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STABILITY OF A BETA-TYPE FUNCTIONAL EQUATION WITH A RESTRICTED DOMAIN

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Abstract. We obtain the Hyers-Ulam-Rassias stability of a beta-type functional equation

$$f(\varphi(x), \phi(y)) = \psi(x, y) f(x, y) + \lambda(x, y)$$

with a restricted domain and the stability in the sense of R. Ger of the equation

$$f(\varphi(x), \phi(y)) = \psi(x, y) f(x, y)$$

with a restricted domain in the following settings:

$$\mid g(\varphi(x), \phi(y)) - \psi(x, y) g(x, y) - \lambda(x, y) \mid \leq \epsilon(x, y)$$

and

$$\mid \frac{g(\varphi(x), \phi(y))}{\psi(x, y), g(x, y)} - 1 \mid \leq \epsilon(x, y).$$

1. Introduction

In 1940, Ulam [16] raised the following question concerning the stability of homomorphism: given a group G_1, a metric group G_2 with metric $(\cdot, \cdot)$ and $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then a homomorphism $g : G_1 \to G_2$ exists with $d(f(x), g(x)) \leq \epsilon$ for all $x \in G_1$? The case of approximately additive mappings was solved by D. H. Hyers [5] under the assumption that G_1 and G_2 are Banach spaces. Th. M. Rassias [14] proved the substantial generalization of the result

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of Hyers and also P. Găvruta obtained a further generalization of the Hyers-Ulam-Rassias theorem (see also [3, 7]). Later, many Rassias and Găvruta type theorems concerning the stability of different functional equations were obtained by numerous authors (see, for instance, [1, 2, 4, 6, 12-14]). In this paper we deal with a beta-type functional equation

$$f(\varphi(x), \phi(y)) = \psi(x, y) f(x, y) + \lambda(x, y). \tag{1}$$

The gamma functional equation and the beta functional equation are example of the equation (1). S. M. Jung [9, 10] investigated the stability of the gamma functional equation. His results have been generalized to the framework of generalized gamma and beta functional equations (see also [8, 11, 15]). The aim of the present note is to give the stability theorem of the equation (1) with a restricted domain, and the stability in the sense of R. Ger of the equation

$$f(\varphi(x), \phi(y)) = \psi(x, y) f(x, y). \tag{2}$$

Our results are generalizations of theorems established in [8]-[11]. Throughout this paper, let $\varphi : (0, \infty) \to R$ and $\phi : (0, \infty) \to R$ be strictly increasing functions, $\lambda : (0, \infty) \times (0, \infty) \to R$ and $\epsilon : (0, \infty) \times (0, \infty) \to (0, \infty)$ be some functions, and let n_1, n_2 be given nonnegative real numbers. Note that for some function φ, $\varphi^0(x) = x$ and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$ for all x.

2. Hyers-Ulam-Rassias stability of the functional equation (1)

In the following theorem we investigate the Hyers-Ulam-Rassias stability for equations of the form (1) with a restricted domain. This result is a generalization of Theorem 1 in [8].

THEOREM 1. *If a function $g : (0, \infty) \times (0, \infty) \to R$ satisfies the inequality*

$$\mid g(\varphi(x), \phi(y)) - \psi(x, y) g(x, y) - \lambda(x, y) \mid < \epsilon(x, y) \tag{3}$$

for all $x > n_1$, $y > n_2$ and a function $\psi : (0, \infty) \times (0, \infty) \to (0, \infty)$ with

$$w(x, y) := \sum_{k=0}^{\infty} \frac{\epsilon(\varphi^k(x), \phi^k(y))}{\prod_{j=0}^{k} \mid \psi(\varphi^j(x), \phi^j(y)) \mid} < \infty$$

for all $x > n_1$, $y > n_2$, *then there exists a unique function* $f : (0,\infty) \times (0,\infty) \to (0,\infty)$

$$f(\varphi(x), \phi(y)) = \psi(x,y)g(x,y) + \lambda(x,y)$$

for all $x, y > 0$ *and*

$$\mid g(x,y) - f(x,y) \mid < w(x,y)$$

for all $x > n_1$ *and* $y > n_2$.

