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DIMENSION FOR A CANTOR-LIKE SET WITH OVERLAPS

Mi Ryeong Lee, Jung Ju Park and Hung Hwan Lee

Abstract. In this paper we define a Cantor-like set K with overlaps in $\mathbb{R}^1$. We find the correlation dimension of the set K without two conditions: the control of placements of basic sets constructing K and the thickness of K being greater than 1.

1. Introduction

In order to explain fractal sets, dimension so-called, the Hausdorff dimension, box dimension or correlation dimension of fractal sets has been studied by various authors. In recent, because of advantages of dimension calculations and smaller value than the Hausdorff dimension, they have investigated about the correlation dimension of fractal sets. So far, it has been usually done the study about the correlation dimension on similar sets or self-similar sets with overlaps in $\mathbb{R}^1$ ([1], [3], [7], [8]).

In [1], [3] and [8], they have studied the correlation dimension defined by the energy theory. For a self-similar set with overlaps in $\mathbb{R}^1$ in [8], they obtained the correlation dimension of the set under the condition of the thickness of the set being greater than 1. In [3] they defined a Cantor set with overlaps which is a generalization of self-similar sets with overlaps in $\mathbb{R}^1$, and they obtained the correlation dimension of the defined set under the above definition and the same condition.

On the other hand, in [5], for a given probability measure on $\mathbb{R}^1$, we have known that the value of the correlation dimension defined by the energy theory is less than or equal to the value defined by a partition into $\mathbb{R}^1$. Also in [4], we can show that two definitions about the correlation dimension are equivalent for a self-conformal probability measure.

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In [7], without the condition of thickness, using the equivalent definition of the correlation dimension, they introduced the correlation dimension on Rams theorem of a self-similar set with overlaps in $\mathbb{R}^1$.

In present paper, we consider a generalized Cantor-like set K with overlaps of sets in [3], [7] and [8]. We omit two conditions which play important roles in the results in [3] and [8]: the control of placement of basic sets constructing K except for the first and last sets ([3]) and the thickness of K being greater than 1([3], [8]).

In this paper, in order to obtain the correlation dimension of K, we use the energy theory for the upper bound and the equivalent definition in [4] and [7] for the lower bound. Finally we show that the facts in [3], [7] and [8] is also true from our result.

2. Preliminaries

Let us introduce the construction of a Cantor-like set with overlaps in $\mathbb{R}^1$. Consider $X \equiv [0,\ 1]$. Fix an integer number $l(\geq 3)$. Suppose that a sequence of mappings $\{f_{i_1,i_2,\cdots,i_n} : (i_1,\cdots,i_n) \in \{1,2,\ \cdots,\ l\}^n$ for $n = 1,2,\cdots\}$ and numbers $0 < r_1, r_2, \cdots, r_l < 1$ are given such that (i) $f_{i_1,i_2,\cdots,i_n} : X \longrightarrow X$ is defined as $f_{i_1,\cdots,i_n}(x) = r_{i_n}x + t_{i_1,\cdots,i_n}$ for some $t_{i_1,\cdots,i_n} \in \mathbb{R}$ and each $i_j \in \{1,2,\cdots,l\}(j = 1,2,\cdots,n)$ (ii) for any $n \geq 1$, a basic set $X_{i_1,i_2,\cdots,i_n} \equiv f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_1,i_2,\cdots,i_n}(X)$ contains l-basic sets $X_{i_1,i_2,\cdots,i_n,1}$, $I_{i_1,i_2,\cdots,i_n,2}$ and $I_{i_1,i_2,\cdots,i_n,l}$, so that the left-hand ends of $I_{i_1,i_2,\cdots,i_n}$ and $I_{i_1,i_2,\cdots,i_n,1}$ and the right-hand ends of $I_{i_1,i_2,\cdots,i_n}$ and $I_{i_1,i_2,\cdots,i_n,l}$ coincide.

Set

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\cdots,i_n)\in\{1,\cdots,l\}^n} X_{i_1,i_2,\cdots,i_n}.$$

We call this K a *Cantor-like set with overlaps.*

REMARK 2.1. (1) We note the following control ([3]) in the construction of the set K: (iii) there exists a constant $0 < c < \frac{1}{2}$ such that $f_{i_1,i_2,\cdots,i_n}(X) \subset [c, 1-c]$ for $i_n \neq 1, l(n = 1,2,\cdots)$. That is, in this paper the restriction of placement of basic sets except for the first and last sets is eliminated.

