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OPENLY SEMIPRIMITIVE PROJECTIVE MODULE

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ABSTRACT. In this paper, a left module over an associative ring with identity is defined to be openly semiprimitive (strongly semiprimitive, respectively) by the zero intersection of all maximal open fully invariant submodules (all maximal open submodules which are fully invariant, respectively) of it. For any projective module, the openly semiprimitivity of the projective module is an equivalent condition of the semiprimitivity of endomorphism ring of the projective module and the strongly semiprimitivity of the projective module is an equivalent condition of the endomorphism ring of the projective module being a subdirect product of a set of subdivisions of division rings.

0. Introduction

Assume that ring R is any associative ring with identity. The ring of all R-endomorphisms on a left R-module ${}_RM$, denoted by $End_R(M)$, will be written on the right side of M as right operators on ${}_RM$, that is, ${}_RM_{End_R(M)}$ will be considered in this paper.

A submodule L of a left $R-$module ${}_RM$ is said to be *fully invariant* if $Lf \leq L$ for each $f \in End_R(M)$. For any subset J of $End_R(M) = S$, let $ImJ = MJ = \sum_{f\in J} Imf = \sum_{f\in J} Mf$ be the sum of images of endomorphisms in J. Also we call N an *open* submodule if $N = N^o$, where $N^o = \sum_{f\in S, Imf\leq N} Imf$ is the sum of all images of endomorphisms contained in N.

DEFINITION 0.1. For an open submodule $P \leq M$ of a left $R-$module ${}_RM$, P is said to be a *maximal open fully invariant* submodule of ${}_RM$

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if it satisfies that $P \leq L \leq M$ for any open fully invariant submodule $L \leq M$ implies that $P = L$ or $L = M$.

For an open submodule $P \leq M$ of a left $R-$module ${}_RM$, P is said to be a *maximal open* submodule of ${}_RM$ if it satisfies that $P \leq L \leq M$ for any open submodule $L \leq M$ implies that $P = L$ or $L = M$.

Generally, a maximal open submodule P of any left $R-$module ${}_RM$ which is fully invariant does not mean a maximal open fully invariant submodule of it. In fact, we may have three kinds of open submodules of ${}_RM$ (if it is projective, it has), that is, a maximal open submodule which is not fully invariant, a maximal open fully invariant submodule, and a maximal open submodule which is fully invariant in ${}_RM$.

For example, over the integer ring $\mathbb{Z}$, the left $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Z} \oplus \mathbb{Z}$ has a maximal open submodule $p\mathbb{Z} \oplus \mathbb{Z}$ (prime p) which is not fully invariant and it has also a maximal open fully invariant submodule $p\mathbb{Z} \oplus p\mathbb{Z}$. In the $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ the maximal open fully invariant submodule $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus 0$ and the maximal open(but not fully invariant) submodule $0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ have a maximal ideal $I^{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus 0}$ and a maximal left(not a two-sided) ideal $I^{0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3}$ of the endomorphism ring of a projective module ${}_{\mathbb{Z}}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Regarding the polynomial ring ${}_{\mathbb{Z}[x]}\mathbb{Z}[x]$ as a left $\mathbb{Z}[x]-$module, then it is a projective $\mathbb{Z}[x]-$module which has a non-open maximal submodule $x\mathbb{Z}[x] + p\mathbb{Z}$(prime p) with a fully invariant submodule $(x\mathbb{Z}[x] + p\mathbb{Z})^o = (p\mathbb{Z})\mathbb{Z}[x]$ of it. For any field F, a direct product ${}_FF^n$(for any natural number $n \in \mathbb{N}$) has a maximal open submodule $0 \times F^{n-1}$ which is not fully invariant. In a non-projective module ${}_{\mathbb{Z}}\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$, a non-fully invariant submodule $\{\overline{0}, \overline{1/p}, \cdots, \overline{(p-1)/p}, \cdots, \overline{1/p^n}, \cdots, \overline{(p-1)/p^n}\} \oplus \mathbb{Z}(p^\infty)$ has a non-fully invariant interior $(\{\overline{0}, \overline{1/p}, \cdots, \overline{(p-1)/p}, \cdots, \overline{1/p^n}, \cdots, \overline{(p-1)/p^n}\} \oplus \mathbb{Z}(p^\infty))^o = 0 \oplus \mathbb{Z}(p^\infty)$ for any natural number $n \in \mathbb{N}$.

A left $R-$module ${}_RM$ is said to be *openly simple* in case there are only trivial images of any endomorphism, that is, for every $f \in End_R(M)$ the image Imf is either M or 0.

THEOREM 0.2 [10]. **(Generalized Schur's Lemma I)**
Every openly simple projective module has a division endomorphism ring.

LEMMA 0.3 [6]. *If a left $R-$module ${}_RM$ has a division endomorphism ring, then ${}_RM$ is cyclic.*

PROPOSITION 0.4 [8]. *A left R-module M is a subdirect product of a class $\mathcal{U}$ of left R-modules if and only if the P-reject of M in $\mathcal{U}$ is zero.*

A ring R is said to be a *subdirect sum* of the rings $\{R_\gamma\}_{\gamma\in I}$ if there is a monomorphism

$$\phi : R \to \prod_I R_\gamma \text{ such that } R\phi\pi_\gamma = R_\gamma,$$

where $\pi_\gamma : \prod_I R_\gamma \to R_\gamma$ is the canonical projection, for each $\gamma \in I$.

DEFINITION 0.5 [9]. For a non-unit endomorphism p on any left $R-$module ${}_RM$, p is said to be *prime* if $p = fg$, for $f, g \in End_R(M)$, then f is right invertible or g is *left retractable* in $End_R(M)$, where an endomorphism $g \in End_R(M)$ is said to be *left retractable* if there is $g' \in End_R(M)$ such that $g'g : {}_RM \xrightarrow{g'} {}_RM \xrightarrow{g} {}_RM$ has the identity restriction to Img, that is, $g'g\,|_{Img} = 1_{Img}$.

