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PROPERTIES ON TYPES OF PRIMITIVE NEAR-RINGS

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ABSTRACT. Throughout this paper, we will consider that R is a near-ring and G an R-group. We initiate the study of monogenic, strongly monogenic R-groups, 3 types of nonzero R-groups and their basic properties. At first, we investigate some properties of D.G. (R, S)-groups, faithful R-groups, monogenic R-groups, simple and R-simple R-groups. Next, we introduce modular right ideals, t-modular right ideals and 3 types of primitive near-rings. The purpose of this paper is to investigate some properties of primitive types near-rings and their characterizations.

1. Introduction

Let R be a (left) near-ring. If R has a unity 1, then R is called *unitary*. If 0 is the neutral element of the group $(R, +)$, then the left distributivity law implies the identity $a0 = 0$ for all $a \in R$. However, $0a$ is not equal to 0, in general. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R. A near-ring R with $(R, +)$ is abelian is called an *abelian* near-ring.

We consider the following notations: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ is called the *zero symmetric part* of R,

$$R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a, \text{ for all } r \in R\}$$

which is called the *constant part* of R, and

$$R_d = \{a \in R \mid a \text{ is distributive}\}$$

is called the *distributive part* of R.

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We note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring. From the Pierce decomposition theorem, we get

$$R = R_0 \oplus R_c$$

as additive groups. So every element $a \in R$ has a unique representation of the form $a = b + c$, where $b \in R_0$ and $c \in R_c$.

One defines a subset H of R such that $RH \subseteq H$ is called a *left R-subset* of R, a subset H of R such that $HR \subseteq H$ is called a *right R-subset* of R, and a left and right R-subset H is said to be a (*two-sided*) *R-subset* of R.

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings $f,\ g$ in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ (called the *pointwise addition of maps*) and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G),\ +,\ \cdot)$ becomes a near-ring. It is called the *near-ring of self maps* on G.

Also, if we define the set

$$M_0(G) := \{f \in M(G) \mid of = o\}$$

for additive group G with identity o, then $(M_0(G),\ +,\ \cdot)$ is a zero symmetric near-ring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if for all $a,\ b \in R$, (i) $(a+b)\theta = a\theta + b\theta$ and (ii) $(ab)\theta = a\theta b\theta$.

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R,\ +,\ \cdot) \longrightarrow (M(G),\ +,\ \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write that xr (right scalar multiplication in R) for $x(\theta_r)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary.* Thus an R-group is an additive group G satisfying (i) $x(a + b) = xa + xb$, (ii)

$x(ab) = (xa)b$ and (iii) $x1 = x$ (If R has a unity 1), for all $x \in G$ and $a,\ b \in R$. Sometimes, we abbreviate this R-group G simply as G_R.

We note that R itself is an R-group called the *regular R-group.*

Moreover, naturally, every group G has an $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf. Also, $M(G)$ is a regular $M(G)$-group.

An R-group G with the property that for each $x,\ y \in G$ and $a \in R$, $(x+y)a = xa + ya$ is called a *distributive R-group*, and also an R-group G with $(G,\ +)$ is abelian is called an *abelian R-group*. For example, if $(G,\ +)$ is abelian, then $M(G)$ is an abelian near-ring and moreover, G is an abelian $M(G)$-group, on the other hand, every distributive near-ring R is a distributive R-group. We will make an exhibition that the existence of distributive abelian R-groups in this chapter 2.

We denote that the neutral element of G as o, this is different from the neutral element 0 of the near-ring R, also we write the trivial groups (or ideals) of G and R are $\{o\} =: o$ and $\{0\} =: 0$ respectively.

A representation θ of R on G is called *faithful* if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful R-group*, or that R *acts faithfully* on G.

For an R-group G, a subgroup T of G such that $TR \subseteq T$ is called an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that $(N+x)a - xa \subseteq N$ for all $x \in G,\ a \in R$. The R-ideals of the regular group R are precisely the right ideals of R. Also, a subgroup V of G such that $VR \subseteq V$ is called an *R-subset* of G.

Let $G,\ T$ be two additive groups (not necessarily abelian). Then the set

$$M(G,\ T) := \{f \mid f : G \longrightarrow T\}$$

of all maps from G to T becomes an additive group under pointwise addition of maps. Since $M(T)$ is a near-ring of self maps on T, we note that $M(G,\ T)$ is an $M(T)$-group with a scalar multiplition:

$$M(G,\ T) \times M(T) \longrightarrow M(G,\ T)$$

defined by $(f,\ g) \longmapsto f \cdot g$, where $x(f \cdot g) = (xf)g$ for all $x \in G$.