(2) We notice that in particular, if we assume that the condition $f_{i_1,\cdots,i_n} = f_{i_n}$ for all $n \geq 1$, *i.e.* $t_{i_1,\cdots,i_n} = t_{i_n}$ and $r_{i_n} = r$, for $i_j \in \{1,2,\cdots,l\}$, then the family $\{f_i\}_{i=1}^{l}$ gives a one parameter family of self-similar iterated function system in [6] and [7] in $\mathbb{R}^1$.

We adopt notations used in [3] and [8].

Put $\Sigma = \{1, 2, \cdots, l\}^{\mathbb{N}}$. For $\mathbb{i} = (i_1, i_2, \cdots) \in \Sigma$, we define an onto map Π from Σ to K as

$$\Pi(\mathbf{i}) = \bigcap_{n=1}^{\infty} f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_1,i_2,\cdots,i_n}(X).$$

For $\mathbf{i} = (i_1, i_2, \cdots, i_n, \cdots)$, $\mathbf{j} = (j_1, j_2, \cdots, j_m, \cdots) \in \Sigma$, write $\mathbf{i} \wedge \mathbf{j} = i_1, i_2, \cdots, i_n$, where $n = \min\{k : i_{k+1} \neq j_{k+1}$ for $k \geq 1\}$. If $i_1 \neq j_1$ then $\mathbf{i} \wedge \mathbf{j} = 0$. Define a metric ρ on Σ as $\rho(\mathbf{i}, \mathbf{j}) = r^{\mathbf{i} \wedge \mathbf{j}}$, where $r^{i_1,\cdots,i_n} = r_{i_1} \cdot r_{i_2} \cdots r_{i_n}$. Write $[\mathbf{i}_n]$ for a cylinder set, $[i_1, \cdots, i_n] = \{\mathbf{j} \in \Sigma : i_k = j_k$ for $1 \leq k \leq n$, $i_{n+1} \neq j_{n+1}\}$. The metric ρ_2 on $\Sigma \times \Sigma$ is defined as $\rho_2((\mathbf{i}, \mathbf{j}), (\mathbf{i}', \mathbf{j}')) = \max\{\rho(\mathbf{i}, \mathbf{i}'), \rho(\mathbf{j}, \mathbf{j}')\}$.

For any $\epsilon > 0$, we say that $[\mathbb{i}_n]$ is an *ϵ-cylinder* if $r^{i_1,\cdots,i_n} \leq \epsilon < r^{i_1,\cdots,i_{n-1}}$. The set $[\mathbf{i}_n, \mathbf{j}_m] \equiv [\mathbf{i}_n] \times [\mathbf{j}_m]$ is an *ϵ-cylinder* in $\Sigma^2 \equiv \Sigma \times \Sigma$ if both $[\mathbf{i}_n]$ and $[\mathbf{j}_m]$ are ϵ-cylinders in Σ. The set of ϵ-cylinders in Σ is denoted by $\mathcal{H}_\epsilon$. The collection of ϵ-cylinders in Σ^2, $\mathcal{H}^2_\epsilon \equiv \mathcal{H}_\epsilon \times \mathcal{H}_\epsilon$ provides a disjoint cover of Σ^2 by sets of diameter ϵ ([3], [8]).

The number $s > 0$ with $\sum_{i=1}^{l} r_i^s = 1$ is called the *similarity dimension* ([3]). Consider the probability measure μ on Σ with weights $(r_1^s, r_2^s, \cdots, r_l^s)$, *i.e.* $\mu([\mathbf{i}_n]) = r_{i_1}^s \cdot r_{i_2}^s \cdots r_{i_n}^s$ for any $n \geq 1$. Define the *push-down measure* $\nu = \mu \circ \Pi^{-1}$ on Λ and let $\mu_2 = \mu \times \mu$. Then ν and μ_2 are probability measures on Λ and Σ^2, respectively.

Denote the diameter of a set A by $|A|$.

REMARK 2.2. ([3]) (1) For an ϵ-cylinder $[\mathbf{i}_n]$, $r_0\epsilon \leq |[\mathbf{i}_n]| \leq \epsilon$ where $r_0 = \min\{r_1, \cdots, r_l\}$.

(2) The measures of ϵ-cylinders $[\mathbf{i}_n]$ and $[\mathbf{i}_n, \mathbf{j}_m]$ satisfy $(r_0)^s\epsilon^s < \mu([\mathbf{i}_n]) \leq \epsilon^s$ and $(r_0)^{2s}\epsilon^{2s} \leq \mu_2([\mathbb{i}_n, \mathbb{j}_m]) \leq \epsilon^{2s}$.

