidemeister orbit sets. Suppose $f : G \to G$ is an endomorphism.

DEFINITION 1.6 [6]. Let $n > 0$ be a given integer. Then f acts on the Reidemeister set $\mathcal{R}(f^n)$ by $[\gamma]_{f^n} \overset{f}{\mapsto} [f(\gamma)]_{f^n}$. The f-orbit of a Reidemeister class $[\gamma]_{f^n}$ will be called a Reidemeister n-orbit, denoted by $[\gamma]_f^{(n)}$. The set of all such Reidemeister f-orbits will be called the Reidemeister n-orbit set of f, denoted by $\mathcal{RO}^{(n)}(f)$.

The length of the orbit $[\gamma]_f^{(n)}$ is the smallest integer $\ell > 0$ such that $[\gamma]_{f^n} = [f^\ell(\gamma)]_{f^n}$. Clearly ℓ divides n.

REMARK 1.7 [6]. $\mathcal{RO}^{(n)}(f)$ is the set of equivalence classes in G of the following equivalence relation: $g, g' \in G$ are equivalent if and only if

$$(*) \qquad g' = \gamma f^i(g) f^n(\gamma^{-1}) \quad \text{for some } i \geq 0 \text{ and } \gamma \in G.$$

Let $H \subset G$ be an f-invariant normal subgroup. The short exact sequence

$$1 \to H \xrightarrow{i} G \xrightarrow{q} \bar{G} \to 1$$

gives us an exact sequence in the category of pointed sets

$$(\mathcal{RO}^{(n)}(f_H), [1]_{f_H}^{(n)}) \xrightarrow{i_*} (\mathcal{RO}^{(n)}(f), [1]_f^{(n)}) \xrightarrow{q_*} (\mathcal{RO}^{(n)}(\bar{f}), [1]_{\bar{f}}^{(n)}) \to 1$$

of Reidemeister orbit sets.

The following lemma is very useful for computing Reidemeister orbit sets.

LEMMA 1.8 [6]. *Suppose $n > 0$ and $g \in G$ are given. Suppose the orbit $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$ has length ℓ, and let $m := n/\ell$. We have a commutative diagram of exact sequences in the category of pointed sets:*

$$\begin{array}{ccccc}
(\mathcal{RO}^{(m)}(f^\ell), [g]_{f^\ell}^{(m)}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\bar{f}^\ell), [\bar{g}]_{\bar{f}^\ell}^{(m)}) & \longrightarrow & 1 \\
\sigma \downarrow & & \downarrow \bar{\sigma} & & \\
(\mathcal{RO}^{(n)}(f), [g]_f^{(n)}) & \xrightarrow{q_*} & (\mathcal{RO}^{(n)}(\bar{f}), [\bar{g}]_{\bar{f}}^{(n)}) & \longrightarrow & 1,
\end{array}$$

where the vertical maps σ and $\bar{\sigma}$ are induced by inclusions, and they are surjective.

Furthermore, σ restricts to a bijection

$$\sigma : q_*^{-1}([\bar{g}]_{\bar{f}^\ell}^{(m)}) \to q_*^{-1}([\bar{g}]_{\bar{f}}^{(n)}).$$

2. Alternative approach on the mapping torus

DEFINITION 2.1. Motivated by the fundamental group of the mapping torus, we define, for an endomorphism $f : G \to G$, the suspension group

$$\widetilde{G}_f := \langle G, z \mid z^{-1}gz = f(g),\ \forall g \in G \rangle.$$

Note that when f is an automorphism, $\widetilde{G}_f$ is a semi-direct product $\mathbf{Z} \ltimes G$.

REMARK 2.2. The natural inclusion $\iota : G \to \widetilde{G}_f$ is not necessarily injective. Its kernel is $\cup_{j>0}\ker(f^j)$. Every elements in $\widetilde{G}_f$ can be written in the form $z^i g z^{-j}$ with $g \in G$ and $i, j \geq 0$.

The following fact is taken from the paper [5].