Let G and T be two R-groups. Then the mapping $f : G \longrightarrow T$ is called a *R-group homomorphism* if for all $x,\ y \in G$ and $a \in R$, (i) $(x+y)f = xf + yf$ and (ii) $(xa)f = (xf)a$. In this paper, we call that the mapping $f : G \longrightarrow T$ with the condition $(xa)f = (xf)a$ is an *R-map* (or *R-homogeneous map* [7]).

Also, we can replace R-group homomorphism by R-group monomorphism, R-group epimorphism, R-group isomorphism, R-group endomorphism and R-group automorphism, if these terms have their usual meanings as for modules ([1]).

A near-ring R is called *distributively generated* (briefly, *D.G.*) by S if

$$(R,+) = gp < S >= gp < R_d >$$

where S is a semigroup of distributive elements in R.

In particular, $S = R_d$ (this is motivated by the set of all distributive elements of R is multiplicatively closed and contain the unity of R if it exists), where $gp < S >$ is a group generated by S, we denote this D.G. near-ring R which is generated by S is (R,S).

On the other hand, the set of all distributive elements of $M(G)$ are obviously the set $End(G)$ of all endomorphisms of the group G, that is,

$$(M(G))_d \ = \ End(G)$$

which is a semigroup under composition, but not yet a near-ring. Here we denote that $E(G)$ is the D.G. near-ring generated by $End(G)$, that is,

$$E(G) = (M(G)),\ End(G)).$$

Obviously, $E(G)$ is a subnear-ring of $(M_0(G),+,\cdot)$. Thus we say that $E(G)$ is the *endomorphism near-ring* of the group G.

Let (R,S) and (T,U) be D.G. near-rings. Then a near-ring homomorphism

$$\theta : (R,S) \longrightarrow (T,U)$$

is called a *D.G. near-ring homomorphism* if $S\theta \subseteq U$. Clearly, any near-ring epimorphism $\theta : (R,S) \longrightarrow (T,U)$ is a D.G. near-ring homomorphism.

Note that a semigroup homomorphism $\theta : S \longrightarrow U$ is a D.G. near-ring homomorphism if it is a group homomorphism from $(R,+)$ to $(T,+)$ ([5], [6]).

For the remainder concepts and results on near-rings and R-groups, we refer to J. D. P. Meldrum [9], and G. Pilz [10].

2. Some properties on monogenic R-Groups

There is a module like concept as follows: Let (R,S) be a D.G. near-ring. Then an additive group G is called a *D.G.* (R,S)*-group* if there exists a D.G. near-ring homomorphism

$$\theta : (R, S) \longrightarrow (M(G), End(G)) = E(G)$$

such that $S\theta \subseteq End(G)$. If we write that xr instead of $x(\theta_r)$ for all $x \in G$ and $r \in R$, then an D.G. (R, S)-group is an additive group G satisfying the following conditions:

$$x(rs) = (xr)s, \ x(r+s) = xr + xs,$$

for all $x \in G$ and all $r, s \in R$, and

$$(x+y)s = xs + ys,$$

for all $x, y \in G$ and all $s \in S$.

Such a homomorphism θ is called a *D.G. representation* of (R, S) on G. This D.G. representation is said to be *faithful* if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful D.G. (R, S)-group.* For any R-group G, we define also the set

$$M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r, \text{ for all } x \in G, \ r \in R\}$$

of all R-maps on G.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G, more specially, if G is monogenic and for each $x \in G$, $xR = o$ or $xR = G$, then G is called a *strongly monogenic R-group.*

For any group G, $M(G)$-group G and $M_0(G)$-group G are strongly monogenic which are appeared in G. Pilz [10]. It is clearly proved that $G \neq 0$ if and only if $GR \neq 0$ for any monogenic or strongly monogenic R-group G.

LEMMA 2.1. *Let R be a near-ring and G an R-group. Then we have the basic concepts:*

(1) *If I is a right ideal of R, then $IR_0 \subseteq I$.*
(2) *If A is an R-ideal of G, then A is an R_0-subgroup of G.*

From this useful lemma, we obtain the following several properties.