PROOF. Let $w_n : (0,\infty) \times (0,\infty) \to (0,\infty)$ and $f_n : (0,\infty) \times (0,\infty) \to R$ be functions defined by

$$w_n(x,y) := \sum_{k=0}^{n-1} \frac{\epsilon(\varphi^k(x), \phi^k(y))}{\prod_{j=0}^{k} \mid \psi(\varphi^j(x), \phi^j(y)) \mid}$$

for each positive integer n and for all $x, y > 0$ and

$$f_n(x,y) := \frac{g(\varphi^n(x), \phi^n(y))}{\prod_{j=0}^{n-1} \psi(\varphi^j(x), \phi^j(y))} - \sum_{k=0}^{n-1} \frac{\lambda(\varphi^k(x), \phi^k(y))}{\prod_{j=0}^{k} \psi(\varphi^j(x), \phi^j(y))}$$

for each positive integer n and for all $x, y > 0$, respectively. By (3) we have

$$\begin{aligned} & \mid f_{n+1}(x,y) - f_n(x,y) \mid \\ = & \frac{1}{\mid \prod_{j=0}^{n} \psi(\varphi^j(x), \phi^j(y)) \mid} \mid g(\varphi^{n+1}(x), \phi^{n+1}(y)) \\ & - \psi(\varphi^n(x), \phi^n(y)) g(\varphi^n(x), \phi^n(y)) - \lambda(\varphi^n(x), \phi^n(y)) \mid \\ \leq & \frac{\epsilon(\varphi^n(x), \phi^n(y))}{\mid \prod_{j=0}^{n} \psi(\varphi^j(x), \phi^j(y)) \mid} \end{aligned}$$

for each position integer n and for all $x > n_1$ and $y > n_2$.

Now we use induction on n to prove

$$\mid f_n(x,y) - g(x,y) \mid \leq w_n(x,y)$$

for all positive integer n and for all $x > n_1$ and $y > n_2$. For the case $n = 1$ the above inequality is an immediate consequence of (3). Assume that it holds true some n. Then

$$\begin{aligned} & \mid f_{n+1}(x,y) - g(x,y) \mid \\ \leq & \mid f_{n+1}(x,y) - f_n(x,y) \mid + \mid f_n(x,y) - g(x,y) \mid \\ \leq & w_{n+1}(x,y) \end{aligned}$$

for all $x > n_1$, and $y > n_2$.

We claim that $\{f_n(x,y)\}$ is a Cauchy sequence. Indeed, for $n > m$, $x > n_1$ and $y > n_2$ we have

$$\begin{aligned} &\mid f_n(x,y) - f_m(x,y) \mid \\ &\leq \sum_{j=m}^{n-1} \mid f_{j+1}(x,y) - f_j(x,y) \mid \\ &\leq \sum_{k=m}^{n-1} \frac{\epsilon(\varphi^k(x), \phi^k(y))}{\prod_{j=0}^{k} \mid \psi(\varphi^j(x), \phi^j(y)) \mid} \\ &= w_n(x,y) - w_m(x,y) \to 0 \end{aligned}$$

as $m \to \infty$. Hence we can define a function $\tilde{f} : (n_1, \infty) \times (n_2, \infty) \to R$ by

$$\tilde{f}(x,y) = \lim_{n\to\infty} f_n(x,y).$$

Since $f_n(\varphi(x), \phi(y)) = \psi(x,y) f_{n+1}(x,y) + \lambda(x,y)$, we have

$$\tilde{f}(\varphi(x), \phi(y)) = \psi(x,y)\tilde{f}(x,y) + \lambda(x,y)$$

for all $x > n_1$ and $y > n_2$ and

$$\begin{aligned} &\mid \tilde{f}(x,y) - g(x,y) \mid = \lim_{n\to\infty} \mid f_n(x,y) - g(x,y) \mid \\ &\leq \lim_{n\to\infty} w_{n+1}(x,y) \\ &= w(x,y) \end{aligned}$$

for all $x > n_1$ and $y > n_2$.

