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A NOTE ON CERTAIN QUOTIENT SPACES OF BOUNDED LINEAR OPERATORS

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ABSTRACT. Suppose X is a closed subspace of $Z = (\sum_{n=1}^{\infty} Z_n)_p$ $(1 < p < \infty,\ \dim Z_n < \infty)$. We investigate an isometrically isomorphic embedding of $L(X)/K(X)$ into $L(X,Z)/K(X,Z)$, where $L(X,Z)$ (resp. $L(X)$) is the space of the bounded linear operators from X to Z (resp. from X to X) and $K(X,Z)$ (resp. $K(X)$) is the space of the compact linear operators from X to Z (resp. from X to X).

1. Introduction and preliminaries

If X and Y are Banach spaces, $L(X,Y)$ (resp. $K(X,Y)$) will denote the Banach space of all bounded linear operators (resp. compact linear operators) from X to Y. If $X = Y$, then we simply write $L(X)$ (resp. $K(X)$). An interesting problem is the proximinal property of $K(X,Y)$ in $L(X,Y)$. Many authors [1, 3-7] have studied this problem and found examples of Banach spaces X and Y for which $K(X,Y)$ is proximinal in $L(X,Y)$. Recall that a closed subspace J of a normed space F is called a proximinal subspace if for each $x \in F \setminus J$ there exists $y \in J$ such that $\|x - y\| = \inf\{\|x - j\| : j \in J\}$, that is, the distance $d(x, J)$ from x to J is attained at y.

In this paper we restrict ourselves to an ℓ_p-sum $Z = (\sum_{n=1}^{\infty} Z_n)_p$ $(1 < p < \infty,\ \dim Z_n < \infty)$ and a closed space X of Z. The proximinality of $K(X,Z)$ in $L(X,Z)$ was already solved positively [1, 6]. Now our interest is to see how $T \in L(X,Z)$ determines $d(T, K(X,Z))$, the norm of $T + K(X,Z)$ in the quotient space $L(X,Z)/K(X,Z)$. In Proposition 2.4, for given $T \in L(X,Z)$ we will write the distance $d(T, K(X,Z))$ in terms

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of T. In Theorem 2.5, using Proposition 2.4 we find a condition under which the map $T + K(X) \to T + K(X, Z)$ is an isometric isomorphism from $L(X)/K(X)$ into $L(X,Z)/K(X,Z)$.

Suppose $\{Z_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces. For $1 \leq p < \infty$, ℓ_p-sum $(\sum_{n=1}^{\infty} Z_n)_p$ of Z_n's is the Banach space of sequences $z = (z_1, z_2, \cdots)$, $z_n \in Z_n$, with the norm $\|z\| = (\sum_{n=1}^{\infty} \|z_n\|^p)^{1/p} < \infty$. For $p = \infty$, ℓ_∞-sum $(\sum_{n=1}^{\infty} Z_n)_\infty$ of Z_n's is defined similarly by sequences $z = (z_1, z_2, \cdots)$, $z_n \in Z_n$, with the norm $\|z\| = \sup_n\{\|z_n\|\} < \infty$.

For each m, the map $P_m : (\sum_{n=1}^{\infty} Z_n)_p \to (\sum_{n=1}^{\infty} Z_n)_p$ defined by $P_m(z) = (z_1, z_2, \cdots, z_m, 0, 0, \cdots)$, $z = (z_1, z_2, \cdots) \in (\sum_{n=1}^{\infty} Z_n)_p$ is a norm one projection. These projections are called the natural projections on $(\sum_{n=1}^{\infty} Z_n)_p$.

For $1 \leq p < \infty$, the dual space $(\sum_{n=1}^{\infty} Z_n)_p^*$ is $(\sum_{n=1}^{\infty} Z_n^*)_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. The adjoint operator P_n^* of the natural projection P_n on $(\sum_{n=1}^{\infty} Z_n)_p$ turns out to be the natural projection on $(\sum_{n=1}^{\infty} Z_n^*)_q$. Therefore, if every Z_n is reflexive, $(\sum_{n=1}^{\infty} Z_n)_p$ is also reflexive for $1 < p < \infty$.

In the rest of this article, unless otherwise specified Z_n will always denote a finite dimensional Banach space and $1 < p < \infty$. P_n will denote the natural projection on $(\sum_{n=1}^{\infty} Z_n)_p$. If Y is a Banach space, then B_Y will denote the closed unit ball of Y. $\mathbb{N}$ will denote the set of the natural numbers.

2. Results

We start with a proposition whose proof seems to be more or less obvious. However, we still include a proof.

PROPOSITION 2.1. *Let Y be a Banach space and $Z = (\sum_{n=1}^{\infty} Z_n)_p$, $1 \leq p < \infty$. If $T \in L(Y, Z)$, then $d(T, K(Y, Z)) = \lim_{n\to\infty} \|T - P_nT\|$.*

PROOF. Let $\alpha = d(T, K(Y, Z))$ and $\varepsilon > 0$. We choose $S \in K(Y, Z)$ such that $\alpha + \varepsilon > \|T - S\|$. Since $S(B_Y)$ has the compact closure, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ and all $y \in B_Y$

$$\|(I - P_n)S(y)\| < \varepsilon.$$

Thus, for all $n \geq m$, $\|S - P_nS\| \leq \varepsilon$ and

$$\begin{aligned} \alpha + \varepsilon > \|T - S\| &\geq \|T - P_nS\| - \|S - P_nS\| \\ &\geq \|T - P_nT\| - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} \|T - P_nT\| \leq \alpha$.