PROPOSITION 2.3. *Two Reidemeister classes $[g]_{f^n}$ and $[g']_{f^n}$ in $\mathcal{R}(f^n)$ are in the same f-orbit if and only if the elements gz^{-n} and $g'z^{-n}$ are conjugate in $\widetilde{G}_f$.*

In other words, there is an injection

$$\alpha^{(n)} : \mathcal{RO}^{(n)}(f) \to \widetilde{G}_{fc}, \quad [g]_f^{(n)} \mapsto [gz^{-n}]_c,$$

where $\widetilde{G}_{fc}$ is the set of conjugacy classes in $\widetilde{G}_f$.

PROOF. It is also directly proved as follows: Suppose $[g]_f^{(n)} = [g']_f^{(n)} \in \mathcal{RO}^{(n)}(f)$. By Remark 1.7, there exists $\gamma \in G$ and $i \geq 0$ such that

$$g' = \gamma f^i(g) f^n(\gamma^{-1}) = \gamma z^{-i} g z^i z^{-n} \gamma^{-1} z^n,$$

so we have $g'z^{-n} = (\gamma z^{-i})(gz^{-n})(\gamma z^{-i})^{-1}$.

On the other hand, suppose that gz^{-n} and $g'z^{-n}$ are conjugate in $\widetilde{G}_f$. Then $g'z^{-n} = (z^i \delta z^{-j})(gz^{-n})(z^i \delta z^{-j})^{-1}$ for some $\delta \in G$ and $i, j \geq 0$, so we get

$$\delta f^j(g) f^n(\delta^{-1}) = z^{-i} g' z^i = f^i(g').$$

This shows that $[g']_f^{(n)} = [f^i(g')]_f^{(n)} = [g]_f^{(n)} \in \mathcal{RO}^{(n)}(f)$. □

Now let $f : G \to G$ be an endomorphism and $H \subset G$ be an f-invariant normal subgroup, as in the previous section. The short exact sequence

$$1 \to H \xrightarrow{i} G \xrightarrow{q} \bar{G} \to 1$$

induces an exact sequence

$$1 \to \widetilde{H} \xrightarrow{\tilde{i}} \widetilde{G}_f \xrightarrow{\tilde{q}} \widetilde{G}_{\bar{f}} \to 1,$$

where

$$\widetilde{G}_{\bar{f}} := \langle \bar{G}, z \mid z^{-1}\bar{g}z = \bar{f}(\bar{g}),\ \forall \bar{g} \in \bar{G} \rangle,$$

the homomorphism $\tilde{q} : \widetilde{G}_f \to \widetilde{G}_{\bar{f}}$ is given by $\tilde{q}(z) = z$ and $\tilde{q}(g) = \bar{g} = q(g)$. The kernel of $\tilde{q}$ is $\widetilde{H} = \{z^j h z^{-j} \mid h \in H, j \geq 0\}$.

REMARK 2.4. (Observation) The natural inclusion $H \xrightarrow{\iota} \widetilde{H}$ is injective (resp. surjective) if $f : G \to G$ is injective (resp. surjective). Hence $H \xrightarrow{\iota} \widetilde{H}$ is an isomorphism if $f : G \to G$ is an automorphism.

Now we will apply the concept of the length of orbits to the mapping torus. Let $f : G \to G$ be an endomorphism. If $\ell | n$, then by definition 2.1 the suspension group for f^ℓ is

$$\widetilde{G}_{f^\ell} := \langle G, z \mid z^{-1}gz = f^\ell(g),\ \forall g \in G \rangle.$$

Applying Proposition 2.3 to the endomorphism f^ℓ, we have

PROPOSITION 2.5. *If $m = n/\ell$, then there is an injection*

$$\alpha^{(m)} : \mathcal{RO}^{(m)}(f^\ell) \to \widetilde{G}_{f^\ell c}, \quad [g]^{(m)}_{f^\ell} \mapsto [gz^{-m}]_c.$$

PROOF. Suppose $[g]^{(m)}_{f^\ell} = [g']^{(m)}_{f^\ell} \in \mathcal{RO}^{(m)}(f^\ell)$. By Remark 1.7, there exists $\gamma \in G$ and $i \geq 0$ such that

$$g' = \gamma (f^\ell)^i(g)(f^\ell)^m(\gamma^{-1}) = \gamma(z^{-i}gz^i)(z^{-m}\gamma^{-1}z^m),$$

so we have $g'z^{-m} = (\gamma z^{-i})(gz^{-m})(\gamma z^{-i})^{-1}$.