LEMMA 0.6 [9]. *For any (quasi-)projective module ${}_RM$, we have the following:*

(1) *For any $f, g \in End_R(M)$ with $Imf \leq Img$, then there is some*

$$h \in End_R(M) \text{ such that } f = hg.$$

(2) *For any monomorphism $f \in End_R(M)$, if Imf is a maximal open submodule of ${}_RM$, then f is a prime endomorphism.*

It is well-known that for any projective $R-$module ${}_RM$, the Jacobson radical $Rad(R)$ of R has the property ([1], [3], [5]):

$$Rad(R)M = Rad(M) \neq M.$$

LEMMA 0.7 [3]. *For any projective left $R-$module ${}_RM$, the Jacobson radical $Rad(End_R(M))$ of the endomorphism ring $End_R(M)$ is, in fact,*

$$\begin{aligned} Rad(End_R(M)) &= \{f \in End_R(M) \mid Imf \leq Rad(M)\} \\ &= \{f \in End_R(M) \mid Imf \text{ is small in } {}_RM\}. \end{aligned}$$

More precisely let's rewrite the above result:

$$\begin{aligned} Rad(End(M)) &= Hom_R(M, Rad(M)) \\ &= I^{Rad(M)} = I^{Rad(M)^o} \\ &= \{f \in End_R(M) \mid Imf \leq Rad(M)\} \\ &= \{f \in End_R(M) \mid Imf \leq Rad(M)^o\} \\ &= \{f \in End_R(M) \mid Imf \text{ is small in } {}_RM\}. \end{aligned}$$

THEOREM 0.8 [5]. *Given a ring S, each of the following subsets of S is equal to the Jacobson radical $Rad(S)$ of S.*

(J_1) *The intersection of all maximal left (right) ideals of S;*
(J_2) *The intersection of all left (right) primitive ideals of S;*
(J_3) *$\{f \in S \mid rfs$ is quasi-regular for all $r, s \in S\}$;*
(J_4) *$\{f \in S \mid rf$ is quasi-regular for all $r \in S\}$;*
(J_5) *$\{f \in S \mid fs$ is quasi-regular for all $s \in S\}$;*
(J_6) *The union of all the quasi-regular left (right) ideals of S;*
(J_7) *The union of all the quasi-regular ideals of S;*
(J_8) *The unique largest superfluous left (right) ideal of S.*

Moreover, $(J_3), (J_4), (J_5), (J_6)$, and (J_7) also describe the Jacobson radical $Rad(S)$ if "quasi-regular" is replaced by "left quasi-regular" or by "right quasi-regular".

Any maximal ideal of a ring means a both-sided (two-sided) ideal of the ring. We are going to deal with all projective left $R-$modules without any, or with some maximal open fully invariant submodules, or with some maximal open submodules which are fully invariant of projective left $R-$modules in the following three sections.

1. Semiprimitive projective module

Any ring R is called *semiprimitive* if the Jacobson radical $Rad(R)$ of R is zero, that is, $Rad(R) = \cap J_\alpha = 0$ for which J_α ($\alpha \in \Lambda$) is a maximal ideal of the ring R. Similarly a left $R-$module ${}_RM$ is said to be *semiprimitive* if the Jacobson radical $Rad(M)$ of ${}_RM$ is zero, that is, $Rad(M) = \cap K_\alpha = 0$, for which K_α ($\alpha \in \Lambda$) is a maximal submodule of the module ${}_RM$. It is easily observed that for any projective module ${}_RM$, the condition of $Rad(M)^o = 0$ if and only if the endomorphism ring $End_R(M)$ is semiprimitive.

In a special case of a projective module ${}_RM$ in which no maximal open fully invariant submodule exists, ${}_RM$ can have the semiprimitive endomorphism ring $End_R(M)$ of ${}_RM$, even though each projective module has at least one maximal submodule of it. This case will be studied in this section.

It is well-known that any $R-$projective module is a direct summand of a free $R-$module.

LEMMA 1.1. *For any free $R-$module ${}_RF$ and for any proper (left) ideal J of the endomorphism ring $End_R(F)$, we have the proper image*

ImJ of J.

PROOF. Since there exists a non-empty set X such that

$$F \simeq \oplus_X R = R^{(X)} \text{ (a direct sum of } X - \text{copies of } R)$$

if J is any proper (left) ideal of $End_R(F)$ with $ImJ = F$, then for a mapping $\chi_x x : X \to F$, where any characteristic function χ_x defined by $(y)\chi_x = 1_R$, the identity of ring R if $y = x$ and $(y)\chi_x = 0$ if $y \neq x$, there exists an endomorphism

$$h_x \in J \text{ such that } x \in Imh_x = Rx \leq ImJ = M.$$

Because of

$$ImJ = \sum_{f_\alpha \in J} Mf_\alpha = \sum_{f_\alpha \in J} Imf_\alpha = M,$$

there is an endomorphism $f_\alpha \in J$ in J such that $Imf_\alpha \geq Rx = Imh_x$, for each $x \in X$.

On the other hand, considering the following diagram:

$$\begin{array}{ccccc} & & & & {}_RF \\ & \exists t_x \swarrow & & & \downarrow h_x \\ {}_RF & & \xrightarrow{f_\alpha \in J} & Imf_\alpha & \longrightarrow 0, \end{array}$$

we have that $h_x = t_x f_\alpha \in J$ for some endomorphism $t_x : {}_RF \to {}_RF$ since every free R–module is R–projective.

Considering the direct sum $\oplus_x \chi_x x : X \to F$ (in fact, this sum is a set function) and the function $\chi_x x$ shown in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\iota} & F \simeq \oplus_X R = R^{(X)} \\ {\scriptstyle \chi_x x,\ \oplus_x \chi_x x}\downarrow & & \swarrow \exists h_x,\ \oplus h_x = h = 1_F \text{ resp.} \\ F \simeq \oplus_X R = R^{(X)}, & & \end{array}$$

where $\iota : X \hookrightarrow {}_RF$ is the inclusion mapping, there exists a unique R–endomorphism $h : F \to F$ such that $\iota h = \oplus_x \chi_x x$, here in fact, we have $h = 1_F = \oplus h_x$.

