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THE SPACE OF FOURIER HYPERFUNCTIONS AS AN INDUCTIVE LIMIT OF HILBERT SPACES

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ABSTRACT. We research properties of the space of measurable functions square integrable with weight $\exp(2\nu|x|)$, and those of the space of Fourier hyperfunctions. Also we show that the several embedding theorems hold true, and that the Fourier-Lapace operator is an isomorphism of the space of strongly decreasing Fourier hyperfunctions onto the space of analytic functions extended to any strip in $\mathbb{C}^n$ which are estimated with the aid of a special exponential function $\exp(\mu|x|)$.

§0. Introduction

We introducted the following in [5]:

Let $F_{(h,\nu)}$ be the space of continuously differentiable functions $\varphi(x)$ for which the norm

$$(0.1)\qquad |\varphi|_{(h,\nu)} = \sup_{x\in\mathbb{R}^n,\alpha} \frac{|\partial^\alpha\varphi(x)|\exp(\nu|x|)}{h^{-|\alpha|}\alpha!},\ h>0,\ \nu\in\mathbb{R}$$

is finite. Then the spaces $F_{(h,\nu)}$ the (continuous) embeddings

$$(0.2)\qquad F_{(h,\nu)} \subset F_{(h',\nu')},\ h\geq h'>0,\ \nu\geq\nu'$$

take place.

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By virtue of (0.2), we can define the spaces $\mathcal{G}$ and $\mathcal{M}$ with the aid of the operations of projective and inductive limits:

$$(0.3)\qquad \begin{aligned}\mathcal{G} &= \bigcap_{h,\nu} F_{(h,\nu)},\\ \mathcal{M} &= \bigcap_{h>0} F_{(h,-\infty)},\ F_{(h,-\infty)} = \bigcup_{\nu} F_{(h,\nu)}.\end{aligned}$$

Let the space $\mathcal{G}'$ be a space of continuous linear functionals on $\mathcal{G}$. Since the embeddings (0.2) induce the adjoint embeddings

$$(0.4)\qquad (F_{(h',\nu')})' \subset (F_{(h,\nu)})',\ h \geq h' > 0,\ \nu \geq \nu',$$

The space $\mathcal{G}'$ regarded as a vector space coincides with the union of $(F_{(h,\nu)})'$:

$$(0.5)\qquad \mathcal{G}' = \bigcup_{h,\nu} (F_{(h,\nu)})'.$$

The right-hand space can be equipped with the topology of inductive limit, and in the left-hand space we can introduce the topology of the strong conjugate space of $\mathcal{G}$.

Note that $\mathcal{G}'$ is reflexive and regular inductive limit, which implies the coincidence of the two above-mentioned topologies in $\mathcal{G}'$. The regularity of $\mathcal{G}'$ implies that for each bounded set $B \subset \mathcal{G}'$ there are real numbers h and ν such that $B \subset (F_{(h,\nu)})'$.

In this paper, making use of the same method as in [3], we research the structure of an inductive limit of Hilbert spaces for the space of Fourier hyperfunctions introduced in [5].

In §2, we research properties of the reflexive Hilbert space $H_{(\nu)}$ of measurable functions square integrable with weight $\exp(2\nu|x|)$ (Proposition 2.1). We show that the space $\mathcal{G}$ is dense in $H = H_{<0>}$ (Theorem 2.3). Since the Fourier operator $\mathfrak{F} : \mathcal{G}' \to \mathcal{G}'$ is one-to-one and transforms the subset $\mathcal{G} \subset \mathcal{G}'$ into itself and $\mathcal{G} \subset H_{<s>} \subset \mathcal{G}'$, we introduce the space $H^{(s)}$, which is the image of $H_{<s>}$ under the operator $\mathfrak{F}^{-1}$.

In §3, §4, we introduce the space $H^{(s)}_{<\nu>}$, which consists of Fourier hyperfunctions in $H^{(-\infty)}$ such that $\delta_{s,N}(D)f \in H_{<\nu>}$, $0 \leq \nu < N$ for pseudodifferential operator $\delta_{s,N}(D)$ with symbol $\delta_{s,N}(\zeta) = \exp(s\sum_{k=1}^{n}$

$(N^2+\zeta_k^2)^{1/2}), |Im\zeta_k| < N$. And we research properties of Fourier hyperfunctions in this spaces (Proposition 3.1, 3.2 and 4.3). Also in Proposition 3.3, 4.1 and 4.2 we show the following embedding theorems: For $h,\ \nu > 0$,

$$F_{(h,\pm\nu+\epsilon)} \subset H^{(h/2)}_{<\pm\nu>} \subset F_{(h/4,\pm\nu)},\ \epsilon > 0.$$

Lastly, we introduce the spaces $\mathcal{G}$, $\mathcal{O}\ \mathcal{M}$, $\mathcal{O}'$ and $\mathcal{G}'$ endowed with topologies of projective and injective limits of the Hibert spaces$\{H^{(s)}_{<\nu>}\}$ and show that the Fourier-Lapace transform $\mathfrak{F} : \mathcal{O}' \to \mathcal{M}$ is an isomorphism of vector spaces (Proposition 4.4).

§1. Preliminaries

As a norm in $\mathbb{R}^n_x$ (in $\mathbb{C}^n_z$, resp.) we take $|x| = \sum_{j=1}^n |x_j|$ ($|z| = \sum_{j=1}^n |z_j|$, resp.), and the volume element $dx = dx_1 \cdots dx_n$ is fixed. Put $D = (D_1, \ldots, D_n);\ D_k = i^{-1}\partial_k,\ \partial_k = \partial/\partial x_k,\ k = 1, \ldots, n,\ i = \sqrt{-1}$.

