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THE BARTLE INTEGRAL AND THE CONDITIONAL WIENER INTEGRAL ON $C[0,t]$

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Abstract. In this paper, we give a new formula between the conditional Wiener integral $E(F|X)$, the conditional Wiener integral of F given X, and the integral with respect to a measure-valued measure, a kind of Bartle integral. Using this formula, we give some examples of evaluation of $E(F|X)$.

1. Introduction

In 1975, J. Yeh has introduced the concepts of conditional Wiener integrals $E(F|X)$, the conditional Wiener integral of F given X. He, in his paper [11], derived some inversion formulas for $E(F|X)$ and evaluated some conditional Wiener integrals conditioned by $X(x) = x(t)$ for a fixed t and x in Wiener space. In 1984, K. S. Chang and J. S. Chang extended Yeh's results to the random vector X given by $X(x) = (x(t_1), x(t_2), \cdots, x(t_n))$ in [1]. C. Park and D. L. Skoug presented a quit simple formula for the calculation of conditional Wiener integral in [6].

In this article, we will give a new formula for conditional Wiener integrals $E(F|X)$, which it is represeented by the integral with respect to a measure-valued measure V_X, the so-called Bartle integral, on Wiener space and using our formula, we will give some evaluations of $E(F|X)$.

2. Preliminaries; notations and lemmas

For a positive real number t, let $C[0,t]$ be the collection of all real-valued continuous functions on $[0,t]$ that vanish at 0. The space $C[0,t]$ with the uniform topology is called the Wiener space. Consider the

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Wiener measure space $(C[0,t], \mathcal{B}(C[0,t]), m_w)$, where $\mathcal{B}(C[0,t])$ is the smallest σ-algebra containing all open sets of $C[0,t]$ and m_w is the Wiener measure on $(C[0,t], \mathcal{B}(C[0,t]))$.

For a fixed natural number n, let $X : C[0,t] \to \mathbb{R}^n$ be a Borel measurable function and $Z : C[0,t] \to \mathbb{R}$ be integrable with respect to the Wiener measure m_w. The set function P_X on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ defined by $P_X(E) = m_w(X^{-1}(E))$. Then P_X is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. The conditional Wiener expectation of Z given X, written $E(Z|X)$, is defined to be a real-valued Borel measrable and P_X-integrable function ψ on $\mathbb{R}^n$ such that

$$\int_{X^{-1}(E)} Z(x)dm_w(x) = \int_E \psi(\xi)dP_X(\xi) \tag{2.1}$$

for E in $\mathcal{B}(\mathbb{R}^n)$. The existence of such a function $E(Z|X)$ follows from the Radon-Nikodym theorem.

We can find the following facts in [9, p. 211-216].

REMARK 2.1. (a) If Z_1 and Z_2 are m_w-integrable and if a, b are in $\mathbb{R}$ then

$$E(aZ_1 + bZ_2|X) = aE(Z_1|X) + bE(Z_2|X). \tag{2.2}$$

(b) If Z_1 and Z_2 are m_w-integrable with $Z_1 \le Z_2$ m_w-*a.e.* then

$$E(Z_1|X) \le E(Z_2|X) \quad P_X - a.e. \tag{2.3}$$

(c) (Conditional form of Lebesgue monotone convergence theorem) If $\langle Z_n \rangle$ is a nondecreasing sequence of nonnegative real-valued Borel measurable functions on $C[0,t]$ and if $\langle Z_n \rangle$ converges to Z m_w-*a.e.* then a sequence $\langle E(Z_n|X) \rangle$ converges to $\langle E(Z|X) \rangle$ P_X-*a.e.*

The following lemma follows from Theorem 3 in [11, p. 629].

LEMMA 2.2. *If the Radon-Nikodym derivative* $\frac{dP_X}{dm_L}$ *exists and* $\int_{C[0,t]} exp(i\langle u, X(x)\rangle)Z(x)dm_w(x)$ *is a Lebesgue integrable function of* u *on* $\mathbb{R}^n$ *then*

$$\begin{aligned} &E(Z|X)(\xi)\frac{dP_X}{dm_L}(\xi) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^n} exp(-i\langle u, \xi\rangle)\int_{C[0,t]} exp(i\langle u, X(x)\rangle)Z(x)dm_w(x)dm_L(u) \end{aligned} \tag{2.4}$$

for ξ in $\mathbb{R}^n$. Here $\langle u, \xi\rangle$ is an usual inner product in $\mathbb{R}^n$.

For B in $\mathcal{B}(C[0,t])$ and for E in $\mathcal{B}(\mathbb{R}^n)$, we let

$$[V_X(B)](E) = m_w(B \cap X^{-1}(E)). \tag{2.5}$$

Clearly, $V_X(B)$ is a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ for B in $\mathcal{B}(C[0,t])$ and V_X is a measure-valued measure on $(C[0,t], \mathcal{B}(C[0,t]))$ in the total variation norm sense.

REMARK 2.3. (a) If the Radon-Nikodym derivative $\frac{dP_X}{dm_L}$ exists and E is m_L-null in $\mathbb{R}^n$ then since for B in $\mathcal{B}(C[0,t])$,

$$[V_X(B)](E) \le m_w(X^{-1}(E)) = P_X(E) = 0,$$

for B in $\mathcal{B}(C[0,t])$, $V_X(B)$ is absolutly continuous with respect to m_L.

