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THE HARMONIC DISTRIBUTIONS ON LIE GROUP

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Abstract. Harmonic distribution is the distribution which has the minimal value of functional called energy. In this paper it is shown as a specific distribution of semisimple Lie group whose manifold is compact.

1. Introduction

Let M and N be complete Riemannian manifolds. Assume that M is compact. A smooth map $f : M \to N$ is called harmonic if it is a critical point of the energy functional. We consider when N is a Grassmannian bundle over M and $f : M \to N$ is a smooth map which happens to be a section of this map. If N is a tangent bundle over M with a connection type metric, then f is a harmonic map if and only if it is a parallel section ([4]). And in [8], a distribution is a section of the Grassmannian bundle $G_p(M)$, or the immersion $M \to G_p(M)$ given by the section. We call it the Gauss map of the p-dimensional distribution. Hence the Gauss map of p-dimensional distribution on a Riemannian manifold M is called harmonic if it is a harmonic map from the manifold into its Grassmannian bundle $G_p(M)$. With the above definition, it becomes harmonic distribution when the Gauss map is a harmonic map. The definition of harmonic distributions, or harmonic sections, is used by other authors [4], [5], [6], [8] in a similar context but for a different concept. They call a section harmonic if its vertical energy is stationary with respect to vertical variations. In our case, however, we are more interested in the harmonic map itself, and therefore we have to consider both vertical and horizontal parts. Also in [1], they provide new examples of harmonic distributions when the base manifolds are homogeneous spaces and the integral submanifolds are totally geodesic. In [8], if G is

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a compact semisimple Lie group of rank k, then the Cartan subalgebra of T_eG is not only a minimal but also harmonic. In this paper, it will be shown that any invariant subspace of compact semisimple Lie group is harmonic. In the following section, we will consider the geometry of Grassmannian and obtain a necessary and sufficient condition for any smooth map between Riemannian manifolds to be a harmonic map. We will also obtain the tension field of Gauss map on Riemannian manifolds. In section 3, the proof of the main theorem is presented. We refer to [2], [3], [7] for basic tools and more detailed description of harmonic maps between Riemannian manifolds.

2. Grassmannian bundle and tension field

The Grassmannian bundle $G_p(M)$ is isomorphic to the quotient of the orthonormal frame bundle $O(M)$ by the action of $SO(p) \times SO(n-p)$. The orthonormal frame bundle $\pi : O(M) \to M$ is a fibre bundle with structure group $SO(n)$. Both canonical form and connection form are globally defined on $O(M)$,

$$\theta = (\theta_A), \ \omega = (\omega_{AB}), \ \omega_{AB} = -\omega_{BA}.$$

They form a basis of the cotangent bundle of $O(M)$ and satisfy the structure equation;

$$d\theta_A = -\omega_{AB} \wedge \theta_B$$

$$d\omega_{AB} = -\omega_{AC} \wedge \omega_{CB} + \Omega_{AB}$$

and the curvature forms Ω_{AB}. Let

$$\overline{\pi} : G_p(M) \to M$$

denote the Grassmannian bundle over M of p-dimensional distributions. It is a fibre bundle over M associated to $O(M)$ with standard fibre of the Grassmann manifold

$$G_p(M) = SO(n)/SO(p) \times SO(n-p)$$

on which $SO(n)$ acts on the left by multiplication. Since there exists a local cross section of the principal fibre bundle

$$SO(p) \times SO(n-p) \to SO(n) \to G_p(M),$$

we can pull the canonical forms θ_A and connection form ω_{AB} down to get local forms on $G_p(M)$. Then we can use this method to give a Riemannian metric on $G_p(M)$.

PROPOSITION 1. *The quadratic tensor*

$$ds^2 = \sum_i \theta_i^2 + \sum_{i,\alpha} \omega_{i,\alpha}^2$$

is globally defined metric tensor on $G_p(M)$.

Let $\{u_1, u_2, \cdots, u_n\}$ denote the standard basis of $\mathbb{R}^n$ and for the origin of $G_p(n)$ we choose the subspace of $\mathbb{R}^n$ spanned by $u_1, u_2, \cdots, u_p$, and denote by

$$o = u_1 \wedge \cdots \wedge u_p.$$

If $U \subset G_p(M)$ is an open subset containing o and

$$u : U \to O(M)$$

is any local section , then

$$\{\varphi_B \mid \varphi_B = u^*\theta_A \text{ if } B = A, \varphi_\mu = u^*\omega_{i\alpha} \text{ if } B = \mu = (i\alpha)\}$$

is an orthonormal coframe from ds^2 on U. From the structure equation of $O(M)$, we find that the pull-back by u^* of the forms

$$\varphi_{AB} = \omega_{AB} + \frac{1}{2} R_{i\alpha AB}\omega_{AB},$$

$$\varphi_{\mu B} = \frac{1}{2} R_{i\alpha BA}\theta_A = -\varphi_{B\mu}, \ \mu = (i\alpha),$$

$$\varphi_{\mu\nu} = \delta_{\alpha\beta}\omega_{ij} + \delta_{ij}\omega_{\alpha\beta}, \ \mu = (i\alpha), \ \nu = (j\beta)$$

gives the Levi-Civita connection forms of ds^2 with respect to this orthonormal coframe field.