Recall the *upper box dimension* ([2]) of a bounded set E in a metric space which is denoted by $\overline{\dim}_B E$. That is,

$$\overline{\dim}_B E = \limsup_{\epsilon \to 0} \frac{\log N(E, \epsilon)}{-\log \epsilon},$$

where $N(E, \epsilon)$ is the smallest number of balls of diameter ϵ needed to cover E. From easy calculations (cf. [2]), we get the following result.

PROPOSITION 2.3. ([3]) For $E \subset \Sigma^2$, let $N_\epsilon(E)$ be the number of ϵ-cylinders intersecting E. Then

$$\overline{\dim}_B(E) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(E)}{-\log \epsilon}.$$

We recall the following definition of the *correlation dimension* ([1], [3], [5], [8]) of $A(\subset \mathbb{R}^d)$ with respect to a probability measure η on A;

$$D_2(A,\eta) = \sup\{\alpha \geq 0 : I_\alpha(\eta) < \infty\} = \inf\{\alpha \geq 0 : I_\alpha(\eta) = \infty\},$$

where $I_\alpha(\eta) = \iint_A |x-y|^{-\alpha} d\eta(x)d\eta(y)$ is the α-energy of A with respect to η.

In [4], we can see that for every $\epsilon > 0$, a fixed partition $\mathcal{D}_\epsilon$ of $\mathbb{R}$ into a grid of intervals of length 2ϵ, $\lim_{\epsilon\to 0} \frac{\log \Sigma_{Q\in\mathcal{D}_\epsilon}\eta^2(Q)}{\log \epsilon}$ exits under a self-conformal probability measure η on A. Further, from [5], we obtain that for the such measure η, the limit value is equal to $D_2(A,\eta)$.

Throughout this paper, for K a Cantor-like set with overlaps and ν the push-down probability measure on K, we write $D_2(K)$ for $D_2(K,\nu)$ and $I_\alpha(\nu)$ for $I_\alpha(\nu) = \int_{\Sigma^2} \mid \Pi(\mathbf{i}) - \Pi(\mathbf{j}) \mid^{-\alpha} d\mu_2$.

3. Results

Throughout this paper, let K, Π, μ, ν and s be as in Section 2. We may assume that $0 \leq s \leq 1$.

Set $Z = \{(\mathbf{i},\ \mathbf{j}) \in \Sigma^2 : \Pi(\mathbf{i}) = \Pi(\mathbf{j})\}$ and $A_\epsilon(Z) = \{[\mathbf{i}_n, \mathbf{j}_m] \in \mathcal{H}^2_\epsilon : [\mathbf{i}_n, \mathbf{j}_m] \cap Z \neq \emptyset\}$. Denote $N_\epsilon \equiv N_\epsilon(Z)$ for the cardinality of $A_\epsilon(Z)$.

For the upper bound about the correlation dimension of K, we calculate the α-energy of K for the probability measure ν.

Write $K_{i_1,i_2,\cdots,i_n} = f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_1,\cdots,i_n}(K)$ for all $\mathbf{i} = (i_1, \cdots, i_n, \cdots) \in \Sigma$.

PROPOSITION 3.1.

$$D_2(K) \leq 2s - \overline{\dim}_B Z.$$

PROOF. Suppose $\alpha > 2s - \overline{\dim}_B Z$. Let $[\mathbf{i}_n,\ \mathbf{j}_m] \in A_\epsilon(Z)$. Then $[\mathbf{i}_n,\ \mathbf{j}_m] \cap Z \neq \emptyset$, and so $K_{i_1,i_2,\cdots,i_n} \cap K_{j_1,j_2,\cdots,j_m} \neq \emptyset$. Hence for any $\mathbf{i} \in [\mathbf{i}_n]$ and $\mathbf{j} \in [\mathbf{j}_m]$, using Remark 2.2(1),

$$\begin{aligned} |\Pi(\mathbf{i}) - \Pi(\mathbf{j})| &\leq |K_{i_1,i_2,\cdots,i_n}| + |K_{j_1,j_2,\cdots,j_m}| \\ &= (r^{\mathbf{i}_n} + r^{\mathbf{j}_m})\ |K| \\ &\leq 2\epsilon. \end{aligned}$$

Therefore, by Remark 2.2(2),

$$\begin{aligned} \int_{[\mathbf{i}_n,\ \mathbf{j}_m]} |\Pi(\mathbf{i}) - \Pi(\mathbf{j})|^{-\alpha}\ d\mu_2 &\geq 2^{-\alpha}\epsilon^{-\alpha}\mu_2([\mathbf{i}_n,\ \mathbf{j}_m]) \\ &\geq 2^{-\alpha}\ r_0^{2s}\ \epsilon^{2s-\alpha}. \end{aligned}$$