(b) Let $\mathcal{RM}(\mathbb{R}^n)$ be the space of all finite $\mathbb{R}$-valued measures μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that are absolutly continuous with respect to m_L. Then $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is isomorphic to $L^1(\mathbb{R}^n, m_L)$ and the dual space $\mathcal{RM}(\mathbb{R}^n)^*$ of $\mathcal{RM}(\mathbb{R}^n)$ is isomorphic to $L^\infty(\mathbb{R}^n, m_L)$. For y^* in $\mathcal{RM}(\mathbb{R}^n)^*$, there is a function θ in $L^\infty(\mathbb{R}^n, m_L)$ such that $x^*(\mu) = \int_{\mathbb{R}^n} \theta(s)\, d\mu$ for μ in $\mathcal{RM}(\mathbb{R}^n)$.

(c) For B in $\mathcal{B}(C[0,t])$, the semivariation $||V_X||(B)$ of V_X on B is

$$\begin{aligned} &||V_X||(B) \\ =\ & sup\{|x^*V_X|(B) : x^* \text{is in } \mathcal{RM}(\mathbb{R}^n)^* \text{ and } ||x^*|| \le 1\} \\ =\ & \{|\int_{\mathbb{R}^n} \theta(s) d[V_X(B)](s)| : |\theta(s)| \le 1 \;\; m_L - a.e.\} \\ \le\ & [V_X(B)](\mathbb{R}^n) \\ \le\ & m_w(B) \end{aligned} \tag{2.6}$$

where $|x^*V_X|(B)$ is the total variation on B of real-valued measure x^*V_X.

By [3, Theorem 10, p. 328], we obtain the following lemma.

LEMMA 2.4. (The Dominated convergence theorem for the Bartle integral) *Let $\mathbb{B}$ be a Banach space and let $(Y, \mathcal{C}, \nu)$ be a $\mathbb{B}$-valued measure space. If $\langle f_n \rangle$ is a sequence of ν-Bartle integrable functions that converges $\|\ \nu\ \|$-a.e. to f and if g is a ν-Bartle integrable function such that $|f_n(s)| \le |g(s)|\ \|\ \nu\ \|$-a.e. s for all natural number n then f is ν-Bartle integrable and for E in $\mathcal{C}$,*

$$(Ba) - \int_E f(s)\, d\nu(s) = \lim_{n\to\infty} (Ba) - \int_E f_n(s)\, d\nu(s). \tag{2.7}$$

Hence, $(Ba) - \int_E f(s)\, d\nu(s)$ means the Bartle integral of f with respect to ν, the integral of scalar-valued function f with respect to a vector measure ν.

By [5, Theorem 2.4, p. 162], we have the following lemma.

LEMMA 2.5. *Suppose the Radon-Nikodym derivative* $\frac{dP_X}{dm_L}$ *exists. Then* f *is* V_X*-Bartle integrable if and only if for each* x^* *in* $\mathcal{RM}(\mathbb{R}^n)^*$, f *is* x^*V_X*-integrable and there is an element* $(Ba)-\int_{C[0,t]} f(x)dV_X(x)$ *in* $\mathcal{RM}(\mathbb{R}^n)$ *such that*

$$x^*\left[(Ba)-\int_{C[0,t]} f(x)\ dV_X(x)\right] = \int_{C[0,t]} f(x)\ dx^*V_X(x) \tag{2.8}$$

for each x^* *in* $\mathcal{RM}(\mathbb{R}^n)^*$.

For a convenience, we let

$$W(n;s_1,s_2,\cdots,s_n;u_1,u_2,\cdots,u_n) = \left[(2\pi)^n\prod_{j=1}^{n}(s_j-s_{j-1})\right]^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{s_j-s_{j-1}}\right\} \tag{2.9}$$

for $0=s_0<s_1<s_2<\cdots<s_n\le t$. Here, we let $u_0=0$.

Now, we establish a formula for the Bartle integral with respect to V_X, which play the impotant role in this paper.

LEMMA 2.6. *Let* $0=t_0<s_1<s_2<\cdots<s_{m_1}=t_1<s_{m_1+1}<\cdots<s_{m_2}=t_2<\cdots<s_{m_{n-1}}=t_{n-1}<s_{m_{n-1}+1}<\cdots<s_{m_n}=t_n=t$ *and let* $X: C[0,t]\to\mathbb{R}^n$ *be a function with* $X(s)=(x(t_1),x(t_2),\cdots,x(t_n))$. *Let* $f:\mathbb{R}^{m_n}\to\mathbb{R}$ *be a Borel measurable function and let* $F: C[0,t]\to\mathbb{R}$ *be a function with* $F(x)=f(x(s_1),x(s_2),\cdots,x(s_{m_n}))$. *Then for* E *in* $\mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned}&\left[\int_{C[0,t]} F(x)\ dV_X(x)\right](E)\\ &\overset{*}{=}\int_E^{(n)}\!\!\cdots\int\int_{\mathbb{R}^{m_1-1}}^{(m_1-1)}\!\!\cdots\int\cdots\int_{\mathbb{R}^{m_n-1}}^{(m_n-1)}\!\!\cdots\int W(m_n;s_1,\cdots,s_{m_n};u_1,\cdots,u_{m_n})\\ &\quad f(u_1,u_2,\cdots,u_{m_n})\ d\Big(\prod_{j=1}^{m_n-n}m_L\Big)(s_1,s_2,\cdots,s_{m_1-1},s_{m_1+1},\cdots,s_{m_2-1},\\ &\quad s_{m_2+1},\cdots,s_{m_n-1})\ d\Big(\prod_{j=1}^{n}m_L\Big)(s_{m_1},s_{m_2},\cdots,s_{m_n}).\end{aligned} \tag{2.10}$$

Here $\stackrel{*}{=}$ means that the existence of one side implies that of the other with the equality.