A distribution on a Riemannian manifold assigns to every point $x \in M$, a subspace $\mathcal{D}_x$ of its tangent space T_xM, where $\mathcal{D}_x$ depends smoothly on x. If $\mathcal{D}$ is a distribution on M with $\dim \mathcal{D}_x = p$, then we can immerse M into the Grassmannian bundle $G_p(M)$ of M by sending each $x \in M$ to $\mathcal{D}_x$. We denote this map by ψ. Consider the local orthonormal frame of M, such that $e_i \in \mathcal{D}$, $(1 \leq i \leq p)$ and $e_\alpha \in \mathcal{D}^\perp$, $(p+1 \leq \alpha \leq n)$. We denote this frame by e_A, $(1 \leq A \leq n)$; it is a section of $O(M)$. Pulling back the canonical form and connection form through this section, we get the local coframe ω_A and Riemannian connection form ω_{AB} on M. Using the Christofell symbols, we can write

$$\omega_{AB} = \sum_C \gamma_{ABC}\omega_C.$$

From proposition 1, The induced metric on $G_p(M)$ is

$$ds_G^2 = \sum_i \omega_i^2 + \sum_{i,\alpha} \omega_{i,\alpha}^2,$$

and its pullback via ψ is

$$\psi^*(ds_G^2) = (\delta_{AB} + \sum_i \gamma_{i\alpha A}\gamma_{i\alpha B})\omega_A\omega_B.$$

We calculate the tension field of the mapping $\psi_{\mathcal{D}} : M \to G_p(M)$.

PROPOSITION 2. ([8], Proposition 5) *The tension field of $\psi_{\mathcal{D}}$ is*

$$\tau(\psi_{\mathcal{D}}) = \sum_{i,\alpha,A,B} R_{i\alpha AB}\ \gamma_{i\alpha B} \otimes E_A + \sum_{i,\alpha,B} \gamma_{i\alpha BB} \otimes E_{i\alpha}$$

where $\gamma_{i\alpha BC}\omega_C = d\gamma_{i\alpha B} + \gamma_{i\alpha D}\omega_{DB} + \gamma_{j\alpha B}\omega_{ji} + \gamma_{i\beta B}\omega_{\beta\alpha}$.

Above tension field formula is composed of horizontal and vertical parts, that is, $\tau(\psi_{\mathcal{D}}) = \tau^{\mathcal{H}}(\psi_{\mathcal{D}}) + \tau^{\mathcal{V}}(\psi_{\mathcal{D}})$. The horizontal tension field is $\tau^{\mathcal{H}}(\psi_{\mathcal{D}}) = \sum_{i,\alpha,A,B} R_{i\alpha AB}\ \gamma_{i\alpha B}$ and the vertical tension field is $\tau^{\mathcal{V}}(\psi_{\mathcal{D}}) = \sum_{i,\alpha,B} \gamma_{i\alpha BB}$.

3. Main theorem

Let G be a compact semisimple Lie group with rank k of dimension n, its Lie algebra $\mathfrak{g}$, consists of all left invariant vector fields, and let H be a Lie subgroup of dimension p, its Lie algebra $\mathfrak{h}$. According to the theory of root space decomposition we have $\mathfrak{g} = C \oplus C^{\perp}$ with respect to killing form where C is the cartan subalgebra of $\mathfrak{g}$ and $C^{\perp}$ is the root space of the adjoint representation of C. There exists a basis of $\mathfrak{g}$, namely $H_i,\ X_\alpha,\ \alpha \in \triangle$ satisfying;

$$\begin{aligned}
&[H_i,\ X_\alpha] = \alpha(H_i)X_\alpha,\\
&[X_\alpha,\ X_\beta] = N_{\alpha\beta}X_{\alpha+\beta},\\
&N_{\alpha\beta} = 0 \text{ if } \alpha + \beta \notin \triangle,\\
&[X_\alpha,\ X_{-\alpha}] \in C,
\end{aligned}$$

where $\triangle$ is the root of $\mathfrak{g}$. Let $\{e_1,\ e_2,\ \cdots\ , e_k,\ e_{k+1},\ \cdots\ , e_n\}$ be an orthonormal basis of $\mathfrak{g}$.