On the other hand, since $P_nT \in K(Y,Z)$, $\|T - P_nT\| \geq \alpha$ for all n. Therefore, $\lim_{n\to\infty} \|T - P_nT\| = \alpha$. □

REMARK. In the above proposition, with a small modification in the proof, we can replace Z by any Banach space E with a Schauder basis.

LEMMA 2.2. *Suppose X is a closed subspace of $(\sum_{n=1}^{\infty} Z_n)_p$, and $\{h_n\}_{n=1}^{\infty}$ is a sequence in X^* such that $\|h_n\| = 1$ for all n and $h_n \to 0$ weakly as $n \to \infty$. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $h_n(x_n) = 1 = \|x_n\|$ for all n, then $x_{n_k} \to 0$ weakly as $k \to \infty$ for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.*

PROOF. Since B_X is weakly compact [2, p.245], without loss of generality we may assume that $x_n \to x \in B_X$ weakly as $n \to \infty$. Writing $x_n = x + y_n$ for all n (where $y_n \to 0$ weakly as $n \to \infty$), we have $1 = \lim_{n\to\infty} \|x_n\|^p = \lim_{n\to\infty}(\|x\|^p + \|y_n\|^p)$. On the other hand, since $1 = h_n(x_n) = h_n(x) + h_n(y_n)$ and $h_n(x) \to 0$ as $n \to \infty$, $1 = \lim_{n\to\infty} h_n(x_n) = \lim_{n\to\infty} h_n(y_n) \leq \liminf_n \|y_n\|$. Therefore, $x = 0$ and $x_n \to 0$ weakly as $n \to \infty$. □

LEMMA 2.3. *Suppose X is a closed subspace of $(\sum_{n=1}^{\infty} Z_n)_p$, and suppose $\{h_n\}_{n=1}^{\infty}$ is a sequence in X^* such that $\|h_n\| = 1$ for all n and $h_n \to 0$ weakly as $n \to \infty$. If Y is a closed subspace of X with $\dim(X/Y) < \infty$, then there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ such that $\|h_{n_k}|_Y\| \to 1$ as $k \to \infty$.*

PROOF. Since B_X is weakly compact, we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ in B_X such that $h_n(x_n) = 1 = \|x_n\|$ for all n. Then by Lemma 2.2, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to 0$ weakly as $k \to \infty$. In particular, for each $z^* \in B_{Y^\perp}$, $z^*(x_{n_k}) \to 0$ as $k \to \infty$. Since $(X/Y)^* = Y^\perp$ is finite dimensional, $B_{Y^\perp}$ is compact [2, p.245] and hence $\sup\{|z^*(x_{n_k})| : z^* \in B_{Y^\perp}\} \to 0$ as $k \to \infty$. On the other hand, in view of $(X/Y)^* = Y^\perp$ we have

$$\begin{aligned} d(x_{n_k}, Y) = \|\tilde{x}_{n_k}\| &= \sup\{|f(\tilde{x}_{n_k})| : f \in (X/Y)^*, \|f\| \leq 1\} \\ &= \sup\{|z^*(x_{n_k})| : z^* \in B_{Y^\perp}\}, \end{aligned}$$

where $\tilde{x}_{n_k} = x_{n_k} + Y \in X/Y$.

Therefore, $d(x_{n_k}, Y) \to 0$ as $k \to \infty$. Since $h_{n_k}(x_{n_k}) = 1$ for all k, $\|h_{n_k}|_Y\| \to 1$ as $k \to \infty$. □

PROPOSITION 2.4. *Suppose X is a closed subspace of $Z = (\sum_{n=1}^{\infty} Z_n)_p$. If $T \in L(X,Z)$, then $d(T, K(X,Z)) = \lim n \to \infty \|T|_{X_n}\|$, where $X_n = X \cap (I - P_n)Z$.*

PROOF. Let $\alpha = d(T, K(X,Z))$. Choose $g_n \in (I - P_n^*)Z^*$ such that $\|g_n\| = 1$ and

$$\|T^*(g_n)\| \le \|T^*|_{(I-P_n^*)Z^*}\| < \|T^*(g_n)\| + \frac{1}{n}.$$

By Proposition 2.1, $\|T^*|_{(I-P_n^*)Z^*}\| = \|(T - P_nT)^*\| \to \alpha$ and hence $\|T^*(g_n)\| \to \alpha$ as $n \to \infty$. Since $g_n \to 0$ weakly in Z^* as $n \to \infty$, $T^*(g_n) \to 0$ weakly in X^* as $n \to \infty$. Since $\dim(X/X_n) < \infty$, by Lemma 2.3, for fixed $n \in \mathbb{N}$ there exists a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that $\|T^*(g_{n_k})|_{X_n}\| \to \alpha$ as $k \to \infty$. On the other hand, we have

$$\begin{aligned}
\|T|_{X_n}\| &= \sup_{y \in B_{X_n}} \|Ty\| \\
&= \sup_{y \in B_{X_n}} \sup_{f \in B_{Z^*}} |f(Ty)| \\
&= \sup_{f \in B_{Z^*}} \sup_{y \in B_{X_n}} |(T^*f)y| \\
&= \sup_{f \in B_{Z^*}} \|(T^*f)|_{X_n}\| \\
&\ge \|T^*(g_{n_k})|_{X_n}\| \quad \text{for all } k.
\end{aligned}$$