We denote by $\mathbb{R}^n_\xi$ the dual space of $\mathbb{R}^n_x$. Let $\xi = (\xi_1, \ldots, \xi_n)$ be coordinates in $\mathbb{R}^n_\xi$ such that the duality is expressed by the bilinear form $< x, \xi >= x_1\xi_1 + \cdots + x_n\xi_n$. If $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \cdots, \gamma_n)$ are multi-indices of nonnegative integers, then $|\beta| = \beta_1 + \cdots + \beta_n$, $\beta + \gamma = (\beta_1 + \gamma_1, \cdots, \beta_n + \gamma_n)$, $\beta! = \beta_1! \cdots \beta_n!$, $\xi^\beta = \xi_1{}^{\beta_1} \cdots \xi_n{}^{\beta_n}$, $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$, and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

Let E_1 and E_2 be topological vector spaces embedded in a topological space E. Denote by $E_1 \bigcap E_2$ and $E_1 + E_2$ the subspace of elements of E_1 being contained in E_2 and the space of sums $\varphi_1 + \varphi_2,\ \varphi_1 \in E_1,\ \varphi_2 \in E_2$, respectively. The topologies of E_1 and E_2 induce topologies in $E_1 \bigcap E_2$ and $E_1 + E_2$. In case E_1 and E_2 are Banach spaces, the Banach norm

$$|\varphi, E_1 \bigcap E_2| = |\varphi, E_1| + |\varphi, E_2|$$

and the norm

$$|\varphi, E_1 + E_2| = \inf_{\varphi_1+\varphi_2=\varphi} (|\varphi_1, E_1| + |\varphi_2, E_2|)$$

are defined on $E_1 \bigcap E_2$ and $E_1 + E_2$, respectively.

Let I denote an open unit cube in $\mathbb{R}^n$:

$$I = \{\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \mid |\omega_j| < 1, j = 1, \ldots, n\}$$

and let $I^{(\kappa)},\ \kappa = 1, \ldots, 2^n$, be the vertices of the cube, i.e., the various vector whose coordinates assume the values ± 1.

We introduce some theorems and propositions to need in this paper which are founded in [5].

THEOREM 1.1. *$f(x) \in F_{(h,\nu)}$ if and only if $f(x)$ can be continued holomorpically to the tube domain $D_h = \{x + yi \in \mathbb{C}^n |\ |y_j| < h, j = 1, 2, \cdots, n\}$ such that*

$$|f(x + yi)| \leq C \exp(-\nu|x|).$$

Let $\nu > 0$. Let $F^{(\nu,s)}$ denote the Banach space of functions $\psi(\zeta)$ holomorphic in the tube domain D_ν and having a finite norm

$$|\psi|^{(\nu,s)} = \sup_{\zeta \in D_\nu} \exp(s|\zeta|)|\psi(\zeta)|. \tag{1.1}$$

PROPOSITION 1.2. *The map $F_{(h,\nu)} \to F^{(h,\nu)} : f(x) \to f(x + yi)$ is a topological isomorphism and there are constants $C_1,\ C_2 > 0$ such that*

$$C_1|f|^{(h,\nu)} \leq |f|_{(h,\nu)} \leq C_2|f|^{(h,\nu)}.$$

REMARK. It follows from Proposition 1.2 that

$$\mathcal{M} = \bigcap_{h>0} F^{(h,-\infty)}. \tag{1.2}$$

PROPOSITION 1.3. *For $h, \nu > 0$ we have*

$$F_{(h,\nu)} = \bigcap_{\kappa=1}^{2^n} F_{[h,\nu I^{(\kappa)}]} \tag{1.3}$$

and (0.1) is equivalent to the natural norm of the right-hand space of (1.3):

$$\sum_{\kappa=1}^{2^n} |\varphi|_{[h,\nu I^{(\kappa)}]}. \tag{0.1'}$$

PROPOSITION 1.4.

(i) *$\mathcal{M}$ is a commutative algebra relative to multiplication.*

(ii) *$\mathcal{G}$ is an ideal in $\mathcal{M}$, i.e., the operation of multiplication is defined:*

$$\mathcal{M} \times \mathcal{G} \to \mathcal{G}\ ((a(x), \psi(x)) \to a(x)\psi(x))$$

and this operator is continuous.

THEOREM 1.5. *The spaces $\mathcal{G}$ and $\mathcal{G}'$ are Fourier self-dual:*

$$\mathfrak{F}\mathcal{G} = \mathcal{G},\ \mathfrak{F}\mathcal{G}' = \mathcal{G}'. \tag{1.4}$$

THEOREM 1.6. *If $\varphi \in F_{(h,\nu)}$, $h, \nu > 0$, then for any $\zeta \in D_\nu$, the absolutely convergent Fourier-Laplace integral is defined:*

$$\mathfrak{F} : \varphi(x) \to \hat{\varphi}(\zeta) = (2\pi)^{-n/2} \int \exp(-i < x, \zeta >)\varphi(x)dx,$$

and Parseval's inequalities hold:

$$C_1|\hat{\varphi}|^{(\nu,h)} \leq |\varphi|_{(h,\nu)} \leq C_2|\hat{\varphi}|^{(\nu,h)}. \tag{1.5}$$

THEOREM 1.7. *The Fourier-Laplace transform operator determines an isomorphism*

$$\mathfrak{F}\mathcal{G} = \bigcap_{h,\nu>0} F^{(\nu,h)}.$$

REMARK. Regarded all function in $\mathcal{M}$ ($\mathcal{G}$ resp.) as an entire function estimated with a special exponential function $\exp(\mu|x|)$(with any exponential function $\exp(\mu|x|)$ resp.), $\mathcal{G}$ is an ideal in $\mathcal{M}$, i.e., each element in $\mathcal{M}$ is a multiplier on $\mathcal{G}$.

By virtue of (0.2), we can define the spaces $\mathcal{O}$ with the aid of the operations of projective and inductive limits:

$$\mathcal{O} = \bigcup_{\nu} F_{(\infty,\nu)},\ F_{(\infty,\nu)} = \bigcap_{h>0} F_{(h,\nu)} \tag{1.6}$$

The spaces $\mathcal{O}$ consists of analytic functions extended to $\mathbb{C}^n$ whose functions increase, for $|Rez| \to \infty$, not stronger than an exponential function $\exp(\mu|Rez|)$.

REMARK. Since $\exp(i\sum_{k=1}^n x_k^2)$ belnogs to $\mathcal{M}$, but does not belong to $\mathcal{O}$, $\mathcal{O}$ is a proper subspace of $\mathcal{M}$.

If $f \in F_{(h,\nu)}$, $g \in F_{(h,-\nu+\epsilon)}$, $h,\ \epsilon > 0$, then the bilinear form

$$(f, g) = \int f(x)g(x)dx \tag{1.7}$$

is defined and depends continuously on f and g (in the corresponding topologies). Hence,

$$\mathcal{G} \subset F_{(h,\nu)} \subset (F_{(h,-\nu+\epsilon)})' \subset \mathcal{G}' \tag{1.8}$$

and the embeddings

$$\mathcal{G} \subset \mathcal{O} \subset \mathcal{G}'. \tag{1.9}$$

take place.