$$C = \langle e_1,\ \cdots\ , e_k\rangle,\ C^{\perp} = \langle e_{k+1},\ \cdots\ , e_n\rangle.$$

Consider the distribution generated by left translation of $\mathfrak{h}$. Then we can obtain the Grassmannian bundle $G_p(G)$ i.e $\psi_{\mathfrak{h}} : G \to G_p(G)$. From Proposition 2, the tension field of $\psi_{\mathfrak{h}}$ is

$$\tau(\psi_{\mathfrak{h}}) = \sum_{i,\alpha,A,B} R_{i\alpha AB}\ \gamma_{i\alpha B} \otimes E_A + \sum_{i,\alpha,B} \gamma_{i\alpha BB} \otimes E_{i\alpha}, \tag{1}$$

where $\gamma_{i\alpha BC}\omega_C = d\gamma_{i\alpha B} + \gamma_{i\alpha D}\omega_{DB} + \gamma_{j\alpha B}\omega_{ji} + \gamma_{i\beta B}\omega_{\beta\alpha}$.
With respect to the bi-invariant metric of G, we choose an orthonormal basis e_i, and its dual ω_i. The ω_i satisfy the Maurer-Cartan equation $d\omega_i = -\frac{1}{2}\sum_{j,k} C_{ijk}\omega_j \wedge \omega_k$ where the C_{ijk} are the structure constants of the Lie group. And the connection forms of the Lie group are $\omega_{ij} = \sum_k \gamma_{ijk}\omega_k$ where $\gamma_{ijk} = \frac{1}{2}(C_{ijk} - C_{jik} - C_{kij})$ are the so-called Christoffel's symbols. The fact that the tension field of a distribution on G vanishes will be proved. Throughout this section, the following index conventions are used,

$$1 \leq i,j \leq p \ , \ p+1 \leq \alpha,\beta \leq n \ , \ 1 \leq A,B \leq n.$$

Now we consider a distribution in C and $C^{\perp}$, respectively. First, we can express the formula (2) by using $[\cdot,\cdot]$.

LEMMA 1.

$$\begin{aligned} \tau(\psi_{\mathfrak{h}}) &= \sum_{i,\alpha,A,B} \frac{1}{8}\langle e_\alpha, [e_B, e_i]\rangle\langle e_A, [e_B, [e_i, e_\alpha]]\rangle \otimes E_A \\ &\quad + \sum_{i,j,\alpha,\beta,B} \frac{1}{4}\Big\{\langle e_\alpha, [e_B, e_j]\rangle\langle e_i, [e_B, e_j]\rangle \\ &\quad + \langle e_\beta, [e_B, e_i]\rangle\langle e_\alpha, [e_B, e_\beta]\rangle\Big\} \otimes E_{i\alpha} \end{aligned} \tag{2}$$

PROOF. Since G is a Lie group with bi-invariant metric, the following equations hold,

$$\nabla_X Y = \frac{1}{2}[X,Y], \text{ for all } X,Y \in \mathfrak{g}$$

$$\Leftrightarrow \langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle, \text{ for all } X,Y,Z \in \mathfrak{g}.$$

First, the horizontal part is

$$\gamma_{i\alpha B} = \frac{1}{2}\langle e_\alpha, [e_B, e_i]\rangle, \ R_{i\alpha AB} = \frac{1}{4}\langle e_A, [e_B, [e_i, e_\alpha]]\rangle,$$

and, the vertical part is

$$\gamma_{i\alpha BC}\omega_C = d\gamma_{i\alpha B} + \gamma_{i\alpha D}\omega_{DB} + \gamma_{j\alpha B}\omega_{ji} + \gamma_{i\beta B}\omega_{\beta\alpha}$$

$$\gamma_{i\alpha BC} = d\gamma_{i\alpha B}(e_C) + \gamma_{i\alpha D}\omega_{DB}(e_C) + \gamma_{j\alpha B}\omega_{ji}(e_C) + \gamma_{i\beta B}\omega_{\beta\alpha}(e_C).$$

