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A CONVERGENCE THEOREM FOR FEYNMAN'S OPERATIONAL CALCULUS : THE CASE OF TIME DEPENDENT NONCOMMUTING OPERATORS

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ABSTRACT. Feynman's operational calculus for noncommuting operators was studied via measures on the time interval. We investigate that if a sequence of p-tuples of measures converges to another p-tuple of measures, then the corresponding sequence of operational calculi in the time dependent setting converges to the operational calculus determined by the limiting p-tuple of measures.

1. Introduction

Let X be a separable Banach space over the complex numbers and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X. Fix $T > 0$. For $i = 1, \cdots, p$ let $A_i : [0, T] \to \mathcal{L}(X)$ be maps that are measurable in the sense that $A_i^{-1}(E)$ is a Borel set in [0,T] for any strong operator open set $E \subset \mathcal{L}(X)$. To each $A_i(\cdot)$ we associate a finite continuous Borel measure μ_i on $[0, T]$ and we require that, for each i,

$$r_i = \int_{[0,T]} ||A_i(s)||_{\mathcal{L}(X)} |\mu_i|(ds) < \infty.$$

Given a positive integer p and p positive numbers $r_1, \cdots, r_p$, let $\mathbb{A}(r_1, \cdots, r_p)$ be the space of complex-valued functions of p complex variables $f(z_1, \cdots, z_p)$, which are analytic at $(0, \cdots, 0)$, and are such that their power series expansion

$$f(z_1, \cdots, z_p) = \sum_{m_1, \cdots, m_p = 0}^{\infty} c_{m_1, \cdots, m_p} z_1^{m_1} \cdots z_p^{m_p} \tag{1}$$

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converges absolutely, at least on the closed polydisk $|z_1| \le r_1, \cdots, |z_p| \le r_p$. Such functions are analytic at least in the open polydisk $|z_1| < r_1, \cdots, |z_p| < r_p$.

For $f \in \mathbb{A}(r_1, \cdots, r_p)$ given by (1), we let

$$(2) \qquad ||f|| = ||f||_{\mathbb{A}(r_1,\cdots,r_p)} := \sum_{m_1,\cdots,m_p=0}^{\infty} |c_{m_1,\cdots,m_p}| r_1^{m_1} \cdots r_p^{m_p}.$$

The function on $\mathbb{A}(r_1, \cdots, r_p)$ defined by (2) makes $\mathbb{A}(r_1, \cdots, r_p)$ into a commutative Banach algebra [3].

To the algebra $\mathbb{A}(r_1, \cdots, r_p)$ we associate a disentangling algebra by replacing the z_i's with formal commuting objects $(A_i(\cdot), \mu_i)^{\sim}, i = 1, \cdots, p$. Consider the collection $\mathbb{D}((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})$ of all expressions of the form

$$\begin{aligned} &f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim}) \\ &= \sum_{m_1,\cdots,m_p=0}^{\infty} c_{m_1,\cdots,m_p} ((A_1(\cdot), \mu_1)^{\sim})^{m_1} \cdots ((A_p(\cdot), \mu_p)^{\sim})^{m_p} \end{aligned}$$

where $c_{m_1,\cdots,m_p} \in \mathbb{C}$ for all $m_1, \cdots, m_p = 0, 1, \cdots$, and

$$\begin{aligned} (3) \qquad &||f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})|| \\ &= ||f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})||_{\mathbb{D}((A_1(\cdot),\mu_1)^{\sim},\cdots,(A_p(\cdot),\mu_p)^{\sim})} \\ &:= \sum_{m_1,\cdots,m_p=0}^{\infty} |c_{m_1,\cdots,m_p}| r_1^{m_1} \cdots r_p^{m_p} < \infty \end{aligned}$$

where $r_i = \int_{[0,T]} ||A_i(s)||_{\mathcal{L}(X)} |\mu_i|(ds), \quad i = 1, 2, \cdots, p$.

Rather than using the notation $(A_i(\cdot), \mu_i)^{\sim}$ below, we will often abbreviate to $A_i(\cdot)^{\sim}$, especially when carrying out calculations. We will often write $\mathbb{D}$ in place of $\mathbb{D}(A_1(\cdot)^{\sim}, \cdots, A_p(\cdot)^{\sim})$ or $\mathbb{D}((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})$.