LEMMA 2.2. *For a near-ring R, the following are equivalent:*

(1) *R is a zero symmetric near-ring.*
(2) *Every right ideal of R is an R-subgroup of R.*

PROOF. (1) $\Longrightarrow$ (2) is obtained from the Lemma 2.1 (1).

(2) $\Longrightarrow$ (1) Suppose that every right ideal of R is an R-subgroup of R. Since 0 is clearly a right ideal of R, 0 is an R-subgroup of R. Thus $0R = 0$. This implies that $R = R_0$. $\square$

LEMMA 2.3 ([10]). *For an R-group G, we have the following:*

(1) *For any x in G, xR is an R-subgroup of G.*
(2) *For any R-subgroup A of G, we have that $oR = oR_c \subseteq A$.*

In Lemma 2.3 (2), oR is the smallest R-subgroup of G under all R-subgroups of G, So throughout this paper, we will write that

$$oR = oR_c =: \Omega.$$

We note that if R is zero symmetric, then $\Omega = \{o\} =: o$, and $\Omega = xR_c$ for all $x \in G$.

From this Lemma 2.3 (2), we define the following concepts: An R-group G is called *simple* if G has no non-trivial ideal, that is, G has no ideals except o and G. Similarly, we can define simple near-ring as ring case. Also, R-group G is called *R-simple* if G has no R-subgroups except Ω and G.

LEMMA 2.4. *For an R-group G and A is a subgroup of G, we have the following:*

(1) *A is an R-ideal of G if and only if A is an R_0-ideal of G.*
(2) *A is an R-subgroup of G if and only if A is an R_0-subgroup of G and $\Omega \subseteq A$.*

PROOF. (1) Necessity is obvious. Suppose A is an R_0-ideal of G. Let $a \in A$, $x \in G$ and $r \in R$. Then since $R = R_0 \oplus R_c$, we rewrite that $r = s + t$, where $s \in R_0$ and $t \in R_c$. Thus we have

$$(a+x)r - xr = (a+x)(s+t) - x(s+t) = (a+x)s + (a+x)t - xt - xs$$

Here, since $t \in R_c$, $(a+x)t - xt = t - t = 0$ so that $(a+x)r - xr = (a+x)s - xs$. Also since $s \in R_0$ and A is an R_0-ideal of G, $(a+x)s - xs \in A$, that is $(a+x)r - xr \in A$. Consequently, A is an R-ideal of G.

(2) This statement can be proved as a similar method of the proof of (1). $\square$

Lemma 2.1 (2) and Lemma 2.4 imply the following statement.

PROPOSITION 2.5. *For an R-group G with $\Omega \neq o$, we have the following:*

(1) *$G = \Omega$ if and only if G is strongly monogenic.*
(2) *R_0-simplicity implies simplicity for G.*

PROPOSITION 2.6. *Let G be an abelian D.G (R, S)-group. Then the set $M_R(G) := \{f \in M(G) \mid (xr)f = (xf)r,\ \text{for all } x \in G,\ r \in R\}$ is a subnear-ring of $M(G)$.*

PROOF. Let $f,\ g \in M_R(G)$. For any $x \in G$ and $r \in R$, since R is a D.G. near-ring generated by S, consider that

$$r = \delta_1 s_1 + \delta_2 s_2 + \delta_3 s_3 + \cdots + \delta_n s_n,$$

where $\delta_i = 1$, or -1 and $s_i \in S$ for $i = 1, \cdots, n$. We have that

$$\begin{aligned}
&(xr)(f+g)\\
&= (xr)f + (xr)g = (xf)r + (xg)r\\
&= xf(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) + xg(\delta_1 s_1 + \delta_2 s_2 + \cdots \delta_n s_n)\\
&= xf\delta_1 s_1 + xg\delta_1 s_1 + xf\delta_2 s_2 + xg\delta_2 s_2 + \cdots + xf\delta_n s_n + xg\delta_n s_n\\
&= \delta_1 xfs_1 + \delta_1 xgs_1 + \delta_2 xfs_2 + \delta_2 xgs_2 + \cdots + \delta_n xfs_n + \delta_n xgs_n\\
&= \delta_1(xfs_1 + xgs_1) + \delta_2(xfs_2 + xgs_2) + \cdots + \delta_n(xfs_n + xgs_n)\\
&= \delta_1(xf + xg)s_1 + \delta_2(xf + xg)s_2 + \cdots + \delta_n(xf + xg)s_n\\
&= (xf + xg)\delta_1 s_1 + (xf + xg)\delta_2 s_2 + \cdots + (xf + xg)\delta_n s_n\\
&= (xf + xg)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n)\\
&= (xf + xg)r = x(f+g)r.
\end{aligned}$$