Thus,

$$d\gamma_{i\alpha B}(e_C) = \frac{1}{2}\nabla_{e_C}\langle e_\alpha, [e_B, e_i]\rangle = 0,$$

$$\begin{aligned}\gamma_{i\alpha D}\omega_{DB}(e_C) &= \frac{1}{4}\langle e_\alpha,[e_B,e_i]\rangle\langle e_B,[e_C,e_D]\rangle,\\ \gamma_{j\alpha B}\omega_{ji}(e_C) &= \frac{1}{4}\langle e_\alpha,[e_B,e_j]\rangle\langle e_i,[e_C,e_j]\rangle,\\ \gamma_{i\beta B}\omega_{\beta\alpha}(e_C) &= \frac{1}{4}\langle e_\beta,[e_B,e_i]\rangle\langle e_\alpha,[e_C,e_\beta]\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\gamma_{i\alpha BC} &= \frac{1}{4}\Big\{\langle e_\alpha,[e_B,e_i]\rangle\langle e_B,[e_C,e_D]\rangle + \langle e_\alpha,[e_B,e_j]\rangle\langle e_i,[e_C,e_j]\rangle\\ &\quad +\langle e_\beta,[e_B,e_i]\rangle\langle e_\alpha,[e_C,e_\beta]\rangle\Big\}.\end{aligned}$$

We have

$$\gamma_{i\alpha BB} = \frac{1}{4}\Big\{\langle e_\alpha,[e_B,e_j]\rangle\langle e_i,[e_B,e_j]\rangle + \langle e_\beta,[e_B,e_i]\rangle\langle e_\alpha,[e_B,e_\beta]\rangle\Big\}.$$

In (3), let τ^H be a horizontal component and τ^V vertical component.

$$\tau^H = \sum_{i,\alpha,B}\frac{1}{8}\langle e_\alpha,[e_B,e_i]\rangle\langle e_A,[e_B,[e_i,e_\alpha]]\rangle \tag{3}$$

$$\tau^V = \sum_{j,\beta,B}\frac{1}{4}\Big\{\langle e_\alpha,[e_B,e_j]\rangle\langle e_i,[e_B,e_j]\rangle + \langle e_\beta,[e_B,e_i]\rangle\langle e_\alpha,[e_B,e_\beta]\rangle\Big\}. \tag{4}$$

At first, it will be shown that $\mathfrak{h}$ is harmonic for $\mathfrak{h} \subset C$.

LEMMA 2. *If a distribution $\mathfrak{h}$ is in the Cartan subalgebra i.e, $\mathfrak{h} \subset C(p<k)$, then $\mathfrak{h}$ is harmonic.*

PROOF. Let

$$\mathfrak{h} = \langle e_1,\ e_2,\ \cdots,e_p\rangle,\ \mathfrak{h}^\perp = \langle e_{p+1},\ \cdots,e_k,e_{k+1},\ \cdots,e_n\rangle.$$

First, we consider τ^H,

$$\begin{aligned}\tau^H &= \sum_{i,\alpha,B}\frac{1}{8}\langle e_\alpha,[e_B,e_i]\rangle\langle e_A,[e_B,[e_i,e_\alpha]]\rangle\\ &= -\sum_{i,B}\frac{1}{8}\Big\{\sum_{\alpha_1=p+1}^{k}\langle[e_{\alpha_1},e_i],e_B\rangle\langle e_A,[e_B,[e_i,e_{\alpha_1}]]\rangle\\ &\quad +\sum_{\alpha_2=k+1}^{n}\langle[e_{\alpha_2},e_i],e_B\rangle\langle e_A,[e_B,[e_i,e_{\alpha_2}]]\rangle\Big\}.\end{aligned}$$

Since C is abelian and e_i, e_{α_1} is in C, we have $[e_{\alpha_1}, e_i] = 0$

$$\tau^H = -\sum_{i,B} \frac{1}{8}\Big(\sum_{\alpha_2=k+1}^{n} \langle [e_{\alpha_2}, e_i], e_B\rangle \langle e_A, [e_B, [e_i, e_{\alpha_2}]]\rangle \Big).$$

Since e_{α_2} is a basis of root space and e_i is a basis of the Cartan subalgebra, we have $[e_{\alpha_2}, e_i] = \alpha_2(e_i)e_{\alpha_2}$, $\alpha_2 \in \triangle$. Thus

$$\tau^H = \sum_{i,\alpha_2,B} \frac{1}{8}\langle \alpha_2(e_i)e_{\alpha_2}, e_B\rangle \langle e_A, [e_B, \alpha_2(e_i)e_{\alpha_2}]\rangle, \ \alpha_2 \in \triangle.$$

Since $\langle \alpha_2(e_i)e_{\alpha_2}, e_B\rangle = \alpha_2(e_i)\delta_{\alpha_2 B}$, if $\delta_{\alpha_2 B} = 1$, then $[e_B, \alpha_2(e_i)e_{\alpha_2}] = 0$. Therefore, $\tau^H = 0$. Next,

$$\tau^V = \sum_{j,\beta,B} \frac{1}{4}\Big\{ \langle e_\alpha, [e_B, e_j]\rangle \langle e_i, [e_B, e_j]\rangle + \langle e_\beta, [e_B, e_i]\rangle \langle e_\alpha, [e_B, e_\beta]\rangle \Big\}.$$