Adding and scalar multiplying such expressions coordinatewise, we can easily see that $\mathbb{D}((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})$ is a vector space and that $||\cdot||_{\mathbb{D}}$ defined by (3) is a norm. The normed linear space $(\mathbb{D}((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim}), ||\cdot||_{\mathbb{D}})$ can be identified with the weighted l_1-space, where the weight at the index $(m_1, \cdots, m_p)$ is $r_1^{m_1} \cdots r_p^{m_p}$. It follows that $\mathbb{D}((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})$ is a commutative Banach algebra with identity [7].

We refer to $\mathbb{D}((A_1(\cdot),\mu_1)^{\sim},\cdots,(A_p(\cdot),\mu_p)^{\sim})$ as the disentangling algebra associated with the p-tuple $((A_1(\cdot),\mu_1)^{\sim},\cdots,(A_p(\cdot),\mu_p)^{\sim})$.

For $m=0,1,\cdots$, let S_m denote the set of all permutations of the integers $\{1,\cdots,m\}$, and given $\pi\in S_m$, we let

$$\Delta_m(\pi)=\{(s_1,\cdots,s_m)\in[0,T]^m:0<s_{\pi(1)}<\cdots<s_{\pi(m)}<T\}.$$

Now for nonnegative integers $m_1,\cdots,m_p$ and $m=m_1+\cdots+m_p$, we define

$$C_i(s)=\begin{cases} A_1(s), & if\ i\in\{1,\cdots,m_1\}\\ A_2(s), & if\ i\in\{m_1+1,\cdots,m_1+m_2\}\\ \vdots & \\ A_p(s), & if\ i\in\{m_1+\cdots+m_{p-1}+1,\cdots,m\}\end{cases}$$

for $i=1,\cdots,m$ and for all $0\le s\le T$.

DEFINITION 1. Let $P^{m_1,\cdots,m_p}(z_1,\cdots,z_p)=z_1^{m_1}\cdots z_p^{m_p}$. We define the action of the disentangling map on this monomial by

$$\begin{aligned}&\mathcal{T}_{\mu_1,\cdots,\mu_p}P^{m_1,\cdots,m_p}(A_1(\cdot)^{\sim},\cdots,A_p(\cdot)^{\sim})\\ =&\mathcal{T}_{\mu_1,\cdots,\mu_p}((A_1(\cdot)^{\sim})^{m_1}\cdots(A_p(\cdot)^{\sim})^{m_p})\\ :=&\sum_{\pi\in S_m}\int_{\Delta_m(\pi)}C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\\ &(\mu_1^{m_1}\times\cdots\times\mu_p^{m_p})(ds_1,\cdots,ds_m).\end{aligned}$$

Finally for $f\in\mathbb{D}((A_1(\cdot),\mu_1)^{\sim},\cdots,(A_p(\cdot),\mu_p)^{\sim})$ given by

$$f(A_1(\cdot)^{\sim},\cdots,A_p(\cdot)^{\sim})=\sum_{m_1,\cdots,m_p=0}^{\infty}c_{m_1,\cdots,m_p}(A_1(\cdot)^{\sim})^{m_1}\cdots(A_p(\cdot)^{\sim})^{m_p}$$

we set

$$\begin{aligned}&\mathcal{T}_{\mu_1,\cdots,\mu_p}f(A_1(\cdot)^{\sim},\cdots,A_p(\cdot)^{\sim})\\ :=&\sum_{m_1,\cdots,m_p=0}^{\infty}c_{m_1,\cdots,m_p}\mathcal{T}_{\mu_1,\cdots,\mu_p}P^{m_1,\cdots,m_p}(A_1(\cdot)^{\sim},\cdots,A_p(\cdot)^{\sim}).\end{aligned}$$

We will often use the alternative notation :

$$P^{m_1,\cdots,m_p}_{\mu_1,\cdots,\mu_p}(A_1(\cdot),\cdots,A_p(\cdot))=\mathcal{T}_{\mu_1,\cdots,\mu_p}P^{m_1,\cdots,m_p}(A_1(\cdot)^{\sim},\cdots,A_p(\cdot)^{\sim})$$

and

$$f_{\mu_1,\cdots,\mu_p}(A_1(\cdot),\cdots,A_p(\cdot))=\mathcal{T}_{\mu_1,\cdots,\mu_p}f(A_1(\cdot)^{\sim},\cdots,A_p(\cdot)^{\sim}).$$

The following result is Proposition 2.2 of [7].