Similarly, we have the following equalities:

$$(xr)(-f) = -(xr)f = -(xf)r = x(-f)r$$

and

$$(xr)f \cdot g = ((xr)f)g = ((xf)r)g = (xf)gr = x(f \cdot g)r.$$

Thus $M_R(G)$ is a subnear-ring of $M(G)$. □

In ring and module theory, we obtain the following important structure for near-ring and R-group theory:

COROLLARY 2.7 (C. J. MAXSON [7]). *Let R be a ring and V a right R-module. Than $M_R(V) := \{f \in M(V) \mid (xr)f = (xf)r,$ for all $x \in V,\ r \in R\}$ is a subnear-ring of $M(V)$.*

EXAMPLE 2.8. If R is a distributive unitary near-ring, then R is a ring (See [10, 1.107]). Furthermore, if R is a distributive unitary near-ring, then every unitary R-group is abelian, and a unitary R-module.

PROOF. Let G be a unitary R-group. Then $x(2) = x(1+1) = x+x$, for all $x \in G$. Thus we see that

$$x+y+x+y = (x+y)(2) = x(2)+y(2) = x+x+y+y,$$

for all $x, y \in G$. This implies that $(G,+)$ is abelian. Since $R = S$, the set of all distributive elements as a D.G. near-ring of R, $(x+y)r = xr+yr$, for all $x, y \in G$ and all $r \in R$. Hence G becomes a unitary R-module. □

LEMMA 2.9 ([8]). *Let (R,S) be a D.G. near-ring. Then all R-subgroups and all R-homomorphic images of a D.G. (R,S)-group are also D.G. (R,S)-groups.*

Let G be an R-group and K, K_1 and K_2 be subsets of G. Define

$$(K_1 : K_2) := \{a \in R | K_2 a \subseteq K_1\}.$$

We abbreviate that for $x \in G$, $(\{x\} : K_2) =: (x : K_2)$ $(o : K)$ is called the *annihilator* of K, sometimes denoted it by $A(K)$. Easily, we can drive that G is a faithful R-group if $A(G) = \{0\}$, that is, $(o : G) = \{0\}$.

LEMMA 2.10 ([3]). *Let G be an R-group and K_1 and K_2 subsets of G. Then we have the following conditions:*

(1) *If K_1 is a normal subgroup of G, then $(K_1 : K_2)$ is a normal subgroup of a near-ring R.*
(2) *If K_1 is an R-subgroup of G, then $(K_1 : K_2)$ is an R-subgroup of R as an R-group.*
(3) *If K_1 is an R-ideal of G and K_2 is an R-subset of G, then $(K_1 : K_2)$ is a two-sided ideal of R.*

PROOF. (1) and (2) are proved by G. Pilz [10] and J. D. P. Meldrum [9]. Now, we prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$K_2(ra) = (K_2 r)a \subseteq K_2 a \subseteq K_1,$$

since K_2 is an R-subset of G, $K_2 r \subseteq K_2$ so that $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$k\{(a + r_1)r_2 - r_1 r_2\} = (ka + kr_1)r_2 - kr_1 r_2 \in K_1$$

for all $k \in K_2$, since $K_2 a \subseteq K_1$ and K_1 is an ideal of G. Thus $(K_1 : K_2)$ is a right ideal of R. Consequently, $(K_1 : K_2)$ is a two-sided ideal of R.□

COROLLARY 2.11 ([9], [10]). *Let R be a near-ring and G an R-group.*

(1) *For any $x \in G$, $(o : x)$ is a right ideal of R.*
(2) *For any R-subset K of G, $(o : K)$ is a two-sided ideal of R.*
(3) *For any subset K of G, $(o : K) = \bigcap_{x \in K}(o : x)$.*

REMARK 2.12. For any R-group homomorphism $f : G \longrightarrow T$, we have $(o : G) \subseteq (o : f(G))$. So every monomorphic image of a faithful R-group is also faithful. Moreover, for any R-group isomorphism $f : G \longrightarrow T$, we have $(o : G) = (o : T))$. In this case, G is faithful if and only if T is faithful.

The following statement is proved very easily, but it is important later.