Since $\langle e_\alpha, [e_B, e_j]\rangle \langle e_i, [e_B, e_j]\rangle = -\langle e_\alpha, [e_B, e_j]\rangle \langle [e_i, e_j], e_B\rangle$ and $[e_i, e_j] = 0$, we have

$$\begin{aligned}
\tau^V &= \sum_{\beta,B} \frac{1}{4}\langle e_\beta, [e_B, e_i]\rangle \langle e_\alpha, [e_B, e_\beta]\rangle \\
&= \sum_{B} \frac{1}{4}\Big\{ \sum_{\beta_1=p+1}^{k} \langle e_{\beta_1}, [e_B, e_i]\rangle \langle e_\alpha, [e_B, e_{\beta_1}]\rangle \\
&\quad + \sum_{\beta_2=k+1}^{n} \langle e_{\beta_2}, [e_B, e_i]\rangle \langle e_\alpha, [e_B, e_{\beta_2}]\rangle \Big\}.
\end{aligned}$$

Since e_{β_1}, e_i are bases of C, $\langle e_{\beta_1}, [e_B, e_i]\rangle = -\langle [e_{\beta_1}, e_i], e_B\rangle = 0$,

$$\tau^V = \sum_{\beta_2,B} \frac{1}{4}\langle e_{\beta_2}, [e_B, e_i]\rangle \langle e_\alpha, [e_B, e_{\beta_2}]\rangle.$$

If $e_B \in C$, i.e, $e_B = e_1, e_2, \cdots, e_k$, then $[e_{\beta_2}, e_B] = -\beta_2(e_B)e_{\beta_2} \in C^\perp$. So $\langle [e_{\beta_2}, e_B], e_i\rangle = 0$. If $e_B \in C^\perp$, i.e $e_B = e_{k+1}, e_{k+2}, \cdots, e_n$, then $[e_{\beta_2}, e_B] = N_{\beta_2 B} e_{\beta_2+B} \in C^\perp$, $\beta_2 + B \in \triangle$. So $\langle [e_{\beta_2}, e_B], e_i\rangle = 0$. Hence,

$$\begin{aligned}
\tau^V &= \sum_{\beta_2} \frac{1}{4}\Big\{ \sum_{B_1=1}^{k} \langle e_{\beta_2}, [e_{B_1}, e_i]\rangle \langle e_\alpha, [e_{B_1}, e_{\beta_2}]\rangle \\
&\quad + \sum_{B_2=k+1}^{n} \langle e_{\beta_2}, [e_{B_2}, e_i]\rangle \langle e_\alpha, [e_{B_2}, e_{\beta_2}]\rangle \Big\}.
\end{aligned}$$

We have $\tau^V = 0$. The case that $\mathfrak{h}$ is in $C^\perp$ is considered next.

LEMMA 3. *If a distribution $\mathfrak{h}$ is in the complement of the Cartan subalgebra i.e, $\mathfrak{h} \subset C^{\perp}(p < n-k)$, then $\mathfrak{h}$ is harmonic.*

PROOF. Let

$$\mathfrak{h} = \langle e_{k+1}, e_{k+2}, \cdots, e_{k+p} \rangle = \langle \overline{e_1}, \overline{e_2}, \cdots, \overline{e_p} \rangle$$

$$\mathfrak{h}^{\perp} = \langle e_1, \cdots, e_k, e_{k+p+1}, \cdots, e_n \rangle = \langle \overline{e_{p+1}}, \cdots, \overline{e_{p+k}}, \overline{e_{k+p+1}}, \cdots, \overline{e_n} \rangle.$$

By the same technique as Lemma 1,

$$\begin{aligned} \tau^H &= \sum_{i,\alpha,B} \frac{1}{8} \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_i] \rangle \langle \overline{e}_A, [\overline{e}_B, [\overline{e}_i, \overline{e}_\alpha]] \rangle \\ &= \sum_{i,B} \frac{1}{8} \Big\{ \sum_{\alpha_1=p+1}^{p+k} \langle [\overline{e}_{\alpha_1}, \overline{e}_i], \overline{e}_B \rangle \langle [[\overline{e}_i, \overline{e}_{\alpha_1}], \overline{e}_B], \overline{e}_A \rangle \\ &\quad + \sum_{\alpha_2=p+k+1}^{n} \langle [\overline{e}_{\alpha_2}, \overline{e}_i], \overline{e}_B \rangle \langle [[\overline{e}_i, \overline{e}_{\alpha_2}], \overline{e}_B], \overline{e}_A \rangle \Big\} \\ &= \sum_{i,B} \frac{1}{8} \Big\{ \sum_{\alpha_1=p+1}^{p+k} \langle i(\overline{e}_{\alpha_1}) \overline{e}_i, \overline{e}_B \rangle \langle [i(\overline{e}_{\alpha_1}) \overline{e}_i, \overline{e}_B], \overline{e}_A \rangle \\ &\quad + \sum_{\alpha_2=p+k+1}^{n} \langle N_{\alpha_2 i} \overline{e}_{\alpha_2+i}, \overline{e}_B \rangle \langle [N_{\alpha_2 i} \overline{e}_{\alpha_2+i}, \overline{e}_B], \overline{e}_A \rangle \Big\}. \end{aligned}$$