Now that the proper (left) ideal J contains $1_F = \oplus_{x \in X} h_x$ on the free module ${}_RF = R^{(X)}$ has its image ${}_RF$. Then it follows that $End_R(F) = J$ from the fact of $1_F = \oplus_{x \in X} h_x : {}_RF \to {}_RF$ by the uniqueness of $\oplus h_x$. Therefore this contradicts to the proper (left) ideal J of $End_R(F)$. □

LEMMA 1.2. *For any projective $R-$module ${}_RM$ and for any proper (left) ideal J of the endomorphism ring $End_R(M)$, we have the proper image ImJ of J.*

PROOF. Since a projective module ${}_RM$ has a free module ${}_RF$ such that

$M \oplus M' = {}_RF \simeq R^{(X)}$, for a non-empty set X. For any proper (left) ideal J of $End_R(M)$, let $J' = \langle J \oplus 0, 0 \oplus 1_{M'} \rangle$ be an (a left) ideal of $End_R(F)$ generated by the set $\{J \oplus 0, 0 \oplus 1_{M'}\}$. If J' is a proper (left) ideal of $End_R(F)$, then ImJ is a proper submodule of ${}_RM$. Otherwise if $J' = End_R(F)$, then we can take another free module ${}_RF' = \oplus^F M \oplus M'$ by attaching $F-$copies of M to ${}_RF$ until we get the proper (left) ideal $\langle \oplus^F J \oplus 0, 0 \oplus 1_{M'} \rangle$ of $End_R(F')$ generated by the set $\{\oplus^F J \oplus 0, 0 \oplus 1_{M'}\}$. Then it follows easily that the image ImJ is a proper open fully invariant submodule of ${}_RM$ (of course, the image ImJ is a proper open submodule of ${}_RM$). □

LEMMA 1.3. *For any projective left $R-$module ${}_RM$, we have the following:*

(1) *For any maximal open submodule $P \leq M$,*
$I^P = \{f \in End_R(M) | Imf \leq P\}$ *is a maximal left ideal of* $End_R(M)$.
(2) *For any maximal left ideal J of $End_R(M)$, ImJ is a maximal open submodule of ${}_RM$.*
(3) *There is at least one of maximal open submodules of ${}_RM$.*
(4) *For any maximal open fully invariant submodule $P \leq M$,*
$I^P = \{f \in End_R(M) | Imf \leq P\}$ *is a maximal ideal of* $End_R(M)$.
(5) *For any maximal ideal J of $End_R(M)$, ImJ is a maximal open fully invariant submodule of ${}_RM$.*
(6) *There is at least one of maximal open fully invariant submodules of ${}_RM$.*

PROOF. (1), (2), and (3) are trivial.

(4): For any maximal open fully invariant submodule $P \leq M$, clearly we have an ideal I^P of the endomorphism ring. To show that I^P is a maximal ideal of $End_R(M)$ we let J be any ideal of $End_R(M)$ such that $I^P \subseteq J \subseteq End_R(M)$. Then the image $ImJ = \sum_{j \in J} Imj$ is an open fully invariant submodule of ${}_RM$ such that $P \leq ImJ \leq M$. Then by Lemma 1.2 and by the maximality of the open fully invariant submodule P we have immediately that $ImJ = P$ or $ImJ = M$. Thus it follows

that $I^P = J$ or $J = End_R(M)$. Therefore I^P is a maximal ideal of $End_R(M)$.

(5): Since ${}_RM$ is projective any maximal ideal J has the proper fully invariant image ImJ by Lemma 1.2. To show that ImJ is a maximal open fully invariant submodule of ${}_RM$, we let $K \leq M$ be an open fully invariant submodule of ${}_RM$ such that $ImJ \leq K \leq M$.

Then we have an ideal I^K such that $I^{ImJ} = J \subseteq I^K \subseteq End_R(M)$.

Since J is a maximal ideal of $End_R(M)$ we have that

$$I^{ImJ}, I^K \leq J \text{ if } K \neq M,$$

implying that $K = ImJ$. Otherwise $K = M$ follows. Thus the proper open fully invariant submodule ImJ is maximal among open fully invariant submodules of ${}_RM$.

(6): Since the endomorphism ring $End_R(M)$ has at least one of maximal ideals of it, it follows (6) from the item (5). □

If there is no maximal open fully invariant submodule of any semiprimitive projective module ${}_RM$ with

$$0 = Rad(M) = \cap P_\alpha \ (P_\alpha \text{ is a maximal submodule of } {}_RM),$$

then it follows that

$$Rad(M)^o = \cap P_\alpha^o$$

from the relation of

$$Rad(End_R(M)) = I^{Rad(M)} = I^{Rad(M)^o} = I^{\cap P_\alpha} = \cap I^{P_\alpha} = \cap I^{P_\alpha^o} = I^{\cap P_\alpha^o},$$

and hence we have the following:

PROPOSITION 1.4. *For any projective module ${}_RM$, if ${}_RM$ is semiprimitive, then so is the endomorphism ring $End_R(M)$.*

PROOF. Suppose that

$$0 = Rad(M) = \cap P_\alpha \text{ (each } P_\alpha \text{ is a maximal submodule of } {}_RM).$$

Since P_α^o is a maximal open submodule of ${}_RM$ for any maximal submodule P_α of ${}_RM$ the relation of

$$Rad(End_R(M)) = \cap I^{P_\alpha} = I^{\cap P_\alpha} = I^0 = 0$$

tells that the endomorphism ring $End_R(M)$ is a semiprimitive ring. □

REMARK 1.5. The converse of Proposition 1.4 does not hold, in general. The polynomial ring $\mathbb{Z}[x]$ is not semiprimitive as a $\mathbb{Z}[x]-$module but it has a semiprimitive endomorphism ring $End_{\mathbb{Z}[x]}(\mathbb{Z}[x])$.

PROPOSITION 1.6. *For any semiprimitive projective module ${}_RM$, if there is no non-trivial maximal open fully invariant submodule of ${}_RM$, the endomorphism ring $End_R(M)$ is a local ring with a unique zero maximal ideal $Rad(End_R(M)) = 0$ and hence $End_R(M)$ is semiprimitive.*

PROOF. Suppose that there is no maximal open fully invariant submodule and suppose that

$$0 = Rad(M) = \cap P_\alpha \text{ (each } P_\alpha \text{ is a maximal submodule of } {}_RM).$$

Then there are no non-trivial maximal ideals of $End_R(M)$ by the Lemma 1.3. It is true that the ideal

$$Rad(End_R(M)) = \cap I^{P_\alpha} = I^{\cap P_\alpha} = I^0 = 0$$

is a unique maximal ideal of $End_R(M)$ by the Theorem 0.7. □

There are lots of semiprimitive projective modules having local endomorphism rings with a unique trivial maximal ideal, such as, the projective modules ${}_{\mathbb{Z}}\mathbb{Z}_p \oplus \mathbb{Z}_p$ and any direct sum ${}_{\mathbb{Z}}\mathbb{Z}_p^{(n)}$ of $n-$copies of $\mathbb{Z}_p$ (for a prime number p) have local endomorphism rings since they are semiprimitive projective modules in which there are no non-trivial maximal open fully invariant submodules.