Now we extend the function $\tilde{f}$ to $(0,\infty) \times (0,\infty)$. We define a function $f : (0,\infty) \times (0,\infty) \to R$ by $f(x,y) = \tilde{f}(x,y)$ if $x > n_1$ and $y > n_2$ and

$$f(x,y) = \frac{\tilde{f}(\varphi^k(x), \phi^k(y))}{\prod_{j=0}^{k-1} \psi(\varphi^j(x), \phi^j(y))} - \sum_{j=0}^{k-1} \frac{\lambda(\varphi^j(x), \phi^j(y))}{\prod_{i=0}^{j} \psi(\varphi^i(x), \phi^i(y))}$$

if $0 < x \leq n_1$ or $0 < y \leq n_2$ and k is the smallest natural number satisfying the inequalities $\varphi^k(x) > n_1$ and $\phi^k(y) > n_2$. In the latter case, we have

$$\begin{aligned} f(\varphi(x),\phi(y)) &= \frac{\tilde{f}(\varphi^{k+1}(x),\phi^{k+1}(y))}{\prod_{j=1}^{k}\psi(\varphi^{j}(x),\phi^{j}(y))} - \sum_{j=1}^{k}\frac{\lambda(\varphi^{j}(x),\phi^{j}(y))}{\prod_{i=1}^{j}\psi(\varphi^{i}(x),\phi^{i}(y))} \\ &= \frac{\tilde{f}(\varphi^{k}(x),\phi^{k}(y))}{\prod_{j=1}^{k-1}\psi(\varphi^{j}(x),\phi^{j}(y))} - \sum_{j=1}^{k-1}\frac{\lambda(\varphi^{j}(x),\phi^{j}(y))}{\prod_{i=1}^{j}\psi(\varphi^{i}(x),\phi^{i}(y))} \\ &= \psi(x,y)f(x,y)+\lambda(x,y). \end{aligned}$$

Thus for every $x, y > 0$ we have

$$f(\varphi(x),\phi(y)) = \psi(x,y)f(x,y)+\lambda(x,y).$$

If $h : (0,\infty)\times(0,\infty) \to R$ is a function which satisfies

$$h(\varphi(x),\phi(y)) = \psi(x,y)h(x,y)+\lambda(x,y)$$

for all $x, y > 0$ and $\mid h(x,y) - g(x,y) \mid \leq w(x,y)$ for all $x > n_1$ and $y > n_2$, then

$$\begin{aligned} &\mid f(x,y)-h(x,y)\mid \\ &= \mid f(\varphi(x),\phi(y)) - h(\varphi(x),\phi(y)) \mid \frac{1}{\mid \psi(x,y)\mid} \\ &= \mid f(\varphi^{n}(x),\phi^{n}(y)) - h(\varphi^{n}(x),\phi^{n}(y)) \mid \frac{1}{\prod_{j=0}^{n-1}\mid \psi(\varphi^{j}(x),\phi^{j}(y))\mid} \\ &\leq \frac{1}{\prod_{j=0}^{n-1}\mid \psi(\varphi^{j}(x),\phi^{j}(y))\mid}(\mid f(\varphi^{n}(x),\phi^{n}(y)) - g(\varphi^{n}(x),\phi^{n}(y))\mid \\ &\quad + \mid g(\varphi^{n}(x),\phi^{n}(y)) - h(\varphi^{n}(x),\phi^{n}(y))\mid) \\ &\leq \frac{2w(\varphi_n(x),\phi_n(y))}{\prod_{j=0}^{n-1}\mid \psi(\varphi^{j}(x),\phi^{j}(y))\mid} = 2[w(x,y)-w_n(x,y)] \to 0 \end{aligned}$$

as $n \to \infty$. This implies the uniqueness of f. □

The functional equation

$$g(x+1,y+1) = \frac{xy}{(x+y)(x+y+1)}g(x,y)$$

for all $x, y > 0$ is called "*the beta functional equation*". Since

$$\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)}<\infty$$

and

$$\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{(x+y+2i)(x+y+2i+1)}{(x+i)(y+i)}$$

for each $x, y > 0$, we investigate the stability of the same beta functional equation

$$g(x+1,y+1)^{-1}=\frac{(x+y)(x+y+1)}{xy}g(x,y)^{-1}$$

for all $x, y > 0$. It is well known that the beta function

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

is a solution of the beta functional equation.

COROLLARY 2. *If a mapping* $g:(0,\infty)\times(0,\infty)\to(0,\infty)$ *satisfies the inequality*

$$\mid g(x+1,y+1)^{-1}-\frac{(x+y)(x+y+1)}{xy}g(x,y)^{-1}\mid\le\delta$$

for all $x > n_1$ *and* $y > n_2$, *then there exists a unique function* $f:(0,\infty)\times(0,\infty)\to(0,\infty)$ *such that* f *is a solution of the beta functional equation and satisfies the inequality*

$$\mid g(x,y)^{-1}-f(x,y)^{-1}\mid\le\frac{xy}{(x+y)(x+y+1)}\delta$$

for all $x > n_1$ *and* $y > n_2$.