PROPOSITION 1. *The disentangling map $\mathcal{T}_{\mu_1,\cdots,\mu_p}$ is a bounded linear operator from $\mathbb{D}((A_1(\cdot),\mu_1)^\sim,\cdots,(A_p(\cdot),\mu_p)^\sim)$ to $\mathcal{L}(X)$. In fact, $\|\mathcal{T}_{\mu_1,\cdots,\mu_p}\| \le 1$.*

2. Stability theorem

Let $\{\nu_n\}_{n=1}^\infty$ be a sequence of Borel probability measures on $[0,T]$. We say that ν_n converges weakly to a Borel probability measure ν and write $\nu_n \rightharpoonup \nu$ provided that

$$\int_{[0,T]} b(s)\,\nu_n(ds) \to \int_{[0,T]} b(s)\,\nu(ds)$$

for every bounded continuous function b on $[0,T]$.

PROPOSITION 2. *Let $A_i : [0,T] \to \mathcal{L}(X)$ be continuous for each $i = 1,2,\cdots,p$. Let $\{\mu_{i,n}\}_{n=1}^\infty$ be sequences of continuous Borel probability measures on $[0,T]$ such that $\mu_{i,n} \rightharpoonup \mu_i$ for each i. Then for any nonnegative integers $m_1,\cdots,m_p$ and for any $\phi \in X$*

$$\begin{aligned}&\lim_{n\to\infty} P^{m_1,\cdots,m_p}_{\mu_{1,n},\cdots,\mu_{p,n}}(A_1(\cdot),\cdots,A_p(\cdot))\phi\\ &= P^{m_1,\cdots,m_p}_{\mu_1,\cdots,\mu_p}(A_1(\cdot),\cdots,A_p(\cdot))\phi.\end{aligned}$$

PROOF. $\{\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p}\}$ is a sequence of continuous probability measures on $[0,T]^m$ since each term in the product is a continuous probability measure. And $[0,T]^m$ is separable. By Theorem 3.2 of [1] $\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p} \rightharpoonup \mu_1^{m_1} \times \cdots \times \mu_p^{m_p}$ since $\mu_{i,n} \rightharpoonup \mu_i$ for each i. For each $\phi \in X$, $C_{\pi(m)}(\cdot)\cdots C_{\pi(1)}(\cdot)\phi : [0,T]^m \to X$ is continuous for each $\pi \in S_m$. From Theorem 5.1 of [1] we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\phi\\ &(\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p})(ds_1,\cdots,ds_m)\\ &= \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\phi\\ &(\mu_1^{m_1}\times\cdots\times\mu_p^{m_p})(ds_1,\cdots,ds_m).\end{aligned}$$

Hence the conclusion follows. □

LEMMA 3. *Let* $\mu_1, \cdots, \mu_p, \mu_{1,n}, \cdots, \mu_{p,n}, n = 1, 2, \cdots$ *be continuous probability measures. Suppose for* $i = 1, 2, \cdots, p$

$$\bar{r}_i = \sup\{r_i, r_{i,1}, \cdots, r_{i,n}, \cdots\} < \infty$$

where $r_i = \int_{[0,T]} ||A_i(s)|| \, |\mu_i|(ds)$ *and* $r_{i,n} = \int_{[0,T]} ||A_i(s)|| \, |\mu_{i,n}|(ds)$. *Then for any*

$$f \in \mathbb{A}(\bar{r}_1, \cdots, \bar{r}_p),$$

$$f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim}) \in \mathbb{D}((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})$$

and

$$f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim}) \in \mathbb{D}((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim})$$

for any $n = 1, 2, \cdots$.