The local endomorphism rings of projective modules are not the main concerns in this paper. We concern mainly about projective modules with maximal open fully invariant submodules of projective modules and projective modules with maximal open submodules which are fully invariant. And these will be studied in the next sections.

2. Openly semiprimitive projective module

If a left $R-$module ${}_RM$ has at least one maximal open fully invariant submodule of ${}_RM$, then we need to define an *openly semiprimitive* left $R-$module ${}_RM$ as follows:

DEFINITION 2.1. A left $R-$module ${}_RM$ is said to be openly semiprimitive if it has the zero intersection of all maximal open fully invariant submodules of ${}_RM$, that is, $\cap_\Lambda P_\alpha = 0$, for all maximal open fully invariant submodules $P_\alpha \leq M$ $(\alpha \in \Lambda)$.

REMARK 2.2. The fully invariance of a maximal open submodule inducing a maximal ideal of the endomorphism ring is essential. For example, the maximal open submodule $p\mathbb{Z} \oplus \mathbb{Z}$ being not fully invariant induces a maximal left ideal $I^{p\mathbb{Z}\oplus\mathbb{Z}}$ of the endomorphism ring of the module ${}_{\mathbb{Z}}\mathbb{Z}\oplus\mathbb{Z}$ but the (both-sided) ideal generated by $I^{p\mathbb{Z}\oplus\mathbb{Z}}$ is $End_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z})$.

THEOREM 2.3. *For any projective module ${}_RM$, the following are equivalent:*

(1) ${}_RM$ *is openly semiprimitive;*
(2) $End_R(M)$ *is semiprimitive.*

PROOF. By Lemma 1.3 it is clear. □

COROLLARY 2.4. *If a ring R is semiprimitive, then for any projective module ${}_RM$, so are ${}_RM$ and the endomorphism ring $End_R(M)$.*

PROOF. By the hypothesis we have that $Rad(R)M = Rad(M) = 0$ and by the Lemma 0.7 we have that

$$Rad(End_R(M)) = \{f \in End_R(M) \mid Imf \leq Rad(M)\} = 0.$$

Therefore every projective left module over a semiprimitive ring R and its endomorphism ring $End_R(M)$ are semi-primitive too. □

However the converse of the Corollary 2.4 is not true in general. For example, over the non-semiprimitive ring $\mathbb{Z}_4$ we can find a semiprimitive projective module ${}_{\mathbb{Z}_4}\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Nonetheless there is a clue for any semiprimitive free module over a ring to induce the semiprimitive ground ring and its semiprimitive endomorphism ring.

Let $Ann_l(R) = \{r \in R \mid rR = 0\}$ denote the left annihilator of a ring R.

PROPOSITION 2.5. *For any ring R with $Ann_l(R) = 0$, the following are equivalent:*

(1) R *is semiprimitive;*
(2) ${}_RM$ *is openly semiprimitive for every free $R-$module ${}_RM$;*
(3) $End_R(M)$ *is semiprimitive for every free $R-$module ${}_RM$.*

PROOF. (1) $\Longrightarrow$ (2): Since $Rad(\oplus R) = \oplus Rad(R) = 0$ by 9.19 Proposition in [5] it is trivial.

(2) $\Longrightarrow$ (3): It follows immediately from the fact that each free $R-$module is projective $R-$module and Lemma 0.7.

(3) $\Longrightarrow$ (1): Since for any ring R with identity 1_R, every endormorphism $\rho : {}_RR \to {}_RR$ is uniquely determined by the assignment $1_R \mapsto (1_R)\rho$ and the endomorphism ring $End_R(R)$ is isomorphic to R. Therefore R is a semiprimitive ring. □

Directly it follows that a left $\mathbb{Z}-$free module $\mathbb{Z}^{(\mathbb{N})}$ (a direct sum of $\mathbb{N}-$copies of $\mathbb{Z}$) is openly semiprimitive from $Ann_l(\mathbb{Z}) = 0$ and the semiprimitivity of the ring $\mathbb{Z}$.

REMARK 2.6. The openly semiprimitivity of any free module does not imply the semiprimitivity of it, however the converse is true always by Proposition 1.4 and Theorem 2.3, for example, an openly semiprimitive free module ${}_{\mathbb{Z}[x]}\mathbb{Z}[x]$ is not semiprimitive.

3. Strongly semiprimitive projective module

For a further development of all projective modules whose endomorphism rings are semiprimitive, we have to develop some properties relative to any projective module ${}_RM$ with the zero intersection $\cap P_\alpha = 0$ of all maximal open submodules P_α ($\alpha \in \Lambda$) which are fully invariant.

For conveniences, we need a definition of any module of this type as follows:

DEFINITION 3.1. A left $R-$module ${}_RM$ is said to be *strongly semiprimitive* if the intersection of all maximal open submodules which are fully invariant in ${}_RM$ is zero, more precisely speaking, $\cap P_\alpha = 0$ for all maximal open submodules $P_\alpha \leq M$ which are fully invariant.

If two distinct non-zero maximal open submodules $0 \neq P, 0 \neq Q \leq M$ which are fully invariant of any left $R-$module ${}_RM$ are given, then they are coprime, that is, $P + Q = M$ follows from the maximality of P and that of Q among open submodules of ${}_RM$. Moreover the additive group $Hom_R(M/P, M/Q) = 0$ follows from the fully invariant submodules $P, Q \leq M$ and from the isomorphic $R-$modules

$$M/P = (P+Q)/P \simeq Q/(P \cap Q) \text{ and } M/Q \simeq P/(P \cap Q)$$

with the following commutative diagram:

$$\begin{array}{ccc} M = P + Q & \xrightarrow{\exists\, \overline{f}} & M = P + Q \\ \Big\downarrow \pi_P & & \pi_Q \Big\downarrow \\ M/P \simeq Q/(P \cap Q) & \xrightarrow{f} & P/(P \cap Q) \simeq M/Q, \end{array}$$

where $\pi_P : M \to M/P$ and $\pi_Q : M \to M/Q$ are the canonical projections.