On the other hand, suppose that gz^{-m} and $g'z^{-m}$ are conjugate in $\widetilde{G}_{f^\ell}$. Then

$$\begin{aligned} g'z^{-m} &= (z^i\delta z^{-j})(gz^{-m})(z^i\delta z^{-j})^{-1} \\ &= z^i\delta(z^{-j}gz^j)(z^{-m}\delta^{-1})z^{-i} \\ &= z^i\delta(f^\ell)^j(g)f^n(\delta^{-1})z^{-m}z^{-i} \end{aligned}$$

for some $\delta \in G$ and $i, j \geq 0$, so we get

$$\delta(f^\ell)^j(g)f^n(\delta^{-1}) = z^{-i}g'z^i = (f^\ell)^i(g').$$

This shows that $[g']^{(m)}_{f^\ell} = [(f^\ell)^i(g')]^{(m)}_{f^\ell} = [g]^{(m)}_{f^\ell} \in \mathcal{RO}^{(m)}(f^\ell)$. □

The proposition 2.5 tells us the following:

PROPOSITION 2.6. *Suppose $n > 0$ and $g \in G$ are given. If $m = n/\ell$, then we have a commutative diagram of exact sequences in the category of pointed sets:*

$$\begin{array}{ccccc}
(\mathcal{RO}^{(m)}(f^\ell), [g]_{f^\ell}^{(m)}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\bar{f}^\ell), [\bar{g}]_{\bar{f}^\ell}^{(n)}) & \longrightarrow & 1 \\
\alpha^{(m)} \downarrow & & \downarrow \bar{\alpha}^{(m)} & & \\
(\tilde{G}_{f^\ell c}, [gz^{-m}]_c) & \xrightarrow{\tilde{q}_*} & (\bar{G}_{\bar{f}^\ell c}, [\bar{g}z^{-m}]_c) & \longrightarrow & 1,
\end{array}$$

where the vertical maps are injective. The images of the vertical maps are the subsets of elements whose z-exponents are $-m$.

Indeed, we have a bijection

$$\alpha^{(m)} : q_*^{-1}([\bar{g}]_{\bar{f}^\ell}^{(m)}) \to \tilde{q}_*^{-1}([\bar{g}z^{-m}]_c).$$

PROOF. First we will show that $\alpha^{(m)}(q_*^{-1}([\bar{g}]_{\bar{f}^\ell}^{(m)})) \subset \tilde{q}_*^{-1}([\bar{g}z^{-m}]_c)$. Suppose that $[y]_{f^\ell}^{(m)} \in q_*^{-1}([\bar{g}]_{\bar{f}^\ell}^{(m)})$. Then we have $[\bar{y}]_{\bar{f}^\ell}^{(m)} = [\bar{g}]_{\bar{f}^\ell}^{(m)}$, and hence

$$\begin{aligned}
\bar{y} &= \bar{\gamma}\bar{(f^\ell)}^i(\bar{g})\bar{(f^\ell)}^m(\bar{\gamma}^{-1}) = \bar{\gamma}z^{-i}\bar{g}z^iz^{-m}\bar{\gamma}^{-1}z^m \\
&= (\bar{\gamma}z^{-i})(\bar{g}z^{-m})(\bar{\gamma}z^{-i})^{-1}z^m
\end{aligned}$$

for some $i \geq 0$ and $\bar{\gamma} \in \bar{G}$. Therefore $\tilde{q}_* \circ \alpha^{(m)}([y]_{f^\ell}^{(n)}) = [\bar{g}z^{-m}]_c$.