Denote by $\mathcal{O}'$ the space of continuous linear functionals on $\mathcal{O}$. $\mathcal{O}'$ regarded as a vector space can be identified with the projective limit of the conjugate spaces $(F_{(\infty,\nu)})'$. The latter, when treated as vector spaces, are identified with the inductive limit $\bigcup_h (F_{(h,\nu)})'$. Thus,

$$\mathcal{O}' = \bigcap_{\nu} \big(\bigcup_{h} (F_{(h,\nu)})' \big). \tag{1.10}$$

The space $\mathcal{O}'$ are called the space of strongly decreasing Fourier hyperfunctions.

§2. The spaces $H_{<\nu>}$

Let $H_{<\nu>}$ ($\nu \in \mathbb{R}$) denote the space of measurable functions square integrable with weight $\exp(2\nu|x|)$. The corresponding norm is written

$$\|f\|_{<\nu>} = ((f,f)_{<\nu>})^{1/2} = \|\exp(\nu|x|)f\|_{L^2}. \tag{2.1}$$

Then we see that

$$\mathcal{G} \subset F_{(h,\mu+\epsilon)} \subset H_{<\mu>} \subset (F_{(h,-\mu+\epsilon)})' \subset \mathcal{G}',\ \epsilon > 0.$$

REMARK. If $\exp(\nu|x|)$ in (2.1) is replaced by $\delta_{\nu,N}(x) = \exp(\nu \sum_{k=1}^{n} (N^2 + x_k^2)^2)$, we note that (2.1) is equivalent to the norm $\|\delta_{\nu,N}(x)f\|_{L^2}$.

Let $H_{<0>} = H$. Then the mapping $f \to f\exp(\nu|x|)$ determines an isometric isomorphism of $H_{<\nu>}$ and H. It follows that $H_{<-\nu>}$ and the Banach conjugate space of $H_{<\nu>}$ are isometrically isomorphic, i.e.,

$$(H_{<\nu>})' = H_{<-\nu>}. \tag{2.2}$$

Consequently, $H_{<\nu>}$ is a reflexive Banach space.

We can define in $H_{<\nu>}$ the scalar product

$$(f,g)_{<\nu>} = \int \exp(2\nu|x|)f(x)\overline{g(x)}dx \tag{2.3}$$

to which the norm (2.1) corresponds. In other words, $H_{<\nu>}$ is a reflexive Hilbert space.

PROPOSITION 2.1.

(i) *For $\nu > 0$ we have*

$$H_{<\nu>} = \bigcap_{\kappa=1}^{2^n} H_{[\nu I^{(\kappa)}]} \tag{2.4}$$

and (2.1) *is equivalent to the natural norm*

$$\sum_{\kappa=1}^{2^n} \|f\|_{[\nu I^{(\kappa)}]} \tag{2.5}$$

of the right-hand space of (2.4).

(i′) *For $\nu > 0$ the space $H_{<\nu>}$ consists of those and only those elements of the intersection $\bigcap_{\Gamma\in\nu I} H_{[\Gamma]}$ for which the norm*

$$'\|f\|_{<\nu>} = \sup_{\Gamma\in\nu I} \|f\|_{[\Gamma]} \tag{2.1′}$$

is finite.

(ii) *For $\nu < 0$ we have*

$$H_{<\nu>} = \sum_{\kappa=1}^{2^n} H_{[\nu I^{(\kappa)}]}, \tag{2.6}$$

and (2.1) *is equivalent to the norm*

$$\inf_{f=f_1+\cdots+f_n} \sum_{\kappa} \|f_\kappa\|_{[\nu I^{(\kappa)}]}. \tag{2.7}$$

PROOF. (i) It is obvious that

$$\exp(\nu|x_i|) \le \exp(\nu x_i) + \exp(-\nu x_i) \le 2\exp(\nu|x_i|).$$

Multiplying these inequalities for $i = 1, \ldots, n$ we find

$$\exp(\nu|x|) \le \sum_{\kappa=1}^{2^n} \exp(< \nu I^{(\kappa)}, x >) \le 2^n \exp(\nu|x|), \tag{2.8}$$

which implies that (2.1) and (2.5) are equivalent for $\nu > 0$.

(i′) Let $\Gamma = (\omega_1, \dots, \omega_n)$ and let $|\omega_j| < \nu, j = 1, \dots, n$. Multiplying the inequalities

$$\exp(\omega_i x_i) \le \exp(\nu x_i) + \exp(-\nu x_i) \le 2\exp(\nu|x_i|) \tag{2.9}$$

for $i = 1, \dots, n$. we obtain

$$\exp(<\Gamma, x>) \le \sum_{\kappa=1}^{2^n} \exp(<\nu I^{(\kappa)}, x>) \le 2^n \exp(\nu|x|), \tag{2.10}$$

whence

$$'\|f\|_{(\nu)} \le \sum_{\kappa=1}^{2^n} \|f\|_{[\nu I^{(\kappa)}]} \le 2^n \ \|f\|_{<\nu>}.$$

Conversely, let $'\|f\|_{(\nu)} < \infty$. Given $x \in \mathbb{R}^n$, we take $\Gamma = (\epsilon_1\rho, \dots, \epsilon\rho)$, $\rho < \nu$, in (2.10), where $\epsilon_i = \pm 1$ and the sign of ϵ_i coincides with that of x_i. Then we derive

$$(\int \exp(2\rho|x|)|f|^2 dx)^{1/2} \le \ '\|f\|_{(\nu)}$$

for all $\rho < \nu$. By continuity, this inequality is retained for $\rho = \nu$ as well.

(ii) By virtue of the obvious inequality

$$\exp(\nu|x|) \le \exp(<\Gamma, x>), \ \forall \nu < 0, \ \Gamma = (\omega_1, \dots, \omega_n), \ |\omega_j| \le |\nu|,$$

the spaces $H_{[\nu I^{(\kappa)}]}$, $\kappa = 1, 2, \dots, 2^n$, and, consequently, their linear hull as well are embedded in $H_{(\nu)}$. And if $\psi = \sum_{\kappa=1}^{2^n} \psi^{(\kappa)}$, $\psi^{(\kappa)} \in H_{[\nu I^{(\kappa)}]}$, then, by triangle inequality,

$$\|\psi\|_{<\nu>} \le \sum_{\kappa=1}^{2^n} \|\psi^{(\kappa)}\|_{[\nu I^{(\kappa)}]}.$$

Taking the infimum in the right-hand side over all the representation $\psi = \sum_{\kappa=1}^{2^n} \psi^{(\kappa)}$, we prove that the right-hand side space of (2.6) is embedded into the left-hand side space.