As a result of the above observation a remark is obtained:

REMARK 3.2. We need the following results obtained elementarily:

(1) Any two distinct non-zero maximal open submodules of any module which are fully invariant is coprime.
(2) For any distinct non-zero maximal open submodules $0 \neq P$, $0 \neq Q \leq M$ which are fully invariant, we have the trivial additive group

$$Hom_R(M/P, M/Q) = 0.$$

For any cyclic module ${}_RM/P_\alpha$ with a maximal open submodule $P_\alpha \leq M$ of a projective $R-$module ${}_RM$ which is fully invariant, we have a cyclic $R-$module ${}_RM/P_\alpha$ with a division endomorphism ring $End_R(M/P_\alpha)$ by Theorem 0.2 and Lemma 0.3. And ${}_RM/P_\alpha$ is quasi-injective by an easy computation:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{g} & M/P_\alpha = R(ag) \\ & & f \Big\downarrow & & \swarrow\, \exists\, h \text{ is an automorphism or the zero map } 0 \\ & & M/P_\alpha = R(af), & & \text{for some } (0 \neq) a \in A. \end{array}$$

Then M/P_α is a quasi-injective and projective module, for any openly simple quotient module M/P_α with any maximal open submodule P_α which is fully invariant and for every $\alpha \in \Lambda$.

At last we have the following properties:

NOTE 3.3. For any strongly semiprimitive projective module ${}_RM$ with all distinct maximal open submodules P_α ($\alpha \in \Lambda$) which are fully invariant,

(1) The quotient module ${}_RM/P_\alpha$ is projective, cyclic, and quasi-injective for every $\alpha \in \Lambda$.

(2) The endomorphism ring $End(M/P_\alpha)$ is a division ring, for every $\alpha \in \Lambda$.
(3) The direct product $\prod_\Lambda M/P_\alpha$ of a set of quasi-injective (projective) modules $\{M/P_\alpha\}_\Lambda$ is also quasi-injective (projective).

An additive subgroup A of a division ring D with identity 1_D is called a subdivision of D if $(A \setminus 0, \cdot)$ is a subgroup of the group $(D \setminus 0, \cdot)$ with the same identity 1_D under the restricted operations $+, \cdot$ to $A \times A$.

THEOREM 3.4. *For any projective module ${}_RM$, the following are equivalent:*

(1) ${}_RM$ *is strongly semiprimitive;*
(2) $End_R(M)$ *is a subdirect sum of a set of subdivisions of division rings.*

PROOF. Suppose that ${}_RM$ is a strongly semiprimitive projective module with $\cap P_\alpha = 0$, for all distinct maximal open submodules P_α $(\alpha \in \Lambda)$ of ${}_RM$ which are fully invariant. For each $\alpha \in \Lambda$, we have an openly simple projective left $R-$module ${}_RM/P_\alpha$. Then ${}_RM$ is a subdirect product of a set of openly simple projective modules M/P_α $(\alpha \in \Lambda)$ by the Proposition 0.4. And hence there is a monomorphism

$\phi : M \to \prod M/P_\alpha$ such that the composition

$\phi\pi_\alpha : M \xrightarrow{\phi} \prod M/P_\alpha \xrightarrow{\pi_\alpha} M/P_\alpha$ is an epimorphism for each $\alpha \in \Lambda$.

On the other hand, the endomorphism ring $End_R(M/P_\alpha)$ is a division ring by Theorem 0.2 the Generalized Schur's Lemma I, for each $\alpha \in \Lambda$. Considering the following diagram:

$$\begin{array}{ccccccc} M & \xrightarrow{\phi} & \phi(M) & \xrightarrow{\hookrightarrow} & \prod_\Lambda M/P_\alpha & \xrightarrow{\pi_\alpha} & M/P_\alpha \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f=\phi^{-1}g\phi} & & \downarrow{\scriptstyle \prod_\Lambda h_\alpha} & & \downarrow{\scriptstyle h_\alpha \text{ such that } (\prod h_\alpha)|_{\phi(M)}=f} \\ M & \xrightarrow{\phi} & \phi(M) & \xrightarrow{\hookrightarrow} & \prod_\Lambda M/P_\alpha & \xrightarrow{\pi_\alpha} & M/P_\alpha, \end{array}$$

where $h_\alpha : M/P_\alpha \to M/P_\alpha$ is either a unit or the zero and $\hookrightarrow$ is the inclusion mapping, then we have the following: for any given endomorphism $g : {}_RM \to {}_RM$, precisely speaking, for any endomorphism

$$f = \phi^{-1}g\phi : \phi(M) \xrightarrow{\phi^{-1}} M \xrightarrow{g} M \xrightarrow{\phi} \phi(M),$$

there is an endomorphism $h_\alpha : M/P_\alpha \to M/P_\alpha$ which is a unit or the zero mapping 0, for every $\alpha \in \Lambda$ since $End_R(M/P_\alpha)$ is a division ring. Thus

f is the restriction of $\prod_\Lambda h_\alpha$ to the submodule $\phi(M)$ of $\prod_\Lambda M/P_\alpha$

and

$$D_\alpha = \{h_\alpha \mid f \in End_R(\phi(M)),\ f = (\prod h_\alpha)|_{M/P_\alpha}\} \subseteq End(M/P_\alpha)$$

is a subdivision of the division ring $End(M/P_\alpha)$

$$\text{since } Hom_R(M/P_\alpha, M/P_\beta) = 0\ (\alpha \neq \beta).$$

On the other hand,

$$End_R(M) \simeq End_R(\phi(M)) \hookrightarrow \prod_\Lambda D_\alpha \to D_\alpha \quad (\alpha \in \Lambda)$$

is an epimorphism which implies that the endomorphism ring $End(\phi(M))$ is a subdirect sum of a set of subdivisions D_α of division ring $End(M/P_\alpha)$ $(\alpha \in \Lambda)$.

$$\text{Let } H_\alpha = \prod D'_\alpha, \text{ by taking } D'_\beta = \begin{cases} 0, & \text{if } \beta = \alpha, \\ D_\beta, & \text{if } \beta \neq \alpha, \end{cases}$$

then each H_α $(\alpha \in \Lambda)$ is a maximal ideal of the endomorphism ring $End_R(\phi(M))$ and their intersection

$$Rad(End_R(\phi(M)) = 0$$

is clearly the zero. Therefore $End_R(M)$ is a subdirect product of a set of subdivision rings D_α $(\alpha \in \Lambda)$ and $End_R(M)$ is semiprimitive since $End_R(\phi(M)) \simeq End_R(M)$. The proof of the converse is easy by the elementary computation, which will not be written here. □