Suppose $[z^syz^{-t}]_c \in \tilde{q}_*^{-1}([\bar{g}z^{-m}]_c)$ for $s, t \geq 0$ and $y \in G$. Then we have

$$z^s\bar{y}z^{-t} = (z^i\bar{\delta}z^{-j})(\bar{g}z^{-m})(z^i\bar{\delta}z^{-j})^{-1} = z^i\bar{\delta}\bar{(f^\ell)}^j(\bar{g})\bar{(f^\ell)}^m(\bar{\delta}^{-1})z^{-(m+i)}$$

for some $i, j \geq 0$ and $\delta \in G$. Thus $s = i$, $t = m + i$ and $\bar{y} = \bar{\delta}\bar{(f^\ell)}^j(\bar{g})\bar{f}^n(\bar{\delta}^{-1})$. So

$$q_*([y]_{f^\ell}^{(n)}) = [\bar{y}]_{\bar{f}^\ell}^{(m)} = [\bar{\delta}\bar{(f^\ell)}^j(\bar{g})\bar{f}^n(\bar{\delta}^{-1})]_{\bar{f}^\ell}^{(m)} = [\bar{g}]_{\bar{f}^\ell}^{(m)}$$

and

$$\alpha^{(m)}([y]_{f^\ell}^{(m)}) = [yz^{-m}]_c = [z^syz^{-t}]_c.$$

This shows that the restriction map $\alpha^{(m)}$ is surjective. □

A conjugacy class is a Reidemeister class of the identity automorphism. By Lemma 1.5 we have

PROPOSITION 2.7. *Suppose $n > 0$ and $g \in G$ are given. Suppose $m = n/\ell$. Let $\varphi_{m,g} : \widetilde{G}_{f^\ell} \to \widetilde{G}_{f^\ell}$ be defined by $\varphi_{m,g} : y \mapsto gz^{-m}yz^m g^{-1}$, $\forall y \in \widetilde{G}_{f^\ell}$, and let $\varphi'_{m,g} : \widetilde{H} \to \widetilde{H}$ and $\bar{\varphi}_{m,\bar{g}} : \widetilde{G}_{\bar{f}^\ell} \to \widetilde{G}_{\bar{f}^\ell}$ be its restriction and projection respectively. Then we have a commutative diagram of exact sequences in the category of pointed sets:*

$$\begin{array}{ccccccc} (\mathcal{R}(\varphi'_{m,g}),[1]_{\varphi'_{m,g}}) & \xrightarrow{\tilde{i}_*} & (\mathcal{R}(\varphi_{m,g}),[1]_{\varphi_{m,g}}) & \xrightarrow{\tilde{q}_*} & (\mathcal{R}(\bar{\varphi}_{m,\bar{g}}),[1]_{\bar{\varphi}_{m,\bar{g}}}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle (gz^{-m})_*} & & \downarrow{\scriptstyle (\bar{g}z^{-m})_*} & & \\ & & (\widetilde{G}_{f^\ell c},[gz^{-m}]_c) & \xrightarrow{\tilde{q}_*} & (\bar{G}_{\bar{f}^\ell c},[\bar{g}z^{-m}]_c) & \longrightarrow & 1, \end{array}$$

where the vertical maps are bijections. In particular, the middle one gives a bijection

$$(gz^{-m})_* : \tilde{q}_*^{-1}([1]_{\bar{\varphi}_{m,\bar{g}}}) \to \tilde{q}_*^{-1}([\bar{g}z^{-m}]_c).$$

PROOF. Applying Lemma 1.5 to the identity automorphism $1_{\widetilde{G}_{f^\ell}} : \widetilde{G}_{f^\ell} \to \widetilde{G}_{f^\ell}$, then we can easily get the conclusion. □