To prove the opposite embedding we construct a system of functions $\chi^{(\kappa)}$, $\kappa = 1, \dots, 2^n$, $\chi^{(\kappa)} \ge 0$, possessing the following properties:

(a) $\sum_{\kappa=1}^{2^n} \chi^{(\kappa)}(x) = 1$

(b) If $\psi \in H_{<\nu>}$, then $\chi^{(\kappa)}\psi \in H_{[\nu I^{(\kappa)}]}$, and $\|\chi^{(\kappa)}(x)\psi\|_{[\nu I^{(\kappa)}]} \le const\|\psi\|_{<\nu>}$.

The embedding of the left-hand space of (2.6) into the right-hand side space is a trivial consequence of (a) and (b).

If $x_{i_l}, \dots, x_{i_k} \geq 0$ and $x_{i_{k+1}}, \dots, x_{i_n} < 0$, let $(\epsilon_1, \dots, \epsilon_n)$ be the coordinates of a vertex $I^{(\kappa)}$, where $\epsilon_{i_1} = \dots = \epsilon_{i_k} = 1$ and $\epsilon_{i_{k+1}} = \dots = \epsilon_{i_n} = -1$. Then we put

$$\chi^{(\kappa)}(x) = \prod_{l=1}^{k} \exp(-x_{i_l}^2) \prod_{l=k+1}^{n} (1 - \exp(-x_{i_l}^2)). \tag{2.11}$$

It is obvious that (a) is fulfilled. Since

$$< I^{(\kappa)}, x >= \sum_{l=1}^{k} x_{i_l} - \sum_{l=k+1}^{n} x_{i_l} = |x|,$$

it is clear that (b) holds, whence the proposition is proved. □

For $\delta_{s,N}(\zeta) = \exp(s \sum_{k=1}^{n} (N^2 + \zeta_k^2)^{1/2})$, $|Im\zeta_k| < \nu < N$ we denote by $H_{(s)}^{<\nu>}$ the space of functions $\psi(\zeta)$ holomorphic in the tube domain D_ν and possessing a finite norm

$$\|\psi\|_{(s)}^{<\nu>} = \sup_{|\omega_j| < \nu, j=1,\dots,n} (\int |\delta_{2s,N}(\xi + i\omega)| |\psi(\xi + i\omega)|^2 d\xi)^{1/2}. \tag{2.12}$$

Note that the Fourier-Laplace operator is defined in the spaces $H_{<\nu>}$ for any $\zeta \in D_\nu$ if $\nu > 0$.

THEOREM 2.2. *The Fourier-Laplace operator determines the isomorphism*

$$\mathcal{F} H_{<\nu>} = H^{<\nu>} \tag{2.13}$$

under which the norm (2.1′) *goes into* (2.12) (*for* $s = 0$).

THEOREM 2.3. *$\mathcal{G}$ is dense in H.*

PROOF. First of all, we show that $\mathcal{G}$ is dense in $F_{(\infty,0)} \bigcap H$ relative to the norm $\| \cdot \|_{L^2}$.

Let $f \in F_{(\infty,0)} \bigcap H$. Then we see that $f(x) \exp(-j^{-1} \sum_{k=1}^{n} x_k^2) \in \mathcal{G}$. It follows from the Lebesgue's dominated convergence theorem that

$$\lim_{j \to \infty} \|f(x) \exp(-j^{-1} \sum_{k=1}^{n} x_k^2) - f(x)\| = 0.$$

Therefore it follows.

Next, we show that $F_{(\infty,0)} \bigcap H$ is dense in H.

Let $f \in H$, and let $\varphi_\epsilon(x) = \pi^{-n/2}\epsilon^{-n}\exp(-\sum_{k=1}^{n}(x_k/\epsilon)^2))$. Then we obtain from Young's inequality that $\|f * \varphi_\epsilon\| \leq \|f\|$. We can easily show from Theorem 1.1 that $f_\epsilon(x) = f * \varphi_\epsilon(x)$ belongs to $F_{(\infty,0)} \bigcap H$.

Since C_0^0 is dense in H, for every $\eta > 0$ there exists a function $\tilde{f} \in C_0^0$ such that $\|f - \tilde{f}\| < \eta/2$. Therefore we obtain

$$\begin{aligned}
&\|f_\epsilon - f\| \\
&\leq \|f_\epsilon - \tilde{f}_\epsilon\| + \|\tilde{f}_\epsilon - \tilde{f}\| + \|\tilde{f} - f\| \\
&\leq \|f - \tilde{f}\| + \|\tilde{f}_\epsilon - \tilde{f}\| + \|\tilde{f} - f\| \\
&\leq \eta + \|\tilde{f}_\epsilon - \tilde{f}\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&|\tilde{f}_\epsilon(x) - \tilde{f}(x)| \\
&\leq \pi^{-n/2}\epsilon^{-n}\int |\tilde{f}(x-t) - \tilde{f}(x)| \exp(-\sum_{k=1}^{n}(t_k/\epsilon)^2)dt \\
&= \pi^{-n/2}\epsilon^{-n}\int_{|t_k|\geq\sqrt{\epsilon}} |\tilde{f}(x-t) - \tilde{f}(x)| \exp(-\sum_{k=1}^{n}(t_k/\epsilon)^2)dt \\
&\quad + \pi^{-n/2}\epsilon^{-n}\int_{|t_k|\leq\sqrt{\epsilon}} |\tilde{f}(x-t) - \tilde{f}(x)| \exp(-\sum_{k=1}^{n}(t_k/\epsilon)^2)dt \\
&\leq 2\pi^{-n/2}\epsilon^{-n}\sup_x |\tilde{f}(x)| \int_{|t_k|\geq\sqrt{\epsilon}} \exp(-\sum_{k=1}^{n}(t_k/\epsilon)^2)dt \\
&\quad + \sup_{|t_k|\leq\sqrt{\epsilon}} |\tilde{f}(x-t) - \tilde{f}(x)|.
\end{aligned}$$

Since $\tilde{f}(x)$ is uniformly continuous, $\tilde{f}_\epsilon - \tilde{f} \to 0$ uniformly, so $\|\tilde{f}_\epsilon - \tilde{f}\| \to 0$ as $\epsilon \to 0_+$. Therefore it follows, and hence proves the theorem. □

REMARK. The operator of multiplication by

$$\exp(-\nu\sum_{k=1}^{n}(N^2 + x_k^2)^{1/2})$$

generates an isomorphism of H and $H_{<\nu>}$ under which the spaces $\mathcal{G}$, $F_{(\infty,0)} \bigcap H$ is preserved. Since $\mathcal{G}$ is dense in H, it is dense in $H_{<\nu>}$ as well, and $H_{<\nu>}$ can be regarded as the completion of $\mathcal{G}$ with respect to $\|\cdot\|_{<\nu>}$.