PROPOSITION 3.5. *For any set of openly simple projective modules M_α $(\alpha \in \Lambda$ an indexed set), if $Hom_R(M_\alpha, M_\beta) = 0$ for each $\alpha \neq \beta$ and for any subset Γ of Λ, then we have the following:*

(1) *Any direct product $\prod_\Gamma M_\alpha$ is an openly semiprimitive projective module.*
(2) *The endomorphism ring $End(\prod_\Gamma M_\alpha) \simeq \prod_\Gamma End_R(M_\alpha)$ is a direct product of division rings $End(M_\alpha)$ $(\alpha \in \Gamma)$. Moreover $End_R(\prod_\Gamma M_\alpha)$ is semiprimitive.*
(3) *Each subdirect product of $\{M_\alpha\}_\Gamma$ $(\Gamma \subseteq \Lambda)$ is also an openly semiprimitive projective module.*

(4) *Every endomorphism* $p_\alpha \in End(\prod_\Gamma M_\alpha)$ *of type* $\prod_\Gamma g_\alpha$, *where*

$$p_\alpha = \prod_\Gamma g_\beta : \prod_\Gamma M_\alpha \to \prod_\Gamma M_\alpha \text{ defined by } g_\beta = \begin{cases} 0, & \text{if } \beta = \alpha, \\ \text{non-zero}, & \text{if } \beta \neq \alpha \end{cases}$$

is prime, for every $\alpha \in \Gamma$.

(5) *Every endomorphism* $h : \prod_\Gamma M_\alpha \to \prod_\Gamma M_\alpha$ *can be factored into a product of some prime endomorphisms.*

PROOF. (1) and (2): For each openly simple projective module M_α, we have a division endomorphism ring $End_R(M_\alpha)$ and the zero maximal open fully invariant submodule of M_α ($\alpha \in \Gamma$). Considering the direct product $\prod_\Gamma M_\alpha$ of which

$$\prod_\Gamma K_\beta, \text{ where } K_\beta = \begin{cases} M_\beta, & \text{if } \beta \neq \alpha, \\ 0, & \text{if } \beta = \alpha \end{cases} \quad (\alpha \in \Gamma)$$

is a maximal open fully invariant submodule with the zero intersection

$$\cap_{\alpha\in\Gamma} \prod_\Gamma K_\beta = 0.$$

Hence it follows immediately that $\prod_\Gamma M_\alpha$ is an openly semiprimitive projective R–module. By the fact of $Hom_R(M_\alpha, M_\beta) = 0$ ($\alpha \neq \beta$)

$$\text{we have easily that} \quad End_R(\prod_\Gamma M_\alpha) \simeq \prod_\Gamma End_R(M_\alpha).$$

(3): It is clear that every openly semiprimitive projective module $\prod_\Gamma M_\beta$ whose endomorphism ring is semiprimitive, for any openly simple projective module M_β ($\beta \in \Gamma$) by the above item (1).

(4): By the Lemma 0.3 we have a cyclic module M_α ($\alpha \in \Gamma$) since the endomorphism ring $End(M_\alpha)$ is a division ring, for each openly simple projective module M_α ($\alpha \in \Gamma$). Thus the mapping $p_\alpha : \prod M_\alpha \to \prod M_\alpha$ is clearly a prime endomorphism by elementary computation.

(5): It is clear. □

Here is a practical way to find prime endomorphisms on any strongly semiprimitive projective module.

PROPOSITION 3.6. *For any strongly semiprimitive projective module* ${}_RM$ *with all distinct maximal open submodules* P_α $(\alpha \in \Lambda)$ *which are fully invariant, we have the following properties:*

(1) ${}_RM$ *is openly semiprimitive.*
(2) $End_R(M)$ *is semiprimitive.*
(3) ${}_RM$ *is a subdirect product of a set of openly simple projective modules* $\{M/P_\alpha\}_\Lambda$.
(4) *For such a subdirect product* ${}_RM$ *of a set of openly simple projective modules* M/P_α $(\alpha \in \Lambda)$ *and for a monomorphism*

$$\phi : M \to \prod_\Lambda M/P_\alpha$$

with an epimorphism

$$\phi\pi_\alpha : M \xrightarrow{\phi} \prod_\Lambda M/P_\alpha \xrightarrow{\pi_\alpha} M/P_\alpha \ (\alpha \in \Lambda),$$

where $\pi_\alpha : \prod_\Lambda M/P_\alpha \to M/P_\alpha$ *is the canonical projection, where* $p'_\alpha = \prod_\Gamma g_\beta : \prod_\Lambda M/P_\alpha \to \prod_\Lambda M/P_\alpha$ *defined by* $g_\beta = \begin{cases} 0, & \text{if } \beta = \alpha, \\ \text{non-zero}, & \text{if } \beta \neq \alpha \end{cases}$ *is a prime endomorphism, and where* $p_\alpha = p'_\alpha|_{\phi(M)}$ *is the restriction of* p'_α *to* $\phi(M)$, *for every* $\alpha \in \Lambda$, *we have the following:*

(*i*) *If* $\phi p_\alpha \phi^{-1} \in End_R(M)$, *then* $\phi p_\alpha \phi^{-1}$ *is a prime endomorphism in* $End_R(M)$ $(\alpha \in \Lambda)$.
(*ii*) *Every endomorphism* $h : {}_RM \to {}_RM$ *is factored into a product of some prime endomorphisms* $\phi p_\alpha \phi^{-1} \in End_R(M)$.