Combining Proposition 2.6 and 2.7 we have

COROLLARY 2.8. *Suppose $n > 0$ and $g \in G$ are given. Suppose $m = n/\ell$. Let $\varphi_{m,g} : \widetilde{G}_{f^\ell} \to \widetilde{G}_{f^\ell}$ be as before. Then we have a commutative diagram of exact sequences in the category of pointed sets:*

$$\begin{array}{ccccccc} & & (\mathcal{RO}^{(m)}(f^\ell),[g]^{(m)}_{f^\ell}) & \xrightarrow{q_*} & (\mathcal{RO}^{(m)}(\bar{f}^\ell),[\bar{g}]^{(m)}_{\bar{f}^\ell}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \beta^{(m)}} & & \downarrow{\scriptstyle \bar{\beta}^{(m)}} & & \\ (\mathcal{R}(\varphi'_{m,g}),[1]_{\varphi'_{m,g}}) & \xrightarrow{\tilde{i}_*} & (\mathcal{R}(\varphi_{m,g}),[1]_{\varphi_{m,g}}) & \xrightarrow{\tilde{q}_*} & (\mathcal{R}(\bar{\varphi}_{m,\bar{g}}),[1]_{\bar{\varphi}_{m,\bar{g}}}) & \longrightarrow & 1, \end{array}$$

where the vertical maps are injective, defined by $\beta^{(m)} : [x]^{(m)}_{f^\ell} \mapsto [xg^{-1}]_{\varphi_{m_g}}$. The images of the vertical maps are the subsets of elements whose z-exponents are 0.

In particular, we have a bijection

$$\beta^{(m)} : q_*^{-1}([\bar{g}]^{(m)}_{\bar{f}^\ell}) \to \tilde{q}_*^{-1}([1]_{\bar{\varphi}_{m,\bar{g}}}).$$

PROOF. Define $\beta^{(m)} = (gz^{-m})_*^{-1} \circ \alpha^{(m)}$, then the restriction map is bijective. □

The results in Section 1 can be applied to the endomorphism $\varphi_{m,g}$. We have some additive formulae for Reidemeister orbit sets relative to the Reidemeister sets on suspension groups.

THEOREM 2.9. *Suppose $n > 0$ and $g \in G$ are given. Suppose the orbit $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$ has length ℓ_g, and let $m_g := n/\ell_g$. Let $\varphi_{m_g,g} : \widetilde{G}_{f^{\ell_g}} \to \widetilde{G}_{f^{\ell_g}}$ be as in Proposition 2.7. For any $\varphi_{m_g,g}$-invariant subgroup $\widetilde{K}_g$ of $\widetilde{G}_{f^{\ell_g}}$ such that $\widetilde{K}_g\widetilde{H} = \tilde{q}^{-1}\mathrm{Fix}(\bar{\varphi}_{m_g,\bar{g}})$, let $T_{\varphi_{m_g,g}}(\widetilde{K}_g)$ be as in Proposition 1.2. Then*

$$\sharp\mathcal{RO}^{(n)}(f) \geq \sum_{[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})} \sharp\mathcal{R}\left(\widehat{\varphi'_{m_g,g}} : \widetilde{H}/T_{\varphi_{m_g,g}}(\widetilde{K}_g) \to \widetilde{H}/T_{\varphi_{m_g,g}}(\widetilde{K}_g)\right)$$

and the equality holds if

$$\mathcal{R}(\varphi_{m_g,g}) = \mathcal{R}\left(\widehat{\varphi_{m_g,g}} : \widetilde{G}_{f^{\ell_g}}/[\widetilde{K}_g^{\widetilde{G}_{f^{\ell_g}}}, \widetilde{H}] \to \widetilde{G}_{f^{\ell_g}}/[\widetilde{K}_g^{\widetilde{G}_{f^{\ell_g}}}, \widetilde{H}]\right)$$

for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$.

PROOF. By Lemma 1.3, for any $\varphi_{m_g,g}$-invariant subgroup $\widetilde{K}_g$ of $\widetilde{G}_{f^{\ell_g}}$ such that

$$\widetilde{K}_g\widetilde{H} = \tilde{q}^{-1}\mathrm{Fix}(\bar{\varphi}_{m_g,\bar{g}}),$$

there exists a surjection

$$A_g : \tilde{q}_*^{-1}([1]_{\bar{\varphi}_{m_g,\bar{g}}}) \to \mathcal{R}\left(\widehat{\varphi'_{m_g,g}} : \widetilde{H}/T_{\varphi_{m_g,g}}(\widetilde{K}_g) \to \widetilde{H}/T_{\varphi_{m_g,g}}(\widetilde{K}_g)\right)$$