We have $\tau^H = 0$.

$$\tau^V = \sum_{j,\beta,B} \frac{1}{4} \Big\{ \underbrace{\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j] \rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j] \rangle}_{=I} + \underbrace{\langle \overline{e}_\beta, [\overline{e}_B, \overline{e}_i] \rangle \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_\beta] \rangle}_{=II} \Big\}.$$

Now we will show that the above two components are vanishing, respectively. First $I = 0$.

$$I = \sum_{j,B} \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j] \rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j] \rangle,$$

and when $\alpha = p+1, p+2, \cdots, p+k$, if $\overline{e}_B \in C$, i.e, $\overline{e}_B = e_1, e_2, \cdots, e_k$, then $\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j] \rangle = \langle [\overline{e}_\alpha, \overline{e}_B], \overline{e}_j \rangle = 0$. If $\overline{e}_B \in C^{\perp}$, i.e, $\overline{e}_B = e_{k+1}, e_{k+2}, \cdots, e_n$, then $[\overline{e}_B, \overline{e}_j] = N_{Bj} \overline{e}_{B+j} \in C^{\perp}$, $B+j \in \triangle$. So $\langle \overline{e}_\alpha, N_{Bj} \overline{e}_{B+j} \rangle = 0$. Hence $I = 0$ for $\overline{e}_\alpha = \overline{e}_{p+1}, \overline{e}_{p+2}, \cdots, \overline{e}_{p+k}$. When $\alpha = p+k+1, p+k+2, \cdots, n$,

$$I = \sum_{j,B} \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j] \rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j] \rangle,$$

if $\overline{e}_B \in C$, i.e, $\overline{e}_B = e_1, e_2, \cdots, e_k$, then $[\overline{e}_B, \overline{e}_j] = j(\overline{e}_B)\overline{e}_j$ and

$$I = \sum_{j,B} \langle \overline{e}_\alpha, j(\overline{e}_B)\overline{e}_j\rangle\langle \overline{e}_i, j(\overline{e}_B)\overline{e}_j\rangle = 0.$$

If $\overline{e}_B \in C^\perp$, i.e, $\overline{e}_B = e_{k+1}, e_{k+2}, \cdots, e_n$, then $[\overline{e}_B, \overline{e}_j] = N_{Bj}\overline{e}_{B+j}$, $B \neq j$, $B + j \in \triangle$. So $I = \sum_{j,B} \langle \overline{e}_\alpha, N_{Bj}\overline{e}_{B+j}\rangle\langle \overline{e}_i, N_{Bj}\overline{e}_{B+j}\rangle = 0$. Next we will prove $II = 0$.

$$\begin{aligned} II &= \sum_{\beta,B} \langle \overline{e}_\beta, [\overline{e}_B, \overline{e}_i]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_\beta]\rangle \\ &= -\sum_B \Big\{ \underbrace{\sum_{\beta_1=p+1}^{p+k} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle}_{=II.1} \\ &\quad + \underbrace{\sum_{\beta_2=p+k+1}^{n} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_2}]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_{\beta_2}]\rangle}_{=II.2} \Big\}. \end{aligned}$$

First $II.1 = 0$.

$$II.1 = \sum_{\beta_1,B} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle.$$

If $\overline{e}_B \in C$, then $[\overline{e}_B, \overline{e}_{\beta_1}] = 0$. If $\overline{e}_B \in C^\perp$, then $[\overline{e}_B, \overline{e}_{\beta_1}] = -B(\overline{e}_{\beta_1})\overline{e}_B$. So,

$$\begin{aligned} II.1 &= \sum_{\beta_1} \Big\{ \sum_{\overline{e}_B \in C} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle \\ &\quad + \sum_{\overline{e}_B \in C^\perp} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_{\beta_1}]\rangle \Big\} \\ &= \sum_{\beta_1} \sum_{\overline{e}_B \in C^\perp} \langle \overline{e}_i, B(\overline{e}_{\beta_1})\overline{e}_B\rangle\langle \overline{e}_\alpha, B(\overline{e}_{\beta_1})\overline{e}_B\rangle \\ &= 0. \end{aligned}$$