PROOF. The proof is elementary by Proposition 3.5. □

COROLLARY 3.7. *For any strongly semiprimitive projective module* ${}_RM$ *with all distinct maximal open submodules* P_α $(\alpha \in \Lambda)$ *of* ${}_RM$ *which are fully invariant, if each endomorphism ring* $End_R(M/P_\alpha)$ *is a field, for every* $\alpha \in \Lambda$. *Then we have the following:*

(1) ${}_RM$ *is openly semiprimitive.*
(2) $End_R(M)$ *is a commutative and semiprimitive ring.*
(3) *Every endomorphism is factored into a product of some prime endomorphisms.*

PROOF. It is clear by Proposition 3.6. □

The next example shows factorizations to be noticed. Practically, if a strongly projective $\mathbb{Z}-$module is taken; ${}_{\mathbb{Z}}\mathbb{Z}_{30} \simeq {}_{\mathbb{Z}}\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ with the zero intersection of all maximal open submodules which are fully invariant was given, then the endomorphism $\rho(2^2 \cdot 3) : {}_{\mathbb{Z}}\mathbb{Z}_{30} \to {}_{\mathbb{Z}}\mathbb{Z}_{30}$ being a left multiplication by $2^2 \cdot 3$ has factorizations and a prime factorization

$$\begin{aligned}
&\rho(2^2 \cdot 3) \\
=&\phi \begin{pmatrix} \rho(12) : \mathbb{Z}_2 \to \mathbb{Z}_2 & 0 & 0 \\ 0 & \rho(12) : \mathbb{Z}_3 \to \mathbb{Z}_3 & 0 \\ 0 & 0 & \rho(12) : \mathbb{Z}_5 \to \mathbb{Z}_5 \end{pmatrix} \phi^{-1} \\
=&\phi \begin{pmatrix} \rho(4) = 0 & 0 & 0 \\ 0 & \rho(4) \neq 0 & 0 \\ 0 & 0 & \rho(4) \neq 0 \end{pmatrix} \begin{pmatrix} \rho(3) \neq 0 & 0 & 0 \\ 0 & \rho(3) = 0 & 0 \\ 0 & 0 & \rho(3) \neq 0 \end{pmatrix} \phi^{-1} \\
=&(\phi A \phi^{-1})^2 (\phi B \phi^{-1}) = \phi A^2 B \phi^{-1} \text{ (a product of prime endomorphisms)}
\end{aligned}$$

(with noting that

$$\rho(2^2 \cdot 3) \neq \phi \begin{pmatrix} \rho(4) & 0 & 0 \\ 0 & \rho(4) & 0 \\ 0 & 0 & \rho(3) \end{pmatrix} \begin{pmatrix} \rho(3) & 0 & 0 \\ 0 & \rho(3) & 0 \\ 0 & 0 & \rho(4) \end{pmatrix} \phi^{-1}$$

because of

$$\phi \begin{pmatrix} \rho(4) & 0 & 0 \\ 0 & \rho(4) & 0 \\ 0 & 0 & \rho(3) \end{pmatrix} \phi^{-1} \notin End_{\mathbb{Z}}(\mathbb{Z}_{30})$$

and because of

$$\phi \begin{pmatrix} \rho(3) & 0 & 0 \\ 0 & \rho(3) & 0 \\ 0 & 0 & \rho(4) \end{pmatrix} \phi^{-1} \notin End_{\mathbb{Z}}(\mathbb{Z}_{30}) \text{)}$$

where

$$A = \begin{pmatrix} \rho(2) : \mathbb{Z}_2 \to \mathbb{Z}_2 & 0 & 0 \\ 0 & \rho(2) : \mathbb{Z}_3 \to \mathbb{Z}_3 & 0 \\ 0 & 0 & \rho(2) : \mathbb{Z}_5 \to \mathbb{Z}_5 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \rho(3) : \mathbb{Z}_2 \to \mathbb{Z}_2 & 0 & 0 \\ 0 & \rho(3) : \mathbb{Z}_3 \to \mathbb{Z}_3 & 0 \\ 0 & 0 & \rho(3) : \mathbb{Z}_5 \to \mathbb{Z}_5 \end{pmatrix}$$

which is factored into a product $\phi A^2 B \phi^{-1}$ of prime endomorphisms, for a monomorphism

$$\phi : {}_{\mathbb{Z}}\mathbb{Z}_{30} \to {}_{\mathbb{Z}}(\mathbb{Z}_{30})/(2(\mathbb{Z}_{30})) \times (\mathbb{Z}_{30})/(3(\mathbb{Z}_{30})) \times (\mathbb{Z}_{30})/(5(\mathbb{Z}_{30}))$$

such that $\phi\pi_\alpha$ is an epimorphism ($\alpha \in \Lambda = \{1, 2, 3\}$).

It may sound strange to factor out any endomorphism on the integer ring $\mathbb{Z}$, but any endomorphism $\rho(a) : {}_{\mathbb{Z}}\mathbb{Z} \to {}_{\mathbb{Z}}\mathbb{Z}$ which is a left multiplication by a can be factored out, in many ways such as

$$\begin{aligned}\rho(a) &= \rho(p_1^{k_1})\rho(p_2^{k_2}) \cdots \rho(p_n^{k_n}) \quad \text{(a product of endomorphisms)} \\ &= \rho(p_1)^{k_1}\rho(p_2)^{k_2} \cdots \rho(p_n)^{k_n} \quad \text{(a product of prime endomorphisms)}\end{aligned}$$

where $a = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ is factored into a product of prime numbers p_i and some non-negative integer k_i ($i = 1, 2, \ldots, n$). Here is, considering a given monomorphism $\phi : {}_{\mathbb{Z}}\mathbb{Z} \to \prod_{\text{prime } p} \mathbb{Z}/p\mathbb{Z}$, then $\rho(2^2 \cdot 3) = \phi X^2 Y \phi^{-1}$ is factored into a product of prime endomorphisms $\phi X \phi^{-1} \in End_{\mathbb{Z}}(\mathbb{Z})$ and $\phi Y \phi^{-1} \in End_{\mathbb{Z}}(\mathbb{Z})$, where X and Y are diagonal matrices such that

$$X = (x_{ij})_{ij} \text{ with } x_{ij} = \begin{cases} \rho(2), & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

and

$$Y = (y_{ij})_{ij} \text{ with } y_{ij} = \begin{cases} \rho(3), & \text{if } i = j. \\ 0, & \text{if } i \neq j. \end{cases}$$

One can easily observe that any endomorphism on $\prod_{\text{prime } p} \mathbb{Z}/p\mathbb{Z}$ of $\rho(p)$ all diagonals and the zeros elsewhere is a prime endomorphism but any endomorphism on $\prod_{\text{prime } p} \mathbb{Z}/p\mathbb{Z}$ of $\rho(p^h q^k)$ all diagonals and the zeros elsewhere is not a prime endomorphism for prime numbers p, q and for $h, k = 1, 2, 3, \cdots$.

However not all endomorphism rings of strongly semiprimitive projective modules are commutative, we remark about factorization with the crucial condition "$\phi p_\alpha \phi^{-1} \in End_R(M)$" to be required as follows:

REMARK 3.8. Prime factorization of an endomorphism on a strongly semiprimitive projective module is not unique, in general.