PROOF. Let $T : C[0,t] \to \mathbb{R}^{m_n}$ be a function with $T(x) = (x(s_1), x(s_2), \cdots, x(s_{m_n}))$. Then for E in $\mathcal{B}(\mathbb{R}^n)$ and for F in $\mathbb{R}^{m_n}$, by the Wiener integration formula,

$$
\begin{aligned}
&\left[\int_{C[0,t]} (\chi_F \circ T)(x)\, dV_X(x)\right](E) \\
&= [V_X(T^{-1}(F))](E) \\
&= m_w(T^{-1}(F) \cap X^{-1}(E)) \\
&= \underset{E}{\int \overset{(n)}{\cdots} \int} \underset{\mathbb{R}^{m_1-1}}{\int \overset{(m_1-1)}{\cdots} \int} \cdots \underset{\mathbb{R}^{m_n-1}}{\int \overset{(m_n-1)}{\cdots} \int} \chi_F(s_1, s_2, \cdots, s_{m_n}) \qquad (2.11)\\
&\quad W(m_n; s_1, \cdots, s_{m_n}; u_1, \cdots, u_{m_n}) f(u_1, u_2, \cdots, u_{m_n}) \\
&\quad d\Big(\prod_{j=1}^{m_n-n} m_L\Big)(s_1, s_2, \cdots, s_{m_1-1}, s_{m_1+1}, \cdots, s_{m_n-1}) \\
&\quad d\Big(\prod_{j=1}^{n} m_L\Big)(s_{m_1}, s_{m_2}, \cdots, s_{m_n}).
\end{aligned}
$$

From the dominated convergence theorem for Lebesgue integral, the equality (2.7) and the usual method as in the general theory of integral, we have our result. □

From the Theorem 2.1 and Theorem 2.3 in [7], we can find the following lemmas.

LEMMA 2.7. *Let $\mathcal{M}(\mathbb{R}^n)$ be the space of $\mathbb{R}$-valued countably additive measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let $f : \Omega \to \mathcal{M}(\mathbb{R}^n)$ be a μ-Bochner integrable function. Then for E in $\mathcal{B}(\mathbb{R}^n)$, $[f(t)](E)$ is a μ-integrable function of t and*

$$\left[(Bo) - \int_\Omega f(t)\, d\mu(t)\right](E) = \int_\Omega [f(t)](E)\, d\mu(t) \qquad (2.12)$$

Here, $(Bo) - \int_\Omega f(t)\, d\mu(t)$ means the Bochner integral of f with respect to μ.

LEMMA 2.8. *Let $\mathbb{B}$ be a separable Banach space, $(\Omega_1, \mathcal{B}_1, \mu_1)$ a real-valued measure space and let $(\Omega_2, \mathcal{B}_2, \mu_2)$ be a $\mathbb{B}$-valued measure space.*

Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be $\mathcal{B}_1 \otimes \mathcal{B}_2$-measurable and $\mu_1 \times \mu_2$-Bartle integrable. Then

$$\text{for } \ ||\mu_2|| - a.e.\ v, \ \ f(u,v) \ \text{ is a } \mu_1\text{-integrable function of } u, \tag{2.13}$$

$$\int_\Omega f(u,v)\ d\mu_1(u) \ \text{ is } \mu_2\text{-Bartle integrable, and} \tag{2.14}$$

$$\begin{aligned} &(Ba)-\int_{\Omega_1\times\Omega_2} f(u,v)\ d\mu_1\times\mu_2(u,v) \\ =\ &(Ba)-\int_{\Omega_2}\left[\int_{\Omega_1} f(u,v)\ d\mu_1(u)\right] d\mu_2(v). \end{aligned} \tag{2.15}$$

Moreover, if for $|\mu_1|$-a.e. u, $f(u,v)$ is a μ_2-Bartle integrable function of v and $(Ba)-\int_{\Omega_2} f(u,v)\ d\mu_2(v)$ is μ_1-Bochner integrable, then

$$\begin{aligned} &(Ba)-\int_{\Omega_1\times\Omega_2} f(u,v)\ d\mu_1\times\mu_2(u,v) \\ =\ &(Bo)-\int_{\Omega_2}\left[(Ba)-\int_{\Omega_2} f(u,v)\ d\mu_2(v)\right] d\mu_1(u) \\ =\ &(Ba)-\int_{\Omega_2}\left[\int_{\Omega_1} f(u,v)\ d\mu_1(u)\right] d\mu_2(v). \end{aligned} \tag{2.16}$$

3. The relation between the Bartle integral and the conditional Wiener integral

In this section, we will find the relation between the conditional Wiener integral $E(F|X)$ and the Bartle integral (Ba)-$\int_{C[0,t]} F(x)dV_X(x)$.