Next $II.2 = 0$.

$$II.2 = \sum_{\beta_2,B} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_2}]\rangle\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_{\beta_2}]\rangle,$$

if $\overline{e}_B \in C$, then $[\overline{e}_B, \overline{e}_{\beta_2}] = \beta_2(\overline{e}_B)\overline{e}_{\beta_2}$. So $\langle \overline{e}_i, [\overline{e}_B, \overline{e}_{\beta_2}]\rangle = \langle \overline{e}_i, \beta_2(\overline{e}_B)\overline{e}_{\beta_2}\rangle = 0$. If $\overline{e}_B \in C^\perp$, then $[\overline{e}_B, \overline{e}_{\beta_2}] = N_{B\beta_2}\overline{e}_{B+\beta_2}$, $B + \beta_2 \in \triangle$. Thus $II.2$ is the following

$$\begin{aligned} II.2 &= \sum_{\beta_2}\Big\{ \sum_{\bar{e}_B\in C} \langle \bar{e}_i, [\bar{e}_B, \bar{e}_{\beta_2}]\rangle\langle \bar{e}_\alpha, [\bar{e}_B, \bar{e}_{\beta_2}]\rangle \\ &\quad + \sum_{\bar{e}_B\in C^\perp} \langle \bar{e}_i, [\bar{e}_B, \bar{e}_{\beta_2}]\rangle\langle \bar{e}_\alpha, [\bar{e}_B, \bar{e}_{\beta_2}]\rangle\Big\} \\ &= \sum_{\beta_2}\Big\{ \sum_{\bar{e}_B\in C} \langle \bar{e}_i, \beta_2(\bar{e}_B)\bar{e}_{\beta_2}\rangle\langle \bar{e}_\alpha, \beta_2(\bar{e}_B)\bar{e}_{\beta_2}\rangle \\ &\quad + \sum_{\bar{e}_B\in C^\perp} \langle \bar{e}_i, N_{B\beta_2}\bar{e}_{B+\beta_2}\rangle\langle \bar{e}_\alpha, N_{B\beta_2}\bar{e}_{B+\beta_2}\rangle\Big\} \\ &= 0 \end{aligned}$$

hence $II = 0$. We have $\tau^V = 0$.

THEOREM 1. *A distribution on a compact semisimple Lie group G is harmonic.*

PROOF. Let $\mathfrak{h}$ be a distribution on G. In cases of $\mathfrak{h} \subset C$ and $\mathfrak{h} \subset C^\perp$, the proof is done by Lemma1,2. It will be shown that $\mathfrak{h}$ is harmonic for $\mathfrak{h} \subset C$ and $\mathfrak{h} \subset C^\perp$. Let

$$\mathfrak{h} = \langle e_1, \cdots, e_r, e_{k+1} \cdots, e_{k+p-r}\rangle = \langle \overline{e_1}, \cdots, \overline{e_r}, \overline{e_{r+1}}, \cdots, \overline{e_p}\rangle,$$

$$\begin{aligned} \mathfrak{h}^\perp &= \langle e_{r+1}, \cdots, e_k, e_{k+(p-r)+1}, \cdots, e_n\rangle \\ &= \langle \overline{e_{p+1}}, \cdots, \overline{e_{p+(k-r)}}, \overline{e_{p+(k-r)+1}}, \cdots, \overline{e_n}\rangle. \end{aligned}$$

Using methods in proof of Lemma 1 and 2, horizontal component is

$$\tau^H = \sum_{i,\alpha,B} \frac{1}{8}\langle[\bar{e}_i, \bar{e}_\alpha], \bar{e}_B\rangle\langle[[\bar{e}_i, \bar{e}_\alpha], \bar{e}_B], \bar{e}_A\rangle.$$

Because $\mathfrak{h}^\perp$ consists of C-component and $C^\perp$-component, we can divide $\mathfrak{h}^\perp$ into two parts. Then

$$\begin{aligned} \tau^H &= \sum_{i,B}\frac{1}{8}\Big\{ \underbrace{\sum_{\overline{e_\alpha}\in C} \langle[\bar{e}_i, \bar{e}_\alpha], \bar{e}_B\rangle\langle[[\bar{e}_i, \bar{e}_\alpha], \bar{e}_B], \bar{e}_A\rangle}_{=I} \\ &\quad + \underbrace{\sum_{\overline{e_\alpha}\in C^\perp} \langle[\bar{e}_i, \bar{e}_\alpha], \bar{e}_B\rangle\langle[[\bar{e}_i, \bar{e}_\alpha], \bar{e}_B], \bar{e}_A\rangle}_{=II}\Big\}. \end{aligned}$$

$$\begin{aligned} I &= \sum_{i,B}\frac{1}{8}\Big(\sum_{\overline{e_\alpha}\in C}\langle[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B\rangle\langle[[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B],\overline{e}_A\rangle\Big) \\ &= \sum_{B,\overline{e_\alpha}\in C}\Big\{\sum_{\overline{e_i}\in C}\langle[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B\rangle\langle[[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B],\overline{e}_A\rangle \\ &\quad+\sum_{\overline{e_i}\in C^\perp}\langle[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B\rangle\langle[[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B],\overline{e}_A\rangle\Big\}. \end{aligned}$$