defined by $A_g([h]_{\varphi_{m_g,g}}) = [p_g(h)]_{\widehat{\varphi'_{m_g,g}}}$ for all $h \in H$, where p_g is the projection $p_g : \widetilde{H} \to \widetilde{H}/T_{\varphi_{m_g,g}}(\widetilde{K}_g)$; A_g is injective whenever $\mathcal{R}(\varphi_{m_g,g}) = \mathcal{R}(\widehat{\varphi_{m_g,g}})$. Now, $\mathcal{RO}^{(n)}(f)$ is the disjoint union of $q_*^{-1}([\bar{g}]_{\bar{f}}^{(n)})$ for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$. By Lemma 1.8 and Corollary 2.8, there is a surjection

$$A_g \circ \beta^{(m)} \circ \sigma^{-1} : q_*^{-1}([\bar{g}]_{\bar{f}}^{(n)}) \to \mathcal{R}(\widehat{\varphi'_{m_g,g}})$$

for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$, which proves the desired inequality. Moreover, if $\mathcal{R}(\varphi_{m_g,g}) = \mathcal{R}(\widehat{\varphi_{m_g,g}})$ for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$, by Lemma 1.2 the equality holds. □

Corollary 2.10. *If* $\mathrm{Fix}(\bar{\varphi}_{m_g,\bar{g}}) = \{1\}$ *for all* $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$, *then*

$$\sharp\mathcal{RO}^{(n)}(f) = \sum_{[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})} \sharp\mathcal{R}(\varphi'_{m_g,g}).$$

Proof. It suffices to define $\widetilde{K}_g = \{1\}$ for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$. □

3. The case of automorphisms

In view of the observation 2.4, it is instructive to start our analysis by assuming that $f : G \to G$ is an automorphism, so that $\iota : H \to \widetilde{H}$ is bijective and $\iota : G \to \widetilde{G}_f$ is injective. Then we have

Proposition 3.1. *Assume that* $f : G \to G$ *is an automorphism. Suppose* $m = n/\ell$. *Let* $\varphi_{m,g} : \widetilde{G}_{f^\ell} \to \widetilde{G}_{f^\ell}$ *be as before. Let* $\psi_g : G \to G$ *be defined by* $\psi_g : y \mapsto gf^n(y)g^{-1}$, $\forall y \in G$, *and let* $\psi'_g : H \to H$ *and* $\bar{\psi}_{\bar{g}} : \bar{G} \to \bar{G}$ *be its restriction and projection respectively. Then we have a commutative diagram of exact sequences in the category of pointed sets:*

$$\begin{array}{ccccccc}
(\mathcal{R}(\psi'_g), [1]_{\psi'_g}) & \xrightarrow{i_*} & (\mathcal{R}(\psi_g), [1]_{\psi_g}) & \xrightarrow{q_*} & (\mathcal{R}(\bar{\psi}_{\bar{g}}), [1]_{\bar{\psi}_{\bar{g}}}) & \longrightarrow & 1 \\
\| & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \bar{\iota}_*} & & \\
(\mathcal{R}(\varphi'_{m,g}), [1]_{\varphi'_{m,g}}) & \xrightarrow{\tilde{i}_*} & (\mathcal{R}(\varphi_{m,g}), [1]_{\varphi_{m,g}}) & \xrightarrow{\tilde{q}_*} & (\mathcal{R}(\bar{\varphi}_{m,\bar{g}}), [1]_{\bar{\varphi}_{m,\bar{g}}}) & \longrightarrow & 1,
\end{array}$$

where the vertical maps are induced by inclusions. Clearly the middle one gives a surjection

$$\iota_* : q_*^{-1}([1]_{\bar{\psi}_{\bar{g}}}) \to \tilde{q}_*^{-1}([1]_{\bar{\varphi}_{m,\bar{g}}}).$$