THEOREM 3.1. *Let $X : C[0,t] \to \mathbb{R}^n$ be a Borel measurable function such that the Radon-Nikodym derivative $\frac{dP_X}{dm_L}$ exists and let $F : C[0,t] \to \mathbb{R}$ be a V_X-Bartle integrable function. Then for E in $\mathcal{B}(\mathbb{R}^n)$,*

$$\begin{aligned} &\left[(Ba)-\int_{C[0,t]} F(x)\ dV_X(x)\right](E) \\ =\ &\int_E E(F|X)(\xi)\ dP_X(\xi). \end{aligned} \tag{3.1}$$

Proof. For B in $\mathcal{B}(C[0,t])$ and for E in $\mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} \chi_B(x)\ dV_X(x)\right](E) \\
&= m_w(B \cap X^{-1}(E)) \\
&= \int_{X^{-1}(E)} \chi_B(x)\ dm_w(x) \\
&= \int_E E(\chi_B|X)(\xi)\ dP_X(\xi).
\end{aligned} \tag{3.2}$$

So, from equality (2.2), for any simple function s on $C[0,t]$, we have

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} s(x)\ dV_X(x)\right](E) \\
&= \int_E E(s|X)(\xi)\ dP_X(\xi).
\end{aligned} \tag{3.3}$$

for E in $\mathcal{B}(\mathbb{R}^n)$.

Let $\langle s_n \rangle$ be an increasing sequence of simple functions on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $\langle s_n \rangle$ converges to F. Then for E in $\mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} F(x)\ dV_X(x)\right](E) \\
&\overset{(1)}{=} \left[\lim_{n\to\infty}(Ba) - \int_{C[0,t]} s_n(x)\ dV_X(x)\right](E) \\
&\overset{(2)}{=} \lim_{n\to\infty}\left[(Ba) - \int_{C[0,t]} s_n(x)\ dV_X(x)\right](E) \\
&\overset{(3)}{=} \lim_{n\to\infty}\int_E E(s_n|X)(\xi)\ dP_X(\xi) \\
&\overset{(4)}{=} \int_E \lim_{n\to\infty} E(s_n|X)(\xi)\ dP_X(\xi) \\
&\overset{(5)}{=} \int_E E(F|X)(\xi)\ dP_X(\xi).
\end{aligned} \tag{3.4}$$

From the definition of Bartle integral, we have Step (1). Step (2) follows from the Vitali-Hahn-Saks theorem. Step (3) results from the equality (3.3). By the monotonic convergence theorem and the equality (2.3), we obtain Step (4). By (c) in Remark 2.1, Step (5) holds. Hence, the proof is finished. □

The following corollary follows from Lemma 2.2.

COROLLARY 3.2. *Under the assumption of Theorem 3.1, if* $\int_{C[0,t]} exp \{i\langle u, X(x)\rangle\}\, dm_w(x)$ *is Lebesgue integrable then*

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} F(x)\, dV_X(x)\right](E) \\
(3.5) \qquad &= \int_E E(F|X)(\xi)\, dP_X(\xi) \\
&= \frac{1}{2\pi} \int_E \int_{\mathbb{R}^n} exp\{-i\langle u, \xi\rangle\} \int_{C[0,t]} exp\{i\langle u, X(x)\rangle\} \\
&\quad F(x)\, dm_w(x) dm_L(u) dm_L(\xi)
\end{aligned}$$

for E *in* $\mathcal{B}(\mathbb{R}^n)$.

THEOREM 3.3. *Let* $0 = t_0 < s_1 < s_2 < \cdots < s_{m_1} = t_1 < s_{m_1+1} < \cdots < s_{m_2} = t_2 < \cdots < s_{m_n} = t_n \leq t$. *Let* $X : C[0,t] \to \mathbb{R}^n$ *be a function with* $X(x) = (x(t_1), x(t_2), \cdots, x(t_n))$ *and let* $f : \mathbb{R}^{m_n} \to \mathbb{R}$ *be Lebesgue integrable. Let* $F : C[0,t] \to \mathbb{R}$ *be a function with* $F(x) = f(x(s_1), x(s_2), \cdots, x(s_{m_n}))$. *Then the conditional Wiener integral* $E(F|X)$ *exists and*

(3.6)

$$\begin{aligned}
&E(F|X)(u_{m_1}, u_{m_2}, \cdots, u_{m_n}) W(n; t_1, t_2, \cdots, t_n; u_{m_1}, u_{m_2}, \cdots, u_{m_n}) \\
&= \underbrace{\int \cdots \int}_{\mathbb{R}^{m_1-1}} \cdots \underbrace{\int \cdots \int}_{\mathbb{R}^{m_n-1}} W(m_n; s_1, s_2, \cdots, s_{m_n}; u_1, \cdots, u_{m_n}) \\
&\quad f(u_1, u_2, \cdots, u_{m_n})\, d\Big(\prod_{j=1}^{m_n-n} m_L\Big)(s_1, s_2, \cdots, s_{m_1-1}, s_{m_1+1}, \cdots, s_{m_n-1}).
\end{aligned}$$

PROOF. From the Wiener integration formula, we have that $\frac{dP_X}{dm_L}(u_{m_1}, u_{m_2}, \cdots, u_{m_n}) = W(n; t_1, t_2, \cdots, t_n; u_{m_1}, u_{m_2}, \cdots, u_{m_n})$. Hence by the equality (2.10) and Theorem 3.1, for E in $\mathcal{B}(\mathbb{R}^n)$,