Therefore, it follows that $\lim_{n\to\infty} \|T|_{X_n}\| \ge \alpha$.

To prove the reversed inequality, let $\varepsilon > 0$. As in the proof of Proposition 2.1, we choose $S \in K(X,Z)$ and $m \in \mathbb{N}$ such that

$$\alpha + \varepsilon > \|T - S\| \quad \text{and} \quad \|S - P_nS\| < \varepsilon \quad \text{for all } n \ge m.$$

A finite rank operator $P_mS : X \to P_mZ$ can be extended to a bounded linear operator $\widetilde{P_mS} : Z \to P_mZ$. Since the adjoint of $\widetilde{P_mS}$ is also compact, we can find $n_0 \in \mathbb{N}$ ($n_0 \ge m$) such that

$$\|\widetilde{P_mS}(I - P_k)\| < \varepsilon \quad \text{for all } k \ge n_0.$$

Therefore, for $k \ge n_0$

$$\|S|_{X_k}\| \le \|(S - P_mS)|_{X_k}\| + \|(P_mS)|_{X_k}\| < 2\varepsilon.$$

It follows that for all $k \ge n_0$

$$\alpha + \varepsilon \ge \|T|_{X_k}\| - \|S|_{X_k}\| \ge \|T|_{X_k}\| - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\alpha \ge \lim_{n\to\infty} \|T|_{X_n}\|$. □

Observe that since X/X_n is finite dimensional X_n is complemented in X. Let Y_n be a subspace of X complementary to X_n, that is $X = Y_n \oplus X_n$ and let Q_n be the projection on X with the range Y_n. If $\liminf_n \|I - Q_n\| = 1$, we get the following result.

THEOREM 2.5. *Let X be a closed subspace of $Z = (\sum_{n=1}^{\infty} Z_n)_p$ and let Q_n be as above. If $\liminf_n \|I - Q_n\| = 1$, then $L(X)/K(X)$ is isometrically embedded into $L(X,Z)/K(X,Z)$.*

PROOF. Let $T \in L(X)$, $\alpha = d(T, K(X,Z))$ and $\beta = d(T, K(X))$. First we will show that $\alpha = \beta$. Since $\alpha \leq \beta$, we only need to show that $\beta \leq \alpha = \lim_{n\to\infty} \|T|_{X_n}\|$. Since $TQ_n \in K(X)$ for every n, writing $T = T(I - Q_n) + TQ_n$ we have

$$\beta = d(T, K(X)) = d(T(I - Q_n), K(X)) \leq \|T(I - Q_n)\|.$$

By passing to a subsequence if necessary, we may assume that $\|I - Q_n\| \to 1$ as $n \to \infty$. Let $\varepsilon > 0$. We choose $m \in \mathbb{N}$ such that $\|I - Q_n\| < 1 + \varepsilon$ for all $n \geq m$. Then for all $n \geq m$

$$\|T(I - Q_n)\| < \|T|_{X_n}\|(1 + \varepsilon)$$

and hence

$$\beta < \|T|_{X_n}\|(1 + \varepsilon).$$

Therefore, $\beta \leq \lim_{n\to\infty} \|T|_{X_n}\|$.

Next, observe that since $K(X) \subseteq K(X,Z)$ the map $\phi\colon L(X,Z)/K(X) \to L(X,Z)/K(X,Z)$ defined by $\phi(T + K(X)) = T + K(X,Z)$ for $T \in L(X,Z)$ is a norm decreasing linear operator. If $T \in L(X)$, Then by the above observation $\|T + K(X)\| = \|T + K(X,Z)\|$ and hence ϕ restricted to $L(X)/K(X)$ is a linear isometry. □

If each E_n is a subspace of Z_n and $X = (\sum_{n=1}^{\infty} E_n)_p$, then projections $\{Q_n\}_{n=1}^{\infty}$ in the above theorem are nothing but the natural projections on $X = (\sum_{n=1}^{\infty} E_n)_p$, and hence $\|I - Q_n\| = \|Q_n\| = 1$. Therefore, we have the following corollary.

COROLLARY 2.6. *Suppose a closed subspace X of $Z = (\sum_{n=1}^{\infty} Z_n)_p$ has the form of $X = (\sum_{n=1}^{\infty} E_n)_p$, where each E_n is a subspace of Z_n. Then $L(X)/K(X)$ is isometrically embedded into $L(X,Z)/K(X,Z)$.*

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