The first term of the equation I vanishes, since $\overline{e}_i, \overline{e}_\alpha$ are in C. By Lemma 2, the second term also vanishes. Thus $I = 0$.

$$\begin{aligned} II &= \sum_{i,B}\frac{1}{8}\Big(\sum_{\overline{e_\alpha}\in C^\perp}\langle[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B\rangle\langle[[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B],\overline{e}_A\rangle\Big) \\ &= \sum_{B,\overline{e_\alpha}\in C^\perp}\Big\{\sum_{\overline{e_i}\in C}\langle[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B\rangle\langle[[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B],\overline{e}_A\rangle \\ &\quad+\sum_{\overline{e_i}\in C^\perp}\langle[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B\rangle\langle[[\overline{e}_i,\overline{e}_\alpha],\overline{e}_B],\overline{e}_A\rangle\Big\}. \end{aligned}$$

By Lemma 1 and 2, $II = 0$. Therefore we have $\tau^H = 0$. The vertical component is

$$\tau^V = \sum_{j,\beta,B}\frac{1}{4}\Big\{\underbrace{\langle\overline{e}_\alpha,[\overline{e}_B,\overline{e}_j]\rangle\langle\overline{e}_i,[\overline{e}_B,\overline{e}_j]\rangle}_{=I}+\underbrace{\langle\overline{e}_\beta,[\overline{e}_B,\overline{e}_i]\rangle\langle\overline{e}_\alpha,[\overline{e}_B,\overline{e}_\beta]\rangle}_{=II}\Big\}$$

$$\begin{aligned} I &= \sum_{j,B}\frac{1}{4}\langle\overline{e}_\alpha,[\overline{e}_B,\overline{e}_j]\rangle\langle\overline{e}_i,[\overline{e}_B,\overline{e}_j]\rangle \\ &= \sum_{B}\frac{1}{4}\Big\{\underbrace{\sum_{\overline{e_j}\in C}\langle\overline{e}_\alpha,[\overline{e}_B,\overline{e}_j]\rangle\langle\overline{e}_i,[\overline{e}_B,\overline{e}_j]\rangle}_{=I.1} \\ &\quad+\underbrace{\sum_{\overline{e_j}\in C^\perp}\langle\overline{e}_\alpha,[\overline{e}_B,\overline{e}_j]\rangle\langle\overline{e}_i,[\overline{e}_B,\overline{e}_j]\rangle}_{=I.2}\Big\} \end{aligned}$$

$$
\begin{aligned}
I.1 &= \sum_{B,\overline{e_j}\in C} \frac{1}{4}\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j]\rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j]\rangle \\
&= \sum_{\overline{e_j}\in C} \frac{1}{4}\Big\{ \sum_{\overline{e_B}\in C} \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j]\rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j]\rangle \\
&\quad + \sum_{\overline{e_B}\in C^\perp} \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j]\rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j]\rangle \Big\}.
\end{aligned}
$$

In the first term of above equation, since $\overline{e}_B, \overline{e}_j$ are in C, $[\overline{e}_B, \overline{e}_j] = 0$. The second term also vanishes, since $\{\overline{e}_\alpha\} \cap \{\overline{e}_i\} = \emptyset$, $[\overline{e}_B, \overline{e}_j] = B(\overline{e}_j)\overline{e}_B$. By Lemma 2,

$$
I.2 = \sum_{B,\overline{e_j}\in C^\perp} \frac{1}{4}\langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_j]\rangle \langle \overline{e}_i, [\overline{e}_B, \overline{e}_j]\rangle = 0.
$$

Because the same reason as I

$$
II = \sum_{\beta,B} \langle \overline{e}_\beta, [\overline{e}_B, \overline{e}_i]\rangle \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_\beta]\rangle = -\sum_{\beta,B} \langle \overline{e}_i, [\overline{e}_B, \overline{e}_\beta]\rangle \langle \overline{e}_\alpha, [\overline{e}_B, \overline{e}_\beta]\rangle = 0.
$$

Hence we have $\tau^V = 0$.

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