(3.7)

$$\begin{aligned}
&\underbrace{\int \cdots \int}_{E} \underbrace{\int \cdots \int}_{\mathbb{R}^{m_1-1}} \cdots \underbrace{\int \cdots \int}_{\mathbb{R}^{m_n-1}} W(m_n; s_1, s_2, \cdots, s_{m_n}; u_1, u_2, \cdots, u_{m_n}) \\
&\quad f(u_1, u_2, \cdots, u_{m_n})\, d\Big(\prod_{j=1}^{m_n-n} m_L\Big)(s_1, s_2, \cdots, s_{m_1-1}, s_{m_1+1}, \cdots, s_{m_n-1}) \\
&\quad d\Big(\prod_{j=1}^{n} m_L\Big)(s_{m_1}, s_{m_2}, \cdots, s_{m_n})
\end{aligned}$$

$$= \left[\int_{C[0,t]} F(x)\, dV_X(x) \right](E)$$

$$= \int \cdots \int_E E(F|X)(\xi) \frac{dP_X}{dm_L}(\xi)\, dm_L(\xi).$$

So, we obtain the equality (3.6). □

THEOREM 3.4. *Let $0 = t_0 < t_1 < t_2 < \cdots < t_n \le t$ be given. Let $X : C[0,t] \to \mathbb{R}^n$ be a function with $X(x) = (x(t_1), x(t_2), \cdots, x(t_n))$. Let k be a fixed natural number. Then $[x(s)]^k$ is a $m_L \times V_X$-Bartle integrable function of (s,x) on $[0,t] \times C[0,t]$.*

PROOF. Let x^* be in $\mathcal{RM}(\mathbb{R}^n)^*$, corresponding to θ in $L^\infty(\mathbb{R}^n, m_L)$. Then

$$\begin{aligned}
&\left| \int_{[0,t]\times C[0,t]} [x(s)]^k \, dx^*(m_L \times V_X)(s,x) \right| \\
&\overset{(1)}{\le} \int_{[0,t]\times C[0,t]} |x(s)|^k \, dm_L \times |x^* V_X|(s,x) \\
&\overset{(2)}{\le} ||x^*|| \int_{[0,t]\times C[0,t]} |x(s)|^k \, dm_L \times ||V_X||(s,x) \\
&\overset{(3)}{\le} ||x^*|| \int_{[0,t]\times C[0,t]} |x(s)|^k \, dm_L \times m_w(s,x) \\
(3.8)\quad &\overset{(4)}{=} ||x^*|| \int_{[0,t]} \int_{C[0,t]} |x(s)|^k \, dm_w(x)\, dm_L(s) \\
&\overset{(5)}{=} ||x^*|| \int_{[0,t]} \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} |u|^k exp\{-\frac{u^2}{2s}\}\, dm_L(u)\, dm_L(s) \\
&\overset{(6)}{=} ||x^*||\, \Gamma(\frac{k+1}{2}) \sqrt{\frac{2^k}{\pi}} \int_{[0,t]} s^{\frac{k}{2}} \, dm_L(s) \\
&\overset{(7)}{=} ||x^*||\, \Gamma(\frac{k+1}{2}) \sqrt{\frac{2^{k+2}}{\pi}} \frac{1}{k+1} \sqrt{t^{k+2}}.
\end{aligned}$$

Step (1), (2) and (7) are clear. Step (3) follows from the inequality (2.6). Step (4) results from Tonelli's theorem. From the Wiener integration formula, we obtain Step (5). Since

$$(3.9)\qquad \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} |u|^k exp\{-\frac{u^2}{2s}\}\, dm_L(u) \;=\; \Gamma(\frac{k+1}{2}) \sqrt{\frac{2^k s^k}{\pi}}$$

[12, p. 442], Step (6) is justified where Γ is the gamma function.

Now, let $F : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ be a measure given by

(3.10)

$$\begin{aligned}
&F(E)\\
&= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j - t_{j-1})\Big](s - t_{i-1})(t_i - s)\Big]^{-\frac{1}{2}}\\
&\quad \Big\{ \int_E exp\Big\{ - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\Big\} \Big(\int_{\mathbb{R}} u^k \, exp\Big\{ - \frac{(u - u_{i-1})^2}{2(s - t_{i-1})}\Big\}\\
&\quad exp\Big\{ - \frac{(u_i - u)^2}{2(t_i - s)}\Big\} \, dm_L(u)\Big) \, d\Big(\prod_{i=1}^{n} m_L\Big)(u_1, u_2, \cdots, u_n)\Big\} \, dm_L(s)
\end{aligned}$$

for E in $\mathcal{B}(\mathbb{R}^n)$.

Then by the Chapman-Kolmogorov equation in [4, p. 36] and the equality (3.9),

$$|F(E)| \le \Gamma(\frac{k+1}{2})\sqrt{\frac{2^k s^k}{\pi}}\frac{1}{k+1}\sqrt{t^{k+2}}, \tag{3.11}$$

so F is well-defined. Moreover,

$$\begin{aligned}
&x^*(F)\\
&\overset{(1)}{=} \int_{\mathbb{R}^n} \theta(u_1, u_2, \cdots, u_n) \, dF(u_1, u_2, \cdots, u_n)\\
&\overset{(2)}{=} \int_{\mathbb{R}^n} \theta(u_1, u_2, \cdots, u_n) \, \frac{dF}{d\prod_{i=1}^{n} m_L}(u_1, u_2, \cdots, u_n)\\
&\quad d\prod_{i=1}^{n} m_L(u_1, u_2, \cdots, u_n)\\
&\overset{(3)}{=} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j - t_{j-1})\Big](s - t_{i-1})(t_i - s)\Big]^{-\frac{1}{2}}\\
&\quad \Big\{ \int_{\mathbb{R}^n} \theta(u_1, u_2, \cdots, u_n) \, exp\Big\{ - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\Big\}
\end{aligned} \tag{3.12}$$

$$\Big(\int_{\mathbb{R}} u^k\, exp\Big\{-\frac{(u-u_{i-1})^2}{2(s-t_{i-1})}\Big\}\, exp\Big\{-\frac{(u_i-u)^2}{2(t_i-s)}\Big\}$$