PROOF. We apply Theorem 2.1 with $\varphi(x)=x+1, \phi(y)=y+1, \lambda(x,y)=0$ and $\psi(x,y)=\frac{(x+y)(x+y+1)}{xy}$. For any $x, y > 0$ we have

$$\sum_{k=0}^{\infty}\prod_{j=0}^{k}\frac{\delta}{\psi(\varphi^j(x),\phi^j(y))}\le\frac{\delta}{\psi(x,y)}\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{2\delta}{\psi(x,y)}.$$

By Theorem 2.1, there exists a unique function $F : (0, \infty) \times (0, \infty) \to (0, \infty)$ such that

$$F(x+1, y+1) = \frac{(x+y)(x+y+1)}{xy} F(x, y)$$

for all $x, y > 0$, and

$$| g(x, y)^{-1} - F(x, y) | \leq \frac{xy}{(x+y)(x+y+1)} 2\delta$$

for all $x > n_1$ and $y > n_2$. Let $F(x, y) = f(x, y)^{-1}$ for all $x, y > 0$. Then we complete the proof of Corollary. □

Consider the Schröder functional equation with two variables

$$f(\varphi(x), \phi(y)) = kf(x, y).$$

For example, $f(x, y) = e^{x+y}$ is a solution of a Schröder functional equation

$$f(x+1, y+1) = e^2 f(x, y).$$

As an application of Theorem 2.1, we can derive the following corollary, concerning the Hyers-Ulam stability of the Schröder functional equation.

COROLLARY 3. *Let $k > 1$ and $\delta > 0$. If a function $g : (0, \infty) \times (0, \infty) \to R$ satisfies the inequality*

$$| g(\varphi(x), \phi(y)) - kg(x, y) | < \delta$$

for all $x > n_1$ and $y > n_2$, then there exists a unique function $f : (0, \infty) \times (0, \infty) \to R$ such that

$$f(\varphi(x), \phi(y)) - kf(x, y) = 0$$

for all $x, y > 0$ and

$$| g(x, y) - f(x, y) | \leq \frac{\delta}{k-1}$$

for all $x > n_1$ and $y > n_2$.

PROOF. By Theorem 2.1 with $\lambda(x, y) = 0$, $\psi(x, y) = k$, and $\epsilon(x, y) = \delta$, we have

$$w(x, y) = \delta \sum_{i=0}^{\infty} \frac{1}{k^{i+1}} = \frac{\delta}{k-1}$$

for all $x, y > 0$, and so we complete the proof of Corollary. □

3. Stability in the sense of R. Ger of the functional equation (2)

The following result is a generalization of Theorem 2 in [8].

THEOREM 4. *Let $g : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a function that satisfies the inequality*

$$(4) \qquad \left| \frac{g(\varphi(x),\phi(y))}{\psi(x,y)g(x,y)} - 1 \right| \le \epsilon(x,y)$$

for all $x > n_1$ and $y > n_2$, where $\epsilon : (0,\infty) \times (0,\infty) \to (0,1)$ and $\psi : (0,\infty) \times (0,\infty) \to (0,\infty)$ are functions such that for all $x > n_1$ and $y > n_2$

$$\alpha(x,y) := \sum_{j=0}^{\infty} \log(1 - \epsilon(\varphi^j(x),\phi^j(y))) < \infty$$

and

$$\beta(x,y) := \sum_{j=0}^{\infty} \log(1 + \epsilon(\varphi^j(x),\phi^j(y))) < \infty.$$

Then there exists a unique function $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that for all $x, y > 0$

$$f(\varphi(x),\phi(y)) = \psi(x,y)f(x,y)$$

and

$$e^{\alpha(x,y)} \le \frac{f(x,y)}{g(x,y)} \le e^{\beta(x,y)}$$

for all $x > n_1$ and $y > n_2$.