LEMMA 2.13 [10]. *Let G be a faithful R-group. Then we have the following conditions:*

(1) *If $(G, +)$ is abelian, then $(R, +)$ is abelian.*
(2) *If G is distributive, then R is distributive.*

From this Lemma, we get the following Proposition:

PROPOSITION 2.14. *If G is a distributive abelian faithful R-group, then R is a ring.*

PROPOSITION 2.15. *Let R be a near-ring and G an R-group. Then we have the following conditions:*

(1) *$A(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/A(G)$-group.*
(2) *For any $x \in G$, we get $xR \cong R/(o : x)$ as R-groups.*

PROOF. (1) By Corollary 2.11 and Lemma 2.10, $A(G)$ is a two-sided ideal of R. We now make G an $R/A(G)$-group by defining, for $x \in R, A(G) + r \in R/A(G)$, the action $x(A(G) + r) = xr$. If $A(G) + r = A(G) + s$, then $r - s \in A(G)$ hence $x(r - s) = 0$ for all x in G, that is to say, $xr = xs$. This tells us that

$$x(A(G) + r) = xr = xs = x(A(G) + s)$$

Thus the action of $R/A(G)$ on G has been shown to be well defined. The verification of the structure of an $R/A(G)$-group is a routine triviality. Finally, to see that G is a faithful $R/A(G)$-group, we note that if $x(A(G) + r) = o$ for all $x \in G$, then by the definition of $R/A(G)$-group structure, we have $xr = o$. Hence $r \in A(G)$, that is, $A(G) + r = A(G)$. This says that only the zero element of $R/A(G)$ annihilates all of G. Thus G is a faithful $R/A(G)$-group.

(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-group epimorphism, so that from the isomorphism theorem for R-groups, since the kernel of ϕ is $(o : x)$, we deduce that

$$xR \cong R/(o : x)$$

as R-groups. □

PROPOSITION 2.16. *If (R, S) is a D.G. near-ring, then every monogenic R-group is a D.G. (R, S)-group.*

PROOF. Let G be a monogenic R-group with x as a generator. Then the map $\phi : r \longmapsto xr$ is an R-group epimorphism from R to G. We see that by Proposition 2.15 (2),

$$G \cong R/A(x),$$

where $A(x) = (0 : x) = Ker\phi$. From the Lemma 2.9, we obtain that G is a D.G. (R, S)-group. □

LEMMA 2.17. *Let G be an R-group. Then G is faithful if and only if for each $x \in G$, $R \cong xR$.*

PROOF. Suppose G is a faithful R-group. Then we can easily define the map $f : a \longmapsto xa$ which is an R-group epimorphism from R to xR as R-groups for each $x \in G$.

To show that f is one-one, if $f(a) = f(b)$ for $a,\ b \in R$, then $xa = xb$, that is, $x(a-b) = o$ for all $x \in G$. This implies that $a-b \in \bigcap_{x\in G}(o : x)$, which is equal to $(o : G) = A(G)$ from Proposition 2.11 (3). Since G is faithful, $a - b = 0$. Hence for all $x \in G$, $R \cong xR$.

Conversely, assume the condition that $R \cong xR$ for all $x \in G$. Consider the map $f : R \longrightarrow xR$ given by $a \longmapsto xa$ is an R-group isomorphism. To show that G is faithful, take any element $a \in A(G)$, that is, $Ga = o$. This implies that for all $x \in G$, $xa = o$, that is, $f(a) = o$. Since f is an R-group isomorphism, $a = 0$.

Consequently, G is faithful. □

The following statement is a generalization of the Proposition 2.14.

PROPOSITION 2.18 [3]. *Let (R, S) be a D.G. near-ring. If G is an abelian faithful D.G. (R, S)-group, then R is a ring.*

The following proposition is useful in chapter 3 to make R-group of type 2 which is the analogous notion of simplicity, and primitive near-ring of type 2 which is the similar notion of primitivity in ring and module theory.