$$dm_L(u)\Big)\, d\Big(\prod_{i=1}^{n} m_L\Big)(u_1,u_2,\cdots,u_n)\Big\}\, dm_L(s)$$

$$\overset{(4)}{=} \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Big(\int_{C[0,t]}[x(s)]^k\, d(x^*V_X)(x)\Big)\, dm_L(s)$$

$$\overset{(5)}{=} \int_{[0,t]}\Big(\int_{C[0,t]}[x(s)]^k\, d(x^*V_X)(x)\Big)\, dm_L(s)$$

$$\overset{(6)}{=} \int_{[0,t]\times C[0,t]}[x(s)]^k\, d(m_L\times x^*V_X)(s,x)$$

$$\overset{(7)}{=} \int_{[0,t]\times C[0,t]}[x(s)]^k\, dx^*(m_L\times V_X)(s,x).$$

Step (1) follows from Remark 2.3 (b). By the Radon-Nikodym theorem, we obtain Step (2). From the definition of a measure F, we have Step (3). Using Lemma 2.6 in above, we can check Step (4). Step (5) is clear. By the Fubini theorem, Step (6) holds. By the general theory of vector measure, Step (7) is justified.

Hence, by Lemma 2.5, we have $[x(s)]^k$ is $m_L\times V_X$-Bartle integrable on $[0,t]\times C[0,t]$ and

$$(3.13)\qquad (Ba)-\int_{[0,t]\times C[0,t]}[x(s)]^k\, d(m_L\times V_X)(s,x) \;=\; F,$$

as desired. □

By the equality (2.12) in Lemma 2.6 in above, we have the following corollaries.

COROLLARY 3.5. *Let $0=t_0<t_1<t_2<\cdots<t_n\le t$ and let $X: C[0,t]\to\mathbb{R}^n$ be a function with $X(x)=(x(t_1),x(t_2),\cdots,x(t_n))$. Then $\int_{[0,t]}[x(s)]^k\, dm_L(s)$ is a V_X-Bartle integrable function of x for any natural number k.*

COROLLARY 3.6. *Let $0=t_0<t_1<t_2<\cdots<t_n\le t$ and let $X: C[0,t]\to\mathbb{R}^n$ be a function with $X(x)=(x(t_1),x(t_2),\cdots,x(t_n))$. Then $(Ba)-\int_{C[0,t]}[x(s)]^k\, dV_X(x)$ is a Bochner integrable function of s for any natural number k.*

PROOF. Let k be a fixed natural number and let x^* be in $\mathcal{RM}(\mathbb{R}^n)^*$, corresponding to θ in $L^\infty(\mathbb{R}^n)$. By the essentially same method as in the proof of Theorem 3.4, we can check that the Radon-Nikodym derivative

$$d\Big((Ba)-\int_{C[0,t]}[x(s)]^k\,dV_X(x)\Big)/dm_L$$
$$=\sum_{i=1}^{n}\chi_{(t_{i-1},t_i)}(s)\Big[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j-t_{j-1})\Big](s-t_{i-1})(t_i-s)\Big]^{-\frac{1}{2}}$$
$$exp\Big\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\Big\}\int_{\mathbb{R}}u^k\,exp\Big\{-\frac{(u-u_{i-1})^2}{2(s-t_{i-1})}\Big\}$$
$$exp\Big\{-\frac{(u_i-u)^2}{2(t_i-s)}\Big\}\,dm_L(u).$$

So,

$$x^*\Big((Ba)-\int_{C[0,t]}[x(s)]^k\,dV_X(x)\Big)$$
$$=\sum_{i=1}^{n}\chi_{(t_{i-1},t_i)}(s)\int_{\mathbb{R}^n}\Big[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j-t_{j-1})\Big](s-t_{i-1})(t_i-s)\Big]^{-\frac{1}{2}}$$
$$exp\Big\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\Big\}\int_{\mathbb{R}}u^k\,exp\Big\{-\frac{(u-u_{i-1})^2}{2(s-t_{i-1})}\Big\}$$
$$exp\Big\{-\frac{(u_i-u)^2}{2(t_i-s)}\Big\}\,dm_L(u)\,d\Big(\prod_{j=1}^{n}\Big)m_L(u_1,u_2,\cdots,u_n)).$$

By the Fubini theorem, $x^*\Big((Ba)-\int_{C[0,t]}[x(s)]^k\,dV_X(x)\Big)$ is measurable of s, that is, $(Ba)-\int_{C[0,t]}x(s)\,dV_X(x)$ is weak measurable of s. Since $\mathcal{RM}(\mathbb{R}^n)$ is separable, by Pettis measurability theorem in [2, Theorem 2, p. 42], $(Ba)-\int_{C[0,t]}|x(s)|^k\,dV_X(x)$ is a strongly measurable function of s. By the similar calculus as in the proof of Theorem 3.4, we have

$$\Big|(Ba)-\int_{C[0,t]}[x(s)]^k\,dV_X(x)\Big| \tag{3.14}$$

$$= sup\{x^*\Big((Ba) - \int_{C[0,t]} [x(s)]^k \, dV_X(x)\Big) :$$
$$x^* \text{ is in } \mathcal{RM}(\mathbb{R}^n)^* \text{ with } ||x^*||_\infty \le 1\}$$
$$\le \Gamma(\frac{k+1}{2})\sqrt{\frac{2^{k+2}}{\pi}}\frac{1}{k+2}\sqrt{t^{k+2}}.$$

So, by Theorem 2 in [2, p. 45], $(Ba) - \int_{C[0,t]}[x(s)]^k \, dV_X(x)$ is a Bochner integrable function of s. □

Using the above results and Lemma 2.6, we obtain the following theorem.