§3. The sructure of a countably Hilbert space in $\mathcal{G}$

Note that $\mathcal{G} \subset H_{<s>} \subset \mathcal{G}'$. Then it follows from Theorem 1.5 that

$$\mathcal{G} \subset \mathfrak{F}^{-1} H_{<s>} \subset \mathcal{G}'.$$

Let

$$H^{(s)} = \mathfrak{F}^{-1} H_{<s>}.$$

Since the composition of $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ is an identity operator in $\mathcal{G}'$, we have

$$\mathfrak{F} H^{(s)} = H_{<s>}.$$

We introduce the norm

$$\|f\|^{(s)} = \|\mathfrak{F} f\|_{<s>}$$

in the space $H^{(s)}$. Thus, $H^{(s)}$ consists of those functions $f \in \mathcal{G}'$, possessing Fourier transforms, which are square summable and whose norm $\|f\|^{(s)}$ is finite.

By pseudodifferential operators (PDO) in $\mathcal{G}$ are meant operators having the form

$$(3.1) \qquad (a(D)\varphi)(x) = (2\pi)^{-n/2} \int \exp(i < x, \xi >) a(\xi) \hat{\varphi}(\xi) d\xi.$$

Such an operator can be written as a composition of three operators:

$$(3.2) \qquad (a(D)\varphi)(x) = \mathfrak{F}^{-1}_{\xi \to x} a(\xi) \mathfrak{F}_{x \to \xi} \varphi.$$

The function $a(\xi)$ is called the symbol of the operator. We now consider the case $\varphi \in \mathcal{G}$ and take, as symbols $a(\xi)$, functions belonging to $\mathcal{M}$. Then, by virtue of Proposition 1.4 (ii) and Theorem 1.5, (3.2) is a composition of three continuous operators transforming $\mathcal{G}$ into itself and hence is a continuous operator from $\mathcal{G}$ into $\mathcal{G}$.

If $f \in \mathcal{G}'$ and $a(\xi) \in \mathcal{M}$, then we define by $a(D)f$ the functional

$$(3.3) \qquad (a(D)f, \varphi) = (f, a(-D)\varphi), \ \forall \varphi \in \mathcal{G}.$$

Fix $\nu > 0$. Each function $a(\zeta) \in F^{(\nu, -\infty)}$ is a multiplier on the space $F^{(\nu', \infty)}$, $\nu' < \nu$, and for $\varphi \in F_{(\infty, \nu')}$, we can define the PDO

$$a(D)\varphi = (2\pi)^{-n/2} \int \exp(i < \xi + i\Gamma, x >) a(\xi + i\Gamma) \hat{\varphi}(\xi + i\Gamma) d\xi, \ \Gamma \in \nu' I.$$

Then Theorem 1.1 and 1.6 imply that the operator

$$a(D): F_{(\infty,\nu')} \to F_{(\infty,\nu')},\ 0 < \nu' < \nu,\ a(\zeta) \in F^{(\nu,-\infty)}$$

is continuous.

We now include the zeroth space $H_{<\nu>}$, $\nu > 0$, in the scale $\{H^{(s)}_{<\nu>}\}$ generated with respect to PDO. To this end we define the symbols

$$\delta_{s,N}(\zeta) = \exp(s\sum_{k=1}^{n}(N^2 + \zeta_k{}^2)^{1/2}). \tag{3.4}$$

Note that if $s \geq 0$, then for $|Im\zeta_k| < \nu < N$,

$$C_1 \exp(s/\sqrt{2}|\zeta|) \leq |\delta_{s,N}(\zeta)| \leq C_2 \exp(s|\zeta|). \tag{3.5}$$

We put

$$H^{(s)}_{<\nu>} = \{f \in H^{(-\infty)} |\ \delta_{s,N}(D)f \in H_{<\nu>},\ 0 \leq \nu < N\} \tag{3.6}$$

and equip this space with the norm

$$\|f\|^{(s)}_{<\nu>} = \|\delta_{s,N}(D)f\|_{<\nu>},\ 0 \leq \nu < N. \tag{3.7}$$

We can introduce in this space a Hilbert scalar product (to which the norm (3.7) corresponds):

$$(f,g)^{(s)}_{<\nu>} = (\delta_{s,N}(D)f, \delta_{s,N}(D)g)_{<\nu>}, 0 \leq \nu < N. \tag{3.8}$$

From (3.6) and Proposition 2.1 we can readily derive other equivalent definitions of the space $H^{(s)}_{<\nu>}$. We have the following

PROPOSITION 3.1. *Let $\nu > 0$. Then the conditions below are equivalent for the Fourier hyperfunctions $f \in H^{(-\infty)}$.*

(i) $\delta_{s,N}(D)f \in H_{<\nu>}$.
(i′) $f = \delta_{-s,N}(D)g,\ g \in H_{<\nu>}$.
(ii) $f \in \bigcap_{\kappa=1}^{2^n} H^{(s)}_{[\nu I^{(\kappa)}]}$.
(ii′) $f \in \bigcap_{\Gamma \in \nu I} H^{(s)}_{[\Gamma]}$, *and the norm*

$$'\|f\|^{(s)}_{<\nu>} = \sup_{\Gamma \in \nu I} \|f\|^{(s)}_{[\Gamma]} \tag{3.7′}$$

is finite.

(iii) *The Fourier transform $\hat{f} \in H_{<-\infty>}$ is continued holomorphically to the cube domain D_ν and belongs to $H^{<\nu>}_{(s)}$.*

The condition (iii) implies that

$$\mathfrak{F}H^{(s)}_{<\nu>} = H^{<\nu>}_{(s)}. \tag{3.9}$$

Indeed, if $f \in H^{(s)}_{<\nu>}$, then we have

$$\begin{aligned} '\|f\|^{(s)}_{<\nu>} &= \sup_{\Gamma\in\nu I} \|f\|^{(s)}_{[\Gamma]} = \sup_{\Gamma\in\nu I} \|\exp(< x, \Gamma >)\delta_{s,N}(D)f\| \\ &= \sup_{\Gamma\in\nu I} \|\delta_{s,N}(\xi + i\Gamma)\hat{f}(\xi + i\Gamma)\| = \|\hat{f}\|^{<\nu>}_{(s)}. \end{aligned}$$

PROPOSITION 3.2. *For $s \geq s'$, $\nu \geq \nu' \geq 0$, we have*

$$H^{(s)}_{<\nu>} \subset H^{(s')}_{<\nu'>}. \tag{3.10}$$

i.e., $\{H^{(s)}_{<\nu>}\}$ is a scale of Banach (Hilbert) spaces.