PROOF. Suppose that

$$f(z_1, \cdots, z_p) = \sum_{m_1, \cdots, m_p = 0}^{\infty} c_{m_1, \cdots, m_p} z_1^{m_1} \cdots z_p^{m_p}$$

such that $\sum_{m_1, \cdots, m_p = 0}^{\infty} |c_{m_1, \cdots, m_p}| \, \bar{r}_1^{m_1} \cdots \bar{r}_p^{m_p} < \infty$. Then

$$\begin{aligned}
&||f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})|| \\
&= \sum_{m_1, \cdots, m_p = 0}^{\infty} |c_{m_1, \cdots, m_p}| \Big[\int_{[0,T]} ||A_1(s)|| \, |\mu_1|(ds) \Big]^{m_1} \cdots \\
&\quad \Big[\int_{[0,T]} ||A_p(s)|| \, |\mu_p|(ds) \Big]^{m_p} \\
&= \sum_{m_1, \cdots, m_p = 0}^{\infty} |c_{m_1, \cdots, m_p}| \, r_1^{m_1} \cdots r_p^{m_p} \\
&\leq \sum_{m_1, \cdots, m_p = 0}^{\infty} |c_{m_1, \cdots, m_p}| \, \bar{r}_1^{m_1} \cdots \bar{r}_p^{m_p} < \infty.
\end{aligned}$$

Hence $f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim}) \in \mathbb{D}\,((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})$. Similarly we can check that $f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim}) \in \mathbb{D}\,((A_1 (\cdot), \mu_{1,n}))^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim})$. □

THEOREM 4. *Let the hypotheses of Proposition 2 be satisfied. Further suppose that for each $i = 1, 2, \cdots, p$ and $n = 1, 2, \cdots$, $\bar{r}_i, r_i, r_{i,n}$ are given as in Lemma 3. Let $\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}}$ denote the disentangling map corresponding to the n^{th} term of sequences of measures. Then for any $f \in \mathbb{A}(\bar{r}_1, \cdots, \bar{r}_p)$, and for any $\phi \in X$,*

$$\lim_{n\to\infty} \mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim})\phi$$
$$= \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})\phi.$$

PROOF. We have

$$\begin{aligned}
&||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim})\phi \\
&\quad - \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})\phi|| \\
&= || \sum_{m_1,\cdots,m_p=0}^{\infty} c_{m_1,\cdots,m_p} P^{m_1,\cdots,m_p}_{\mu_{1,n},\cdots,\mu_{p,n}}(A_1(\cdot), \cdots, A_p(\cdot))\phi \\
&\quad - \sum_{m_1,\cdots,m_p=0}^{\infty} c_{m_1,\cdots,m_p} P^{m_1,\cdots,m_p}_{\mu_1,\cdots,\mu_p}(A_1(\cdot), \cdots, A_p(\cdot))\phi|| \\
&\le \sum_{m_1,\cdots,m_p=0}^{\infty} |c_{m_1,\cdots,m_p}| \, ||P^{m_1,\cdots,m_p}_{\mu_{1,n},\cdots,\mu_{p,n}}(A_1(\cdot), \cdots, A_p(\cdot))\phi \\
&\quad - P^{m_1,\cdots,m_p}_{\mu_1,\cdots,\mu_p}(A_1(\cdot), \cdots, A_p(\cdot))\phi||.
\end{aligned}$$