We have

$$\begin{aligned} I_\alpha(\nu) &= \int_{\sum^2} \mid \Pi(\mathbf{i}) - \Pi(\mathbf{j}) \mid^{-\alpha} d\mu_2 \\ &\geq \sum_{A_\epsilon(Z)} \int_{[\mathbf{i}_n,\ \mathbf{j}_m]} |\Pi(\mathbf{i}) - \Pi(\mathbf{j})|^{-\alpha}\, d\mu_2 \\ &\geq C_1 N_\epsilon\, \epsilon^{2s-\alpha}, \end{aligned}$$

where $C_1 = 2^{-\alpha}(r_0)^{2s}$. Thus, if $\overline{\dim}_B Z = \limsup_{\epsilon\to 0} \frac{\log N_\epsilon}{-\log \epsilon} > 2s - \alpha$, then $I_\alpha(\nu) = \infty$. By the definition of $D_2(K)$, we have $D_2(K) \leq 2s - \overline{\dim}_B Z$. □

THEOREM 3.2. $D_2(K) \geq 2s - \overline{\dim}_B Z$.

PROOF. For $\epsilon > 0$, we consider a partition $\mathcal{D}_\epsilon$ of $\mathbb{R}$ into a grid of intervals of length 2ϵ. The set of centers of such intervals is called $\mathcal{C}_\epsilon$. Put $I_\epsilon(x)$ for interval grid with center x.

Let $\epsilon' > \epsilon$. For $x' \in \mathcal{C}'_\epsilon$, let

$$N_{\epsilon',\epsilon} = \#\{[\mathbf{i}_n] \in \mathcal{H}_\epsilon : X_{\mathbf{i}_n} \cap I_{\epsilon'}(x') \neq \emptyset\}.$$

First, we prove that

$$(1) \qquad 1 < \frac{\sum_{x'\in\mathcal{C}_{\epsilon'}} N^2_{\epsilon',\epsilon}(x')}{N_\epsilon} < 3\gamma^2$$

where $\gamma = \lfloor \frac{2(\epsilon'+\epsilon)}{r_0\epsilon} \rfloor + 1$. To see the second inequality, fix a $x' \in \mathcal{C}'_\epsilon$. If $X_{\mathbf{i}_n} \cap I_{\epsilon'}(x') \neq \emptyset$, then $X_{\mathbf{i}_n} \subset (x' - (\epsilon' + \epsilon),\ x' + (\epsilon' + \epsilon))$. Subdivide the interval $I_{\epsilon'+\epsilon}(x')$ into subintervals of length $r_0\epsilon$. There are exactly γ end points of such intervals. Thus there is such an end point of an interval which is contained in at least $\frac{N_{\epsilon',\epsilon}(x')}{\gamma}$ basic sets corresponding to elements of $\mathcal{H}_\epsilon$.

So there are at least $\frac{N^2_{\epsilon',\epsilon}(x')}{3\gamma^2}$ pairs of basic sets corresponding to elements of $\mathcal{H}_\epsilon$ which intersect each other and which can be associated uniquely with x'.

Next, we prove that there exists a constant $C_2 > 0$ such that

$$(2) \qquad \frac{\sum_{x\in\mathcal{C}_\epsilon} \nu^2(I_\epsilon(x))}{\epsilon^{2s} \cdot N_\epsilon} < C_2.$$

For any $x \in \mathcal{C}_\epsilon$, let x' be the center of interval grid $\mathcal{D}_{\epsilon'}$ which contains x in its interior or as its right end point.

Let x'_L and x'_R be centers of the two neighborhoods of $I_{\epsilon'}(x')$ in $\mathcal{D}'_\epsilon$. From the definition of the measure $\nu = \mu \circ \Pi^{-1}$, we have