PROOF. If $f \in H^{(s)}_{<\nu>}$, then it follows from Proposition 3.1 (i′) that $f = \delta_{-s,N}(D)g$, $g \in H_{<\nu>}$. Therefore we have

$$\begin{aligned} \|f\|^{(s')}_{<\nu'>} &= \|\exp(\nu'|x|)\delta_{s',N}(D)f\| = \|\exp(\nu'|x|)\delta_{-(s-s'),N}(D)g\| \\ &\leq \sum_{\kappa=1}^{2^n} \|\exp(< x, \nu' I^{(\kappa)} >)\delta_{-(s-s'),N}(D)g\| \\ &= \sum_{\kappa=1}^{2^n} \|\delta_{-(s-s'),N}(\xi + i\nu' I^{(\kappa)})\hat{g}(\xi + i\nu' I^{(\kappa)})\| \\ &\leq C\sum_{\kappa=1}^{2^n} \|\hat{g}(\xi + i\nu' I^{(\kappa)})\| = C\sum_{\kappa=1}^{2^n} \|\exp(< x, \nu' I^{(\kappa)} >)g(x)\| \\ &\leq 2^n C\|f\|^{(s)}_{<\nu>}. \end{aligned}$$

□

We shall show that this scale is equivalent to Hölder's scale.

PROPOSITION 3.3. *For every $h, \nu > 0$ we have the embeddings*

$$F_{(h,\nu+\epsilon)} \subset H^{(h/2)}_{<\nu>} \subset F_{(h/4,\nu)},\ \epsilon > 0. \tag{3.11}$$

PROOF. Let $f \in F_{(h,\nu+\epsilon)}$. Then we have

$$\begin{aligned}
\|f\|_{<\nu>}^{(h/2)} &= \|\exp(\nu|x|)\delta_{h/2,N}(D)f\| \\
&\leq \sum_{\kappa=1}^{2^n} \|\exp(< x, \nu I^{(\kappa)} >)\delta_{h/2,N}(D)f\| \\
&= \sum_{\kappa=1}^{2^n} \|\delta_{h/2,N}(\xi + i\nu I^{(\kappa)})\hat{f}(\xi + i\nu I^{(\kappa)})\| \\
&\leq C\sum_{\kappa=1}^{2^n} \|\exp(h/2|\xi + i\nu I^{(\kappa)}|)\hat{f}(\xi + i\nu I^{(\kappa)})\| \\
&\leq C\sum_{\kappa=1}^{2^n}\sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!}\|(\xi + i\nu I^{(\kappa)})^{\alpha}\hat{f}(\xi + i\nu I^{(\kappa)})\| \\
&= C\sum_{\kappa=1}^{2^n}\sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!}\|\exp(< x, \nu I^{(\kappa)} >)D^{\alpha}f\| \\
&\leq C\sum_{\kappa=1}^{2^n}\sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!}\|\exp(\nu|x|)D^{\alpha}f\| \\
&\leq 2^n C\sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!}|f|_{(h,\nu+\epsilon)}h^{-|\alpha|}\alpha!\|\exp(-\epsilon|x|)\|.
\end{aligned}$$

This proves the left-hand embedding of (3.11).

To prove the right-hand embedding of (3.11), from Proposition 1.3 and 3.1 (ii) it suffices to show sobolev's embedding theorem written in the form

$$H^{(h)}_{[\nu I^{(\kappa)}]} \subset F_{[h/2,\nu I^{(\kappa)}]}. \tag{3.12}$$

Let $f \in H^{(h)}_{[\nu I^{(\kappa)}]}$. Then we have

$$\begin{aligned}
&\int |\widehat{D^{\gamma}f}(\xi + i\nu I^{(\kappa)})|d\xi \\
&= \int |\delta_{h,N}(\xi + i\nu I^{(\kappa)})\hat{f}(\xi + i\nu I^{(\kappa)})(\xi + i\nu I^{(\kappa)})^{\gamma}\delta_{-h,N}(\xi + i\nu I^{(\kappa)})|d\xi \\
&\leq \|\delta_{h,N}(\xi + i\nu I^{(\kappa)})\hat{f}(\xi + i\nu I^{(\kappa)})\|\|(\xi + i\nu I^{(\kappa)})^{\gamma}\delta_{-h,N}(\xi + i\nu I^{(\kappa)})\|
\end{aligned}$$

$$= \|\exp(< x, \nu I^{(\kappa)} >)\delta_{h,N}(D)f\|\|(\xi + i\nu I^{(\kappa)})^\gamma \delta_{-h,N}(\xi + i\nu I^{(\kappa)})\|$$
$$\leq C\gamma!(h/2)^{-|\gamma|}\|f\|^{(h)}_{[\nu I^{(\kappa)}]}\|\exp(-(\sqrt{2}-1)h/2|\xi + i\nu I^{(\kappa)}|)\|$$
$$\leq C'\gamma!(h/2)^{-|\gamma|}\|f\|^{(h)}_{[\nu I^{(\kappa)}]}.$$

Since

$$D^\gamma f(x) = (2\pi)^{-n/2}\int \exp(i < x, \xi + i\nu I^{(\kappa)} >)\widehat{D^\gamma f}(\xi + i\nu I^{(\kappa)})d\xi,$$

we obtain

$$|f|_{[h/2,\nu I^{(\kappa)}]} \leq (2\pi)^{-n/2}C'\|f\|^{(h)}_{[\nu I^{(\kappa)}]}.$$

□

From (3.11) it follows automatically that

$$\mathcal{G} = \bigcap_{h,\nu} H^{(h)}_{<\nu>}. \tag{3.13}$$

§4. The space $\mathcal{G}'$ as an inductive limit of Hilbert spaces

For $\nu > 0$ we define $H^{(-s)}_{<-\nu>}$, $s \in \mathbb{R}$ as the Banach conjugate space of $H^{(s)}_{<\nu>}$ and introduce in it the norm of a conjugate space:

$$\|f\|^{(-s)}_{<-\nu>} = \sup\{|(f,g)| \; \|g\|^{(s)}_{<\nu>} \leq 1\}. \tag{4.1}$$

It follows from (2.2) that (4.1) is compatible with (2.1) for $s = 0$. Since $\mathcal{G}$ is dense in $H^{(s)}_{<\nu>}$, the functionals belonging to $H^{(-s)}_{<-\nu>}$ can be regarded as elements of $\mathcal{G}'$.