THEOREM 3.7. *Let* $0 = t_0 < t_1 < t_2 < \cdots < t_n \le t$ *and let* $X : C[0,t] \to \mathbb{R}^n$ *be a function with* $X(x) = (x(t_1), x(t_2), \cdots, x(t_n))$. *Let* k *be a natural number. Then* $[x(s)]^k$ *is* $m_L \times V_X$*-Bartle integrable,* $\int_{[0,t]}[x(s)]^k \, dm_L(s)$ *is* V_X*-Bartle integrable and* $\int_{C[0,t]}[x(s)]^k \, dV_X(x)$ *is* m_L*-Bochner integrable.*

Moreover, for E *in* $\mathcal{B}(\mathbb{R}^n)$

$$\Big[\int_{[0,t]\times C[0,t]} [x(s)]^k \, dm_L \times V_X(s,x)\Big](E)$$
$$= \Big[(Bo) - \int_{[0,t]} \Big((Ba) - \int_{C[0,t]} [x(s)]^k \, dV_X(x)\Big) \, dm_L(s)\Big](E)$$
$$= \int_{[0,t]} \Big[(Ba) - \int_{C[0,t]} [x(s)]^k \, dV_X(x)\Big](E) \, dm_L(s)$$
$$= \Big[(Ba) - \int_{C[0,t]} \Big(\int_{[0,t]} [x(s)]^k \, dm_L(s)\Big) \, dV_X(x)\Big](E).$$

4. Examples

In this section, using the results in Section 3, we will give examples of evaluation for conditional Wiener integral of some Wiener functional.

EXAMPLE 4.1. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n \le t$ and $t_{i-1} < s < t_i$ for some $i = 1, 2, \cdots, n$. Let $f(u) = exp\{-\alpha u^2\}$ where α is a positive real number and $F(x) = f(x(s))$. Then by Theorem 3.3,

$$E(F|X)(u_1, u_2, \cdots, u_n)W(n; t_1, t_2, \cdots, t_n)$$

$$= \left[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j - t_{j-1})\Big](s - t_{i-1})(t_i - s)\right]^{-\frac{1}{2}} \int_{\mathbb{R}} exp\{-\alpha u^2\}$$

$$exp\Big\{ - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\Big\}\ exp\Big\{ - \frac{(u - u_{i-1})^2}{2(s - t_{i-1})}\Big\}$$

$$exp\Big\{ - \frac{(u_i - u)^2}{2(t_i - s)}\Big\}\ dm_L(u)$$

$$= \left[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j - t_{j-1})\Big](s - t_{i-1})(t_i - s)\right]^{-\frac{1}{2}}$$

$$\int_{\mathbb{R}} exp\Big\{ - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\Big\}$$

$$exp\Big\{ - \frac{k}{2(s - t_{i-1})(t_i - s)}\Big(u - \frac{((s - t_{i-1})u_i + (t_i - s)u_{i-1})^2}{k}\Big)\Big\}$$

$$exp\Big\{ - \frac{1}{2k(s - t_{i-1})(t_i - s)}\Big[((s - t_{i-1})u_i + (t_i - s)u_{i-1})^2$$

$$- k((s - t_{i-1})u_i^2 + (t_i - s)u_{i-1}^2)\Big]\Big\}\ dm_L(u),$$

where $k = 2\alpha(s - t_{i-1})(t_i - s) + (t_i - t_{i-1})$. Hence,

$$E(F|X)(u_1, u_2, \cdots, u_n)$$

$$= \Big(2\alpha(s - t_{i-1})(t_i - s) + (t_i - t_{i-1})\Big)^{-\frac{1}{2}}\Big(t_i - t_{i-1}\Big)^{\frac{1}{2}}$$

$$exp\Big\{\frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} - \frac{(u_i - u_{i-1})^2 + 2\alpha((s - t_{i-1})u_i^2 + (t_i - s)u_{i-1}^2)}{4\alpha(s - t_{i-1})(t_i - s) + 2(t_i - t_{i-1})}\Big\}.$$

EXAMPLE 4.2. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n \le t$ and let $X : C[0,t] \to \mathbb{R}^n$ be a function with $X(x) = (x(t_1), x(t_2), \cdots, x(t_n))$. Let $F(x) = \int_0^t x(s)\ dm_L(s)$. Then for E in $\mathcal{B}(\mathbb{R}^n)$

$$\Big[\int_{C[0,t]} F(x)\ dV_X(x)\Big](E)$$

$$\overset{(1)}{=} \Big[\int_{[0,t]}\int_{C[0,t]} x(s)\ dV_X(x)\ dm_L(s)\Big](E)$$

$$\overset{(2)}{=} \Big[\sum_{i=1}^{n}\int_{(t_{i-1},t_i)}\int_{C[0,t]} x(s)\ dV_X(x)\ dm_L(s)\Big](E)$$

$$\overset{(3)}{=} \sum_{i=1}^{n}\int_{(t_{i-1},t_i)}\Big[\int_{C[0,t]} x(s)\ dV_X(x)\Big](E)\ dm_L(s)$$

$$\overset{(4)}{=} \sum_{i=1}^{n}\int_{(t_{i-1},t_i)}\underset{E}{\int\overset{(n)}{\cdots}\int}\int_{\mathbb{R}}\Big[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j-t_{j-1})\Big]$$