PROOF. Let $f_n : (0,\infty) \times (0,\infty) \to R$ be the function defined by for each position integer n,

$$f_n(x,y) := \frac{g(\varphi^n(x),\phi^n(y))}{\prod_{j=0}^{n-1} \psi(\varphi^i(x),\phi^i(y))}.$$

For any $x > n_1$, $y > n_2$ and for all positive integer m, n with $n > m$, we have

$$\begin{aligned}\frac{f_n(x,y)}{f_m(x,y)} &= \frac{g(\varphi^n(x),\phi^n(y))}{g(\varphi^m(x),\phi^m(y))} \cdot \frac{1}{\prod_{j=m}^{n-1} \psi(\varphi^j(x),\phi^j(y))} \\ &= \prod_{j=m}^{n-1} \frac{g(\varphi^{j+1}(x),\phi^{j+1}(y))}{\psi(\varphi^j(x),\phi^j(y))g(\varphi^j(x),\phi^j(y))}.\end{aligned}$$

By (4), we have

$$\begin{aligned} 0 &< 1 - \epsilon(\varphi^j(x), \phi^j(y)) \\ &\leq \frac{g(\varphi^{j+1}(x), \phi^{j+1}(y))}{\psi(\varphi^j(x), \phi^j(y)) g(\varphi^j(x), \phi^j(y))} \\ &\leq 1 + \epsilon(\varphi^j(x), \phi^j(y)) \end{aligned}$$

for all $x > n_1$ and $y > n_2$. Thus for all $x > n_1$ and $y > n_2$ we have

$$\prod_{j=m}^{n-1} (1 - \epsilon(\varphi^j(x), \phi^j(y)) \leq \frac{f_n(x,y)}{f_m(x,y)} \leq \prod_{j=m}^{n-1} (1 + \epsilon(\varphi^j(x), \phi^j(y))$$

or

$$\begin{aligned} &\sum_{j=m}^{n-1} \log(1 - \epsilon(\varphi^j(x), \phi^j(y)) \leq \log f_n(x,y) - \log f_m(x,y) \\ &\leq \sum_{j=m}^{n-1} \log(1 + \psi(\varphi^j(x), \phi^j(y)). \end{aligned}$$

Since these series converge by assumption, $\{\log f_n(x,y)\}$ is a Cauchy sequence for all $x > n_1$ and $y > n_2$. Now we can define

$$L(x,y) := \lim_{n \to \infty} \log f_n(x,y)$$

and

$$\tilde{f}(x,y) = e^{L(x,y)} = \lim_{n \to \infty} f_n(x,y)$$

for all $x > n_1$ and $y > n_2$. It is easy to see that

$$\begin{aligned} \tilde{f}(\varphi(x), \phi(y)) &= \lim_{n \to \infty} f_n(\psi(x), \phi(y)) \\ &= \lim_{n \to \infty} \varphi(x,y) f_{n+1}(\psi(x), \phi(y)) \\ &= \varphi(x,y) \tilde{f}(x,y). \end{aligned}$$

Now we extend the function $\tilde{f}$ to $(0,\infty) \times (0,\infty)$. We defined a function $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ by $f(x,y) = \tilde{f}(x,y)$ if $x > n_1$, and $y > n_2$ and

$$f(x,y) = \frac{\tilde{f}(\varphi^n(x), \phi^n(y))}{\prod_{j=0}^{n-1} \psi(\varphi^j(x), \phi^j(y))}$$

if $0 < x \le n_1$ or $0 < y \le n_2$ and k is the smallest natural number satisfying the inequalities $\varphi^k(x) > n_1$ and $\phi^k(y) > n_2$. In the latter case, we have

$$f(\varphi(x),\phi(y)) = \frac{\tilde{f}(\varphi^{n+1}(x),\phi^{n+1}(y))}{\prod_{j=1}^{n}\psi(\varphi^j(x),\phi^j(y))} = \psi(x,y)f(x,y).$$

Thus for every $x, y > 0$,

$$f(\varphi(x),\phi(y)) = \psi(x,y)f(x,y).$$

Since for every $x > n_1$ and $y > n_2$

$$\frac{f_n(x,y)}{g(x,y)} = \prod_{j=0}^{n-1}\frac{g(\varphi^{j+1}(x),\phi^{j+1}(y))}{\psi(\varphi^j(x),\phi^j(y))g(\varphi^j(x),\phi^j(y))},$$

we get

$$\prod_{j=0}^{n-1}(1-\epsilon(\varphi^j(x),\phi^j(y)) \le \frac{f_n(x,y)}{g(x,y)} \le \prod_{j=0}^{n-1}(1+\epsilon(\varphi^j(x),\phi^j(y)).$$