Note that

$$\begin{aligned}
&|c_{m_1,\cdots,m_p}| \, ||P^{m_1,\cdots,m_p}_{\mu_{1,n},\cdots,\mu_{p,n}}(A_1(\cdot), \cdots, A_p(\cdot))\phi \\
&\quad - P^{m_1,\cdots,m_p}_{\mu_1,\cdots,\mu_p}(A_1(\cdot), \cdots, A_p(\cdot))\phi|| \\
&\le |c_{m_1,\cdots,m_p}| \big[||P^{m_1,\cdots,m_p}_{\mu_{1,n},\cdots,\mu_{p,n}}(A_1(\cdot), \cdots, A_p(\cdot))|| \\
&\quad + ||P^{m_1,\cdots,m_p}_{\mu_1,\cdots,\mu_p}(A_1(\cdot), \cdots, A_p(\cdot))|| \big] \, ||\phi|| \\
&\le |c_{m_1,\cdots,m_p}| [[\int_{[0,T]} ||A_1(s)|| \, |\mu_{1,n}|(ds)]^{m_1} \cdots \\
&\quad [\int_{[0,T]} ||A_p(s)|| \, |\mu_{p,n}|(ds)]^{m_p} + [\int_{[0,T]} ||A_1(s)|| \, |\mu_1|(ds)]^{m_1} \cdots \\
&\quad [\int_{[0,T]} ||A_p(s)|| \, |\mu_p|(ds)]^{m_p}] \, ||\phi|| \\
&= |c_{m_1,\cdots,m_p}| \left[r_{1,n}^{m_1} \cdots r_{p,n}^{m_p} + r_1^{m_1} \cdots r_p^{m_p} \right] ||\phi|| \\
&\le 2|c_{m_1,\cdots,m_p}| \, \bar{r}_1^{m_1} \cdots \bar{r}_p^{m_p} ||\phi||.
\end{aligned}$$

Since $\sum_{m_1,\cdots,m_p=0}^{\infty} |c_{m_1,\cdots,m_p}| \bar{r}_1^{m_1}\cdots \bar{r}_p^{m_p} < \infty$, by Proposition 2 and Lebesgue Dominated Convergence Theorem, we obtain the result. □

THEOREM 5. *Let $A_i : [0,T] \to \mathcal{L}(X)$ be measurable for each $i = 1,2,\cdots,p$. Let $\{\mu_{i,n}\}_{n=1}^{\infty}$ for $i = 1,2,\cdots,p$ be sequences of continuous Borel probability measures on $[0,T]$ such that $\mu_{i,n} \rightharpoonup \mu_i$ for each i. Further assume that $M_i := \sup_{s\in[0,T]} ||A_i(s)|| < \infty$ for each $i = 1,\cdots,p$. Then for any $f \in \mathbb{A}(M_1,\cdots,M_p)$,*

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots,(A_p(\cdot),\mu_{p,n}\widetilde{)})\\ &= \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)}).\end{aligned}$$

PROOF. First we consider $P_{\mu_1,\cdots,\mu_p}^{m_1,\cdots,m_p}(A_1(\cdot),\cdots,A_p(\cdot))$. We see that

$$\begin{aligned}&||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots,(A_p(\cdot),\mu_{p,n}\widetilde{)})\\ &\qquad - \mathcal{T}_{\mu_1,\cdots,\mu_p} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)})||\\ &= ||\sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\,(\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p})\\ &\qquad (ds_1,\cdots,ds_m)\\ &\quad - \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\,(\mu_1^{m_1}\times\cdots\times\mu_p^{m_p})\\ &\qquad (ds_1,\cdots,ds_m)||\\ &\le \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} ||C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})||\\ &\qquad |\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p}(ds_1,\cdots,ds_m) - \mu_1^{m_1}\times\cdots\times\mu_p^{m_p}(ds_1,\cdots,ds_m)|\\ &\le \sum_{\pi\in S_m} M_1^{m_1}\cdots M_p^{m_p}\,|\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p}(\Delta_m(\pi)) -\\ &\qquad \mu_1^{m_1}\times\cdots\times\mu_p^{m_p}(\Delta_m(\pi))|.\end{aligned}$$

Here $\{\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p}\}$ is a sequence of continuous probability measures on $[0,T]^m$. Since $[0,T]^m$ is separable and $\mu_{i,n} \rightharpoonup \mu_i$ for each i, $\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p} \rightharpoonup \mu_1^{m_1}\times\cdots\times\mu_p^{m_p}$ using Theorem 3.2 of [1]. We can apply (v) of Theorem 2.1 of [1] to conclude that

$$|\mu_{1,n}^{m_1}\times\cdots\times\mu_{p,n}^{m_p}(\Delta_m(\pi)) - \mu_1^{m_1}\times\cdots\times\mu_p^{m_p}(\Delta_m(\pi))| \to 0$$

as $n \to \infty$. We therefore conclude

(4)
$$\begin{aligned}&\lim_{n\to\infty} \mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots,(A_p(\cdot),\mu_{p,n}\widetilde{)}) \\ &= \mathcal{T}_{\mu_1,\cdots,\mu_p} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)}).\end{aligned}$$