$$(s-t_{i-1})(t_i-s)\Big]^{-\frac{1}{2}} u\ exp\Big\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\Big\}$$

$$exp\Big\{-\frac{(u-u_{i-1})^2}{2(s-t_{i-1})}-\frac{(u_i-u)^2}{2(t_i-s)}\Big\}$$

$$dm_L(u)\ d\prod_{i=1}^{n} m_L(u_1,u_2,\cdots,u_n)\ dm_L(s)$$

$$\overset{(5)}{=} \underset{E}{\int\overset{(n)}{\cdots}\int}\Big[(2\pi)^n\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j-t_{j-1})\Big]exp\Big\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\Big\}\Big]^{-\frac{1}{2}}$$

$$\sum_{i=1}^{n}\int_{(t_{i-1},t_i)}\Big[2\pi(s-t_{i-1})(t_i-s)\Big]^{-\frac{1}{2}}$$

$$\int_{\mathbb{R}} u\ exp\Big\{-\frac{(u-u_{i-1})^2}{2(s-t_{i-1})}-\frac{(u_i-u)^2}{2(t_i-s)}\Big\}$$

$$dm_L(u)\ dm_L(s)\ d\prod_{i=1}^{n} m_L(u_1,u_2,\cdots,u_n)$$

$$\overset{(6)}{=} \underset{E}{\int\overset{(n)}{\cdots}\int}\Big[(2\pi)^n\Big[\prod_{j=1}^{n}(t_j-t_{j-1})\Big]^{-\frac{1}{2}}exp\Big\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(u_j-u_{j-1})^2}{2(t_j-t_{j-1})}\Big\}\Big]$$

$$(t_i-t_{i-1})^{-\frac{1}{2}}\Big(\sum_{i=1}^{n}\frac{(t_i-t_{i-1})(u_i+u_{i-1})}{2}\Big)exp\Big\{-\frac{(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\Big\}$$

$$d\prod_{i=1}^{n} m_L(u_1,u_2,\cdots,u_n).$$

Step (1) follows from Lemma 2.8 in above. Step (2) and (6) are clear. Step (3) results from Lemma 2.7 in above. From Lemma 2.6, we obtain Step (4). By the Fubini theorem, Step (5) holds.

Hence, by Theorem 3.3,

$$E(F|X)(u_1, u_2, \cdots, u_n) = \sum_{i=1}^{n} \frac{(t_i - t_{i-1})(u_i + u_{i-1})}{2}.$$

This result is exactly same to K. S. Chang and J. S. Chang's result in [1, Theorem 3.1, p. 102].

EXAMPLE 4.3. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n \leq t$ and let $X : C[0,t] \to \mathbb{R}^n$ be a function with $X(x) = (x(t_1), x(t_2), \cdots, x(t_n))$. Let $F(x) = \int_0^t [x(s)]^2 \, dm_L(s)$. By the essentially same method as in Example 4.2, for E in $\mathcal{B}(\mathbb{R}^n)$

$$\Big[\int_{C[0,t]} F(x) \, dV_X(x)\Big](E)$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big[\int_{C[0,t]} [x(s)]^2 \, dV_X(x)\Big](E) \, dm_L(s)$$

$$= \int_E^{(n)}\!\!\cdots\int \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big[(2\pi)^{n+1}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(t_j - t_{j-1})\Big](s - t_{i-1})(t_i - s)\Big]^{-\frac{1}{2}}$$

$$u^2 \, exp\Big\{ - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\Big\} exp\Big\{ - \frac{(u - u_{i-1})^2}{2(s - t_{i-1})} - \frac{(u_i - u)^2}{2(t_i - s)}\Big\}$$

$$dm_L(u) \, dm_L(s) \, d\Big(\prod_{i=1}^{n} m_L\Big)(u_1, u_2, \cdots, u_n)$$

$$= \int_E^{(n)}\!\!\cdots\int \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big[\frac{(s - t_{i-1})(t_i - s)}{t_i - t_{i-1}} + \Big(\frac{(t_i - s)u_{i-1} + (s - t_{i-1})u_i}{t_i - t_{i-1}}\Big)^2\Big]$$

$$dm_L(s) \, W(n; t_1, t_2, \cdots, t_n; u_1, u_2, \cdots, u_n) \, d\Big(\prod_{i=1}^{n} m_L\Big)(u_1, u_2, \cdots, u_n)$$

$$= \int_E^{(n)}\!\!\cdots\int \Big[\sum_{i=1}^{n} \Big(\frac{(t_i - t_{i-1})^2}{6} + \frac{(t_i - t_{i-1})u_{i-1}^2}{3} + \frac{(t_i - t_{i-1})u_i^2}{3}$$

$$+ \frac{(t_i - t_{i-1})u_i u_{i-1}}{3}\Big)\Big] W(n; t_1, t_2, \cdots, t_n; u_1, u_2, \cdots, u_n)$$
$$d\Big(\prod_{i=1}^{n} m_L\Big)(u_1, u_2, \cdots, u_n).$$

Hence,

$$\begin{aligned} &E(F|X)(u_1, u_2, \cdots, u_n) \\ &= \sum_{i=1}^{n} \Big(\frac{(t_i - t_{i-1})^2}{6} + \frac{(t_i - t_{i-1})u_{i-1}^2}{3} + \frac{(t_i - t_{i-1})u_i^2}{3} \\ &\quad + \frac{(t_i - t_{i-1})u_i u_{i-1}}{3} \Big). \end{aligned}$$

This result is exactly same to K. S. Chang and J. S. Chang's result in [1, Theorem 3.3, p. 105].

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