Note that $H^{(h/2)}_{<\nu>} \subset H_{<\nu>}$ and $F_{(h,-\nu+\epsilon)} \subset H_{<-\nu>}$. Therefore each function $f \in F_{(h,-\nu+\epsilon)}$, $\epsilon > 0$, belongs to $H^{(-h/2)}_{<-\nu>}$.

Proposition 4.1. *For $h, \nu > 0$ we have*

$$F_{(h,-\nu+\epsilon)} \subset H^{(h/2)}_{(-\nu)} \tag{4.2}$$

PROOF. Let $f \in F_{(h,-\nu+\epsilon)}$ and $g \in H^{(-h/2)}_{<\nu>}$. Then it follows from Proposition 3.1 (i′) that $g = \delta_{h/2,N}(D)g_0$, $g_0 \in H_{<\nu>}$. We construct a system of functions $\chi^{(\kappa)}$, $\kappa = 1, \dots, 2^n$, $\chi^{(\kappa)} \geq 0$, possessing the following properties:

(a) $\sum_{\kappa=1}^{2^n} \chi^{(\kappa)}(x) = 1$

(b) If $f \in F_{(h,-\nu+\epsilon)}$, then $\chi^{(\kappa)} f \in F_{[h,(-\nu+\epsilon)I^{(\kappa)}]}$, and $|\chi^{(\kappa)} f|_{[h,(-\nu+\epsilon)I^{(\kappa)}]} \leq const|f|_{(h,-\nu+\epsilon)}$.

If $x_{i_1}, \dots, x_{i_k} \geq 0$ and $x_{i_{k+1}}, \dots, x_{i_n} < 0$, let $(\epsilon_1, \dots, \epsilon_n)$ be the coordinates of a vertex $I^{(\kappa)}$, where $\epsilon_{i_1} = \cdots = \epsilon_{i_k} = 1$ and $\epsilon_{i_{k+1}} = \cdots = \epsilon_{i_n} = -1$. Then we put

$$\chi^{(\kappa)}(x) = \prod_{l=1}^{k} \exp(-x_{i_l}^2) \prod_{l=k+1}^{n} (1 - \exp(-x_{i_l}^2)).$$

It is obvious that (a) is fulfilled.

Since $< I^{(\kappa)}, x >= \sum_{l=1}^{k} x_{i_l} - \sum_{l=k+1}^{n} x_{i_l} = |x|$, it follows from the proof of Theorem 1.1 that (b) holds. Then we have

$$\begin{aligned}
&|(f,g)| \\
&\leq \sum_{\kappa=1}^{2^n} |(\chi^{(\kappa)} f, g)| \\
&= \sum_{\kappa=1}^{2^n} |(\exp(-\nu|x|)\delta_{h/2,N}(D)\chi^{(\kappa)} f, \exp(\nu|x|)g_0)| \\
&\leq \sum_{\kappa=1}^{2^n} \| \exp(< -\nu I^{(\kappa)}, x >)\delta_{h/2,N}(D)\chi^{(\kappa)} f\| \| \exp(\nu|x|)g_0\| \\
&\leq \sum_{\kappa=1}^{2^n} \|\delta_{h/2,N}(\xi - i\nu I^{(\kappa)})\widehat{\chi^{(\kappa)} f}(\xi - i\nu I^{(\kappa)})\| \|g\|^{(-h/2)}_{<\nu>} \\
&\leq C \sum_{\kappa=1}^{2^n} \| \exp(h/2|\xi - i\nu I^{(\kappa)}|)\widehat{\chi^{(\kappa)} f}(\xi - i\nu I^{(\kappa)})\| \|g\|^{(-h/2)}_{<\nu>} \\
&\leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!} \|(\xi - i\nu I^{(\kappa)})^{\alpha} \widehat{\chi^{(\kappa)} f}(\xi - i\nu I^{(\kappa)})\| \|g\|^{(-h/2)}_{<\nu>} \\
&\leq C \sum_{\kappa=1}^{2^n} \sum_{\alpha} \frac{(h/2)^{|\alpha|}}{\alpha!} \| \exp(< -\nu I^{(\kappa)}, x >)D^{\alpha}(\chi^{(\kappa)} f)(x)\| \|g\|^{(-h/2)}_{<\nu>}
\end{aligned}$$

$$\leq C\sum_{\kappa=1}^{2^n}\sum_{\alpha}\frac{(h/2)^{|\alpha|}}{\alpha!}|\chi^{(\kappa)}f|_{[h,(-\nu+\epsilon)I^{(\kappa)}]}h^{-|\alpha|}\alpha!\|\exp(-\epsilon|x|)\|\|g\|^{(-h/2)}_{<\nu>}$$
$$\leq C'|f|_{(h,-\nu+\epsilon)}\|g\|^{(-h/2)}_{<\nu>},$$

whence it follows. □

Thus, we have

$$\mathcal{G}\subset F_{(h,-\nu+\epsilon)}\subset H^{(\pm h/2)}_{<-\nu>}\subset\mathcal{G}'\ \forall h,\nu>0, \tag{4.3}$$

where all the embeddings are dense.

PROPOSITION 4.2. *For $h,\nu>0$, we have*

$$H^{(h/2)}_{<-\nu>}\subset F_{(h/4,-\nu)}. \tag{4.4}$$

PROOF. Let $f\in H^{(h/2)}_{<-\nu>}$. Since

$$\begin{aligned}&D^\alpha f(x)\\&=(2\pi)^{-n/2}\int_{Im\zeta_j=\omega_j}\exp(i<x,\zeta>)\zeta^\alpha\hat{f}(\zeta)d\xi,\ \omega_j=-\nu\frac{|x_j|}{x_j}\\&=(2\pi)^{-n/2}\int_{Im\zeta_j=\omega_j}\exp(i<x,\zeta>)\delta_{-h/2,N}(\zeta)\zeta^\alpha\delta_{h/2,N}(\zeta)\hat{f}(\zeta)d\xi,\end{aligned}$$