We now turn to $\mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)})$. For $f \in \mathbb{A}(M_1, \cdots, M_p)$ we have

$$\begin{aligned}&||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots,(A_p(\cdot),\mu_{p,n}\widetilde{)}) \\ &\quad - \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)})|| \\ &=|| \sum_{m_1,\cdots,m_p=0}^{\infty} c_{m_1,\cdots,m_p} \mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots, \\ &\quad (A_p(\cdot),\mu_{p,n}\widetilde{)}) \\ &\quad - \sum_{m_1,\cdots,m_p=0}^{\infty} c_{m_1,\cdots,m_p} \mathcal{T}_{\mu_1,\cdots,\mu_p} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_1\widetilde{)},\cdots, \\ &\quad (A_p(\cdot),\mu_p\widetilde{)})|| \\ &\le \sum_{m_1,\cdots,m_p=0}^{\infty} |c_{m_1,\cdots,m_p}|\, ||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots, \\ &\quad (A_p(\cdot),\mu_{p,n}\widetilde{)}) \\ &\quad - \mathcal{T}_{\mu_1,\cdots,\mu_p} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)})||.\end{aligned}$$

Now

$$\begin{aligned}&|c_{m_1,\cdots,m_p}|\, ||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots,(A_p(\cdot),\mu_{p,n}\widetilde{)}) \\ &\quad - \mathcal{T}_{\mu_1,\cdots,\mu_p} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)})|| \\ &\le |c_{m_1,\cdots,m_p}|\big[||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_{1,n}\widetilde{)},\cdots, \\ &\quad (A_p(\cdot),\mu_{p,n}\widetilde{)})|| + ||\mathcal{T}_{\mu_1,\cdots,\mu_p} P^{m_1,\cdots,m_p}((A_1(\cdot),\mu_1\widetilde{)},\cdots,(A_p(\cdot),\mu_p\widetilde{)})||\big] \\ &\le |c_{m_1,\cdots,m_p}|\Big[\Big[\int_{[0,T]} ||A_1(s)||\,|\mu_{1,n}|(ds)\Big]^{m_1} \cdots \Big[\int_{[0,T]} ||A_p(s)||\,|\mu_{p,n}|(ds)\Big]^{m_p} \\ &\quad + \Big[\int_{[0,T]} ||A_1(s)||\,|\mu_1|(ds)\Big]^{m_1} \cdots \Big[\int_{[0,T]} ||A_p(s)||\,|\mu_p|(ds)\Big]^{m_p}\Big] \\ &\le 2|c_{m_1,\cdots,m_p}| M_1^{m_1} \cdots M_p^{m_p}.\end{aligned}$$

Since $\sum_{m_1,\cdots,m_p=0}^{\infty} |c_{m_1,\cdots,m_p}| M_1^{m_1} \cdots M_p^{m_p} < \infty$, by (4) and Lebesgue Dominated Convergence Theorem, we obtain the result. □

COROLLARY 6. *Assume the same hypotheses as in Theorem 5. Then for any $f \in \mathbb{A}(M_1, \cdots, M_p)$, and for any $\phi \in X$,*

$$\lim_{n\to\infty} \mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim})\phi$$
$$= \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})\phi.$$

PROOF. Let $f \in \mathbb{A}(M_1, \cdots, M_p)$. Then for any $\phi \in X$, we have

$$\begin{aligned} &||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim})\phi \\ &\quad - \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})\phi|| \\ &\le ||\mathcal{T}_{\mu_{1,n},\cdots,\mu_{p,n}} f((A_1(\cdot), \mu_{1,n})^{\sim}, \cdots, (A_p(\cdot), \mu_{p,n})^{\sim}) \\ &\quad - \mathcal{T}_{\mu_1,\cdots,\mu_p} f((A_1(\cdot), \mu_1)^{\sim}, \cdots, (A_p(\cdot), \mu_p)^{\sim})||\ ||\phi|| \to 0 \end{aligned}$$

as $n \to \infty$ by Theorem 5. This finishes the proof. □

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