THEOREM 3.9. *For any strongly semiprimitive projective left $R-$module ${}_RM$ with the zero intersection of all distinct maximal open submodules P_α ($\alpha \in \Lambda$) which are fully invariant, the following are equivalent:*

(1) ${}_RM$ *is openly semiprimitive;*

(2) $Rad(M)^o = 0$;
(3) ${}_RM$ *is a subdirect product of a set of all openly simple projective modules* ${}_RM/P_\alpha$ $(\alpha \in \Lambda)$;
(4) ${}_RM$ *is cogenerated by a set of all openly simple projective modules* ${}_RM/P_\alpha$ $(\alpha \in \Lambda)$;
(5) $End_R(M)$ *is a subdirect sum of a set of subdivisions of all division rings* $End_R(M/P_\alpha)$ $(\alpha \in \Lambda)$;
(6) $End_R(M)$ *is semiprimitive.*

PROOF. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5): It is completed easily.
(5) $\Longrightarrow$ (1): By Theorem 2.3 it is clear. □

We need to simplify these results together with each strongly semiprimitive projective module and semiprimitive endomorphism ring in order to see some kind of generalization:

COROLLARY 3.10. *For a set* $\{N_\alpha\}_\Lambda$ *of strongly semiprimitive projective* $R-$*modules* ${}_RN_\alpha$ *with* $Hom(N_\alpha, N_\beta) = 0$ $(\alpha \neq \beta)$ *and for any subset* Γ *of* Λ *, the subdirect product of a set* $\{N_\alpha\}_\Gamma$ *is strongly semiprimitive.*

PROOF. Suppose that M is a subdirect product of a set of N_α $(\alpha \in \Gamma)$. Then there is an $R-$monomorphism $\phi : M \to \prod_\Gamma N_\alpha$ such that $\phi\pi_\alpha : M \xrightarrow{\phi} \prod_\Gamma N_\alpha \xrightarrow{\pi_\alpha} N_\alpha$ is an epimorphism for each $\alpha \in \Gamma$.

Considering the following diagram:

$$\begin{array}{ccccccc} M & \xrightarrow{\phi} & \prod_\Gamma N_\alpha & \xrightarrow{\pi_\beta} & N_\beta & \longrightarrow & 0 \\ {\scriptstyle \phi\overline{\phi\pi_\alpha}}\downarrow & \swarrow \exists\overline{\phi\pi_\alpha} & & \searrow \pi_\alpha & & & \\ M & \xrightarrow[\phi]{} & \prod_\Gamma N_\alpha & \xrightarrow[\pi_\alpha]{} & N_\alpha & \longrightarrow & 0, \end{array}$$

since any direct product $\prod_\Gamma N_\alpha$ of projective modules N_α $(\alpha \in \Gamma)$ is also projective there is an $R-$homomorphism

$$\overline{\phi\pi_\alpha} : \prod_\Gamma N_\alpha \to M \text{ such that } \overline{\phi\pi_\alpha}\phi\pi_\alpha = \pi_\alpha, \text{ for each } \alpha \in \Gamma.$$

Let $K_\alpha = \phi^{\leftarrow}(\prod_\Gamma Q_\beta)$ be the preimage of $\prod_\Gamma Q_\beta$ under the monomorphism ϕ, where $Q_\beta = 0$ if $\beta \neq \alpha$ and $Q_\beta = N_\alpha$ if $\beta = \alpha$. Then we have an openly simple projective module which is isomorphic to N_α for every $\alpha \in \Gamma$. And let $L_\alpha = \sum_{\beta \neq \alpha} K_\beta$ is a maximal open submodule of M which is fully invariant and their intersection $\cap L_\alpha = 0$ follows, telling that M is a strongly semiprimitive module. □

COROLLARY 3.11. *For any set of strongly semiprimitive projective $R-$modules ${}_RM_\alpha$ with $Hom(M_\alpha, M_\beta) = 0$ $(\alpha \neq \beta$ in an index set $\Lambda)$ and for any subset Γ of Λ, we have the following:*

(1) *Any direct product $\prod_\Gamma M_\alpha$ is strongly semiprimitive.*
(2) *Any direct sum $\oplus_\Gamma M_\alpha$ is strongly semiprimitive.*

PROOF. (1) and (2): It is trivial since each direct product $\prod_\Gamma M_\alpha$ and each direct sum $\oplus_\Gamma M_\alpha$ are subdirect product of the set $\{M_\alpha\}_\Gamma$ of strongly semiprimitive projective modules M_α $(\alpha \in \Gamma)$ with

$$Hom_R(M_\alpha, M_\beta) = 0 \ (\alpha \neq \beta \text{ in } \Gamma),$$

for any subset Γ of Λ. $\prod_\Gamma M_\alpha$ and $\oplus_\Gamma M_\alpha$ are semiprimitive projective modules with the zero intersection of all distinct maximal open submodules being fully invariant which have the semiprimitive endomorphism rings $End_R(\prod_\Gamma M_\alpha)$ and $End_R(\oplus_\Gamma M_\alpha)$. □

References

[1] H. Bass, *Finistic Dimension and a Homological Generalization of Semiprimary Rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.
[2] I. N. Herstein, *Noncommutative rings*, MAA Notes, 1968.
[3] R. Ware and J. Zelmanowitz, *The Jacobson Radical of the Endomorphism Ring of a Projective Module*, Proc. Amer. Math. Soc. **26** (1970), 15–20.
[4] R. Ware, *Endomorphism Rings of Projective Modules*, Trans. Amer. Math. Soc. **55** (1971), 233–256.
[5] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, New York; Spring–Verlag, 1973.
[6] I. Beck, *On Modules whose Endomorphism ring is Local*, Israel J. Math. **29** (1978), no. 4, 393–407.
[7] T. W. Hungerford, *Algebra*, New York; Spring –Verlag, 1984.
[8] S.-S. Bae, *On Reject of Subdirect Product*, Kyungnam University, Thesis Collection **14** (1987), 7–10.
[9] ———, *Certain Discriminations of Prime Endomorphism and Prime Matrix*, East Asian Math. J. **14** (1998 no. 2), 259–268.
[10] ———, *Generalized Schur's Lemmas*, Journal of the Graduate School, Kyungnam University **16** (2001 no. 2), 7–11.

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