we have

$$\begin{aligned}&|D^\alpha f(x)|\\&\leq(2\pi)^{-n/2}\exp(\nu|x|)\|\delta_{-h/2,N}(\zeta)\zeta^\alpha\|\|\delta_{h/2,N}(\zeta)\hat{f}(\zeta)\|\\&\leq(2\pi)^{-n/2}\exp(\nu|x|)\|\exp(-\frac{h}{2\sqrt{2}}|\zeta|)\zeta^\alpha\|\\&\quad\times\|\exp(<x,Im\zeta>)\delta_{h/2,N}(D)f(x)\|\\&\leq(2\pi)^{-n/2}\alpha!(\frac{h}{2\sqrt{3}})^{-|\alpha|}\exp(\nu|x|)\|\exp(-\frac{h}{2}(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}})|\zeta|)\|\|f\|^{(h/2)}_{<-\nu>}\\&\leq(2\pi)^{-n/2}\alpha!(\frac{h}{4})^{-|\alpha|}\exp(\nu|x|)\|\exp(-\frac{h}{2}(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}})|\zeta|)\|\|f\|^{(h/2)}_{<-\nu>},\end{aligned}$$

whence it follows. □

Since the spaces $H^{(s)}_{<\nu>}$, $\nu \geq 0$, form a scale, the spaces $H^{(-s)}_{<-\nu>}$ form the dual scale, and we can consider the inductive limits $H^{(-\infty)}_{<\nu>}$, $H^{(-\infty)}_{<-\nu>}$, and endow them with the natural topology. By virtue of the reflexivity of $H^{(s)}_{<\nu>}$, these limits are regular, and, according to the general properties of regular inductive limits, we have the topological isomorphisms

$$(4.5) \qquad (H^{(\infty)}_{<\nu>})' = H^{(-\infty)}_{<-\nu>},$$

$$(4.6) \qquad \mathcal{G}' = \bigcup_{s,\nu} H^{(s)}_{<\nu>},$$

where the left-hand and right-hand spaces are equipped with topologies of strong conjugate space and inductive limits, respectively.

In view of the duality (4.5), we can define PDO on $H^{(-\infty)}_{<-\nu>}$. Let $a(\zeta) \in F^{(N,-\infty)}$, and for $N > \nu$ we put

$$(4.7) \qquad (a(D)f, g) = (f, a(-D)g), \ \forall g \in H^{(\infty)}_{<\nu>}.$$

On the dense subset $\mathcal{G} \subset H^{(-s)}_{<-\nu>}$ this definition is compatible with the one stated in (3.4). Proposition 3.1 (i ′) and (2.2) imply the isomorphism

$$(4.8) \qquad \delta_{-s,N}(D) H^{(-s)}_{<-\nu>} = H_{<-\nu>}$$

with the equality of norms :

$$(4.1') \qquad \|f\|^{(-s)}_{<-\nu>} = \|\delta_{-s,N}(D)f\|_{<-\nu>}.$$

Proposition 2.1 (ii) and (4.8) imply the following

PROPOSITION 4.3. *For $\nu > 0$ and any $s \in R$ we have*

$$(4.9) \qquad H^{(-s)}_{<-\nu>} = \sum_{\kappa=1}^{2^n} H^{(-s)}_{[-\nu I^{(\kappa)}]},$$

and (4.1) is equivalent to the norm

$$(4.1'') \qquad \inf_{f=f_1+\cdots+f_n} \sum_{\kappa=1}^{2^n} \|f_\kappa\|^{(-s)}_{[-\nu I^{(\kappa)}]}$$

of the right-hand space of (4.9).

(3.11), (4.3) and Proposition 4.2 implies

$$\mathcal{O} = \bigcup_{\nu} H^{(\infty)}_{<\nu>}, \; \mathcal{O}' = \bigcap_{\nu} H^{(-\infty)}_{<\nu>}, \tag{4.10}$$
$$\mathcal{M} = \bigcap_{s} H^{(s)}_{<-\infty>}.$$

Denote by $F^{<\nu>}_{(s)}$ the space of holomorphic functions in the tube domain D_ν and having a finite norm

$$|\psi|^{<\nu>}_{(s)} = \sup_{\zeta \in D_\nu} |\delta_{s,N}(\zeta)||f(\zeta)|. \tag{4.11}$$

It follows from Proposition 1.2 and (3.5) that if $s \geq 0$, then

$$F_{(\nu,s)} \simeq F^{(\nu,s)} \subset F^{<\nu>}_{(s)} \subset F^{(\nu,s/\sqrt{2})} \simeq F_{(\nu,s/\sqrt{2})}, \tag{4.12}$$

and hence (1.2) implies that

$$\mathcal{M} = \bigcap_{\nu} F^{<\nu>}_{(-\infty)}. \tag{1.2'}$$

PROPOSITION 4.4. *The following isomorphisms of vector spaces hold:*

$$\mathcal{F}\mathcal{O}' = \mathcal{M}, \tag{4.13}$$

where $\mathcal{F}$ is a Fourier-Laplace operator.

PROOF. By (3.9) and (4.10) we obtain

$$\mathcal{F}\mathcal{O}' = \bigcap_{\nu} \bigcup_{s} \mathcal{F} H^{(s)}_{<\nu>} = \bigcap_{\nu} \bigcup_{s} H^{<\nu>}_{(s)} = \bigcap_{\nu} H^{<\nu>}_{(-\infty)}.$$

Hence, the proof of (4.13) reduces to the proof of the equivalence of the scales $\{F^{<\nu>}_{(s)}\}$ and $\{H^{<\nu>}_{(s)}\}$ where $\nu > 0$. We shall prove the following.□

PROPOSITION 4.5. *For every $\nu > 0$, the embeddings*

$$F^{<\nu>}_{(s+\epsilon)} \subset H^{<\nu>}_{(s)} \subset F^{<\nu>}_{(s-\epsilon')}, \; \forall \epsilon, \epsilon' > 0, \tag{4.14}$$

take place.

PROOF. If $N > \nu$, then the operator of multiplication by the function $\delta_{r,N}(\zeta)$ generates the isomorphisms

$$F_{(s)}^{<\nu>} \to F_{(s-r)}^{<\nu>},\ H_{(s)}^{<\nu>} \to H_{(s-r)}^{<\nu>}\ (\psi \to \delta_{r,N}(\zeta)\psi).$$

Hence, when proving (4.14) we can confine ourselves to the case of $s > \epsilon'$.

The left-side inclusion in (4.14) is obvious. According to (3.9), (1.5), (4.12) and Proposition 3.3,

$$H_{(s)}^{<\nu>} = \mathfrak{F}H_{<\nu>}^{(s)} \subset \mathfrak{F}F_{(s/2,\nu)} \subset F^{(\nu,s/2)} \subset F_{(s/2)}^{<\nu>}.$$

This proves the proposition. □

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