$$\begin{aligned}\nu(I_\epsilon(x)) &= \nu(I_\epsilon(x) \cap K) \leq \nu(\cup_{[\mathbf{i}_n]\in\mathcal{H}_\epsilon}(I_\epsilon(x) \cap X_{\mathbf{i}_n}))\\ &\leq \mu(\cup\{[\mathbf{i}_n] \in \mathcal{H}_\epsilon : X_{\mathbf{i}_n} \cap I_{\epsilon'}(x') \neq \emptyset\})\\ &\leq \mu(\cup\{[\mathbf{i}_n] \in \mathcal{H}_\epsilon : X_{\mathbf{i}_n} \cap (I_{\epsilon'}(x'_L) \cup I_{\epsilon'}(x') \cup I_{\epsilon'}(x'_R)) \neq \emptyset\})\\ &\leq \sum \mu(\{[\mathbf{i}_n] \in \mathcal{H}_\epsilon : X_{\mathbf{i}_n} \cap (I_{\epsilon'}(x'_L) \cup I_{\epsilon'}(x') \cup I_{\epsilon'}(x'_R)) \neq \emptyset\})\\ &\leq \epsilon^s(N_{\epsilon',\epsilon}(x'_L) + N_{\epsilon',\epsilon}(x') + N_{\epsilon',\epsilon}(x'_R)).\end{aligned}$$

Then we have

$$\nu^2(I_\epsilon(x)) \leq \epsilon^{2s} \cdot 3(N^2_{\epsilon',\epsilon}(x'_L) + N^2_{\epsilon',\epsilon}(x') + N^2_{\epsilon',\epsilon}(x'_R)).$$

Take $\epsilon' = a\epsilon$ for some $a > 2$. Hence we obtain

$$\begin{aligned}\sum_{x\in\mathcal{C}_\epsilon} \nu^2(I_\epsilon(x)) &\leq 3 \cdot \epsilon^{2s} \cdot a \sum_{x'\in\mathcal{C}_{\epsilon'}} (N^2_{\epsilon',\epsilon}(x'_L) + N^2_{\epsilon',\epsilon}(x') + N^2_{\epsilon',\epsilon}(x'_R))\\ &\leq 3 \cdot 3 \cdot a \cdot \epsilon^{2s} \sum_{x'\in\mathcal{C}_{\epsilon'}} N^2_{\epsilon',\epsilon}(x'),\end{aligned}$$

so

$$\frac{\sum_{x\in\mathcal{C}_\epsilon} \nu^2(I_\epsilon(x))}{\epsilon^{2s}\sum_{x'\in\mathcal{C}_{\epsilon'}} N^2_{\epsilon',\epsilon}(x')} < 9a$$

for some $a > 2$. Therefore, using (1), we can obtain that for a constant $C_2 > 0$,

$$\frac{\sum_{x\in\mathcal{C}_\epsilon} \nu^2(I_\epsilon(x))}{\epsilon^{2s} \cdot N_\epsilon} < C_2.$$

Owing to the existence of the limit for the correlation dimension,

$$\lim_{\epsilon\to 0} \frac{\sum_{x\in\mathcal{C}_\epsilon} \nu^2(I_\epsilon(x))}{\log \epsilon} \geq 2s - \overline{\lim_{\epsilon\to 0}} \frac{\log N_\epsilon}{-\log \epsilon}.$$

From Proposition 2.3 and (2), we have the result. □

REMARK 3.3. In Proposition 3.1, we note that we have the result without the condition *(iii)* in [3]. Also in Theorem 3.2, we notice that we can have the result without two conditions of thickness of K being bigger than 1 and the condition *(iii)*. In [3] and [8], under such two conditions, it is hold that $D_2(K) = 2s - \overline{\dim}_B Z$.

Write $\bar{1} = (1, 1, 1, \cdots)$ and $\mathbb{i}_n\bar{1} = (i_1, i_2, \cdots, i_n, 1, 1, \cdots)$ $(n \geq 1)$. Then the left end point of $X_{\mathbf{i}_n}$ is $\Pi(\mathbf{i}_n\bar{1})$ for all $n \geq 1$. Therefore, for $[\mathbf{i}_n, \mathbf{j}_m] \in \mathcal{H}^2_\epsilon$,

$$X_{\mathbf{i}_n} \cap X_{\mathbf{j}_m} \cap K \neq \emptyset \text{ if and only if } |\Pi(\mathbf{i}_n\bar{1}) - \Pi(\mathbf{j}_m\bar{1})| \leq \epsilon.$$

COROLLARY 3.4.

$$\varlimsup_{\epsilon \to 0} \frac{\log \#\{[\mathbf{i}_n, \mathbf{j}_m] \in \mathcal{H}^2_\epsilon : |\Pi(\mathbb{i}_n\bar{1}) - \Pi(\mathbf{j}_m\bar{1})| \leq \epsilon\}}{-\log \epsilon} = 2s - \mathrm{D}_2(K).$$

PROOF. Using Proposition 3.1 and Theorem 3.2, we can have the above result. □

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Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail: hhlee@knu.ac.kr
jjpark66@hanmail.net
lmr67@yumail.ac.kr