Proof. If $[y_1]_{\psi_g} = [y_2]_{\psi_g}$ for $y_1, y_2 \in G$, then there exists $y \in G$ such that

$$\begin{aligned}
y_2 &= yy_1\psi_g(y^{-1}) \\
&= yy_1g(f^\ell)^m(y^{-1})g^{-1} \\
&= yy_1g(z^{-m}y^{-1}z^m)g^{-1} \\
&= yy_1\varphi_{m,g}(y^{-1}).
\end{aligned}$$

Thus we have $[y_1]_{\varphi_{m,g}} = [y_2]_{\varphi_{m,g}}$ in $\mathcal{R}(\varphi_{m,g})$. This shows that ι_* is well-defined.

On the other hand, since the inclusion map $\iota : H \to \widetilde{H}$ is bijective, we may assume that $\widetilde{H}$ is the same as H. If $[h_1]_{\varphi'_{m,g}} = [h_2]_{\varphi'_{m,g}}$ for $h_1, h_2 \in H$, then there exists $h \in H$ such that

$$\begin{aligned} h_2 &= hh_1\varphi'_{m,g}(h^{-1}) \\ &= hh_1(gz^{-m}h^{-1}z^m g^{-1}) \\ &= hh_1\psi'_g(h^{-1}). \end{aligned}$$

Thus $[h_1]_{\psi'_g} = [h_2]_{\psi'_g}$. Therefore we have $\mathcal{R}(\psi'_g) = \mathcal{R}(\varphi'_{m,g})$. □

Combining Corollary 2.10 and Proposition 3.1 we have

COROLLARY 3.2. *Assume that $f : G \to G$ is an automorphism. Suppose $n > 0$ and $g \in G$ are given. Suppose the orbit $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$ has length ℓ_g, and let $m_g := n/\ell_g$. Let $\varphi_{m_g,g} : \widetilde{G}_{f^{\ell_g}} \to \widetilde{G}_{f^{\ell_g}}$ be as in Proposition 2.7. Let ψ_g be as before. If $\mathrm{Fix}(\bar{\varphi}_{m_g,\bar{g}}) = \{1\}$ for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$, then*

$$\sharp\mathcal{RO}^{(n)}(f) = \sum_{[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})} \sharp\mathcal{R}(\psi'_g).$$

PROOF. It is also directly proved as follows: Since $\mathrm{Fix}(\bar{\varphi}_{m,\bar{g}}) = \{1\}$, we have $\mathrm{Fix}(\bar{\psi}_{\bar{g}}) = \{1\}$. Then by [3, Theorem 1.8] we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccccccc} 1 \to & (\mathcal{R}(\psi'_g), [1]_{\psi'_g}) & \xrightarrow{i_*} & (\mathcal{R}(\psi_g), [1]_{\psi_g}) & \xrightarrow{q_*} & (\mathcal{R}(\bar{\psi}_{\bar{g}}), [1]_{\bar{\psi}_{\bar{g}}}) & \to 1 \\ & \| & & \downarrow \iota_* & & \downarrow \bar{\iota}_* & \\ 1 \to & (\mathcal{R}(\varphi'_{m_g,g}), [1]_{\varphi'_{m_g,g}}) & \xrightarrow{\tilde{i}_*} & (\mathcal{R}(\varphi_{m_g,g}), [1]_{\varphi_{m_g,g}}) & \xrightarrow{\tilde{q}_*} & (\mathcal{R}(\bar{\varphi}_{m_g,\bar{g}}), [1]_{\bar{\varphi}_{m_g,\bar{g}}}) & \to 1, \end{array}$$

where the vertical maps are induced by inclusions. In particular, the middle one gives a bijection from $q_*^{-1}([1]_{\bar{\psi}_{\bar{g}}})$ to $\tilde{q}_*^{-1}([1]_{\bar{\varphi}_{m_g,\bar{g}}})$. By Lemma 1.8 and Corollary 2.8, we get the conclusion $|q_*^{-1}([\bar{g}]_{\bar{f}}^{(n)})| = \sharp\mathcal{R}(\psi'_g)$ for all $[\bar{g}]_{\bar{f}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{f})$. □

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