This implies that

$$e^{\alpha(x,y)} \le \frac{f(x,y)}{g(x,y)} \le e^{\beta(x,y)}$$

for all $x > n_1$, and $y > n_2$. Now it remain only to prove the uniqueness of f. Assume that $h : (0,\infty)\times(0,\infty) \to (0,\infty)$ is some solution of equation (2) which satisfies (4). By equation (2),

$$\frac{f(x,y)}{h(x,y)} = \frac{f(\varphi(x),\phi(y))}{h(\varphi(x),\phi(y))} = \frac{f(\varphi^n(x),\phi^n(y))}{g(\varphi^m(x),\phi^m(y))}\frac{g(\varphi^n(x),\phi^n(y))}{h(\varphi^n(x),\phi^n(y))}$$

for any $x, y > 0$ and for all n. Hence we have

$$\frac{e^{\alpha(\varphi^n(x),\phi^n(y))}}{e^{\beta(\varphi^n(x),\phi^n(y))}} \le \frac{f(x,y)}{h(x,y)} \le \frac{e^{\beta(\varphi^n(x),\phi^n(y))}}{e^{\alpha(\varphi^n(x),\phi^n(y))}}$$

for all $x, y > 0$. Note that for any $\alpha(\varphi^n(x),\phi^n(y)) \to 0$ as $n \to \infty$ and $\beta(\varphi^n(x),\phi^n(y)) \to 0$ as $n \to \infty$. Thus, it is obvious that $f(x,y) = h(x,y)$ for all $x, y > 0$. □

COROLLARY 5. *Let $g : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a function which satisfies the inequality*

$$\left| \frac{xy}{(x+y)(x+y+1)} \cdot \frac{g(x,y)}{g(x+1,y+1)} - 1 \right| \le \epsilon(x,y)$$

for all $x, y > n_0$, where $\epsilon : (0,\infty) \times (0,\infty) \to (0,1)$ is a function such that

$$\alpha(x,y) := \sum_{j=0}^{\infty} \log(1 - \epsilon(x+j, y+j))$$

and

$$\beta(x,y) := \sum_{j=0}^{\infty} \log(1 + \epsilon(x+j, y+j))$$

are bounded for all $x, y > n_0$. Then there exists a unique solution

$$f : (0,\infty) \times (0,\infty) \to (0,\infty)$$

of the functional equation (2) with

$$e^{\alpha(x,y)} \le \frac{f(x,y)^{-1}}{g(x,y)^{-1}} \le e^{\beta(x,y)}$$

for all $x, y > n_0$.

PROOF. Follows from Theorem 3.1 for $\varphi(x) = x+1$, $\phi(y) = y+1$,

$$\psi(x,y) = \frac{(x+y)(x+y+1)}{xy},$$

and $G(x,y) = g(x,y)^{-1}$. □

COROLLARY 6. *Let k_1, $k_2 > 1$ and $\delta > 0$. If $g : (0,\infty) \times (0,\infty) \to (0,\infty)$ satisfies the inequality*

$$\left| \frac{g(k_1 x, k_2 y)}{\psi(x,y) g(x,y)} - 1 \right| < \frac{\delta}{x+y}$$

for all $x, y > \frac{\delta}{2}$, then there exists a unique function $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that for all $x, y > 0$

$$f(k_1 x, k_2 y) = \psi(x,y) f(x,y)$$

and

$$e^{\alpha(x,y)} \leq \frac{f(x,y)}{g(x,y)} \leq e^{\beta(x,y)}.$$

PROOF. Let $\varphi(x) = k_1 x$ and $\phi(y) = k_2 y$ and $\epsilon(x,y) = \frac{\delta}{x+y}$. Then

$$\alpha(x,y) = \sum_{j=0}^{\infty} \log(1 - \frac{\delta}{k_1^j x + k_2^j y}) < \infty$$

and

$$\beta(x,y) = \sum_{j=0}^{\infty} \log(1 + \frac{\delta}{k_1^j x + k_2^j y}) < \infty$$

for all x, $y > \frac{\delta}{2}$. By Theorem 3.1, we complete the proof. □

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