PROPOSITION 2.19. *Let G be a monogenic R-group with generator x. Then we have the following properties:*

(1) *For any right ideal I of R, xI is an R-ideal of G.*
(2) *If I is a left R-subgroup of R and xI is an R-ideal of G, then $(xI : x)$ is an ideal of R.*
(3) *If e is a right identity of R and if G is a faithful R-group, then e is a two-sided identity of R.*
(4) *If G is R_0-simple, then either $GR = o$ or G is strongly monogenic.*

PROOF. (1) Let $a \in G$. Then there exists $t \in R$ such that $a = xt$. Thus for each $xy \in xI, r \in R$, and $a \in G$,

$$\begin{aligned}(a + xy)r - ar &= (xt + xy)r - (xt)r = x(t + y)r - x(tr)\\ &= x\{(t + y)r - tr\} \in xI.\end{aligned}$$

In this same method, it is easily shown that xI is an additive normal subgroup of G. Therefore xI is an R-ideal of G.

(2) For any $y \in I$ and $a,\ b \in R$, we obtain the following equality:

$$x\{(y + a)b - ab\} = x(y + a)b - x(ab) = (xy + xa)b - (xa)b) \in xI.$$

Hence $(y+a)b-ab \in (xI : x)$. In this way, we can show that $(xI : x)$ is an additive normal subgroup of R. Consequently, $(xI : x)$ is an ideal of R.

(3) First, let e is a right identity of R and $g = xt$ be any element in G. Then we have the relation that

$$ge = (xt)e = x(te) = xt = g$$

Next, let r be any element of R and g be an arbitrary element in G. Then one gets the following equality that

$$g(er-r) = g(er) + g(-r) = (ge)r - gr = gr - gr = o$$

Thus $(er-r) \in (o : G) = A(G)$. Since G is faithful, above this equality implies that $er - r = 0$, that is, $er = r$. Hence e is a two-sided identity of R.

(4) Suppose that G is R_0-simple and $G = GR \neq o$. Then G has no R-subgroups except $\Omega = o$ and G. Let $x \in G$ and $xR \neq o$. Then since xR is an R-subgroup, moreover an R_0-subgroup by Lemma 2.4 (2) of G, $G = xR$. Hence G is strongly monogenic. □

3. Some properties on types of primitive near-rings

In chapter 2, we studied some properties of monogenic R-groups, faithful R-groups and introduced simplicity and R-simplicity concepts of R-groups.

From now on, we shall introduce several types of R-groups and several types of near-rings. Also, we will investigate some properties of primitivity types of near-rings and their characterizations.

Since 1963, Betsch [2] introduced three primitivity types of near-rings. However, the research has been concentrated mostly to the zero symmetric near-rings. As we showed in introduction, in general, near-rings appear as certain sets of group transformations and zero symmetric near-rings correspond to sets of zero preserving transformations. There are lots of interesting transformation near-rings whose elements do not preserve zero in general, for example, near-rings of affine functions on modules, near-rings of polynomial functions on groups or rings, near-rings of continuous functions on topological groups etc.

Let G be a nonzero R-group. Then there are 3 types of R-groups:

(1) G is said to be of *type 0* if it is simple and monogenic;

(2) G is said to be of *type 1* if it is simple and strongly monogenic;
(3) G is said to be of *type 2* if it is R_0-simple.

In this definitions, the notion of (3) is a generalization of type 2 R-group in G. Pilz [10] and K. Kaarli [4]. In ring and module theory, the R-group of type 2 is the same concept of an irreducible ring module.

The previous Lemma 2.1 and Proposition 2.5 implies the following proposition:

PROPOSITION 3.1. *Let G be an R-group. Then we get the following conditions:*

(1) *G is of type 2 implies G is of type 1 implies G is of type 0.*
(2) *If G is of type 1 or G type 2, then $\Omega = 0$ or $\Omega = G$.*
(3) *If G is unitary R_0-group, then G is of type 1 if and only if G is of type 2.*

For example, if $G = Z_4 = \{0,1,2,3\}$, $R_1 := \{f \in M_0(G) \mid 2f = 0 \text{ } or \text{ } 2\}$, $R_2 := \{f \in M_0(G) \mid 2f = 0\}$ and $R_3 := \{f \in M_0(G) \mid 3f = 0\}$, then we get the following:

(1) R_1-group G is of type 0 but not of type 1;
(2) R_2-group G is of type 1 but not of type 2;
(3) R_3-group G is of type 2.

Now we will consider changes of near-rings, for example, an R-group G_R change into $G_{R/I}$ for some ideal I of R, G_{R_0} and G_{R_c}. This changes will be an important tool in later consideration.

LEMMA 3.2 [10]. *Let G be an additive group, I be an ideal of a near-ring R and $t \in \{0,1,2\}$.*

(1) *If G is an R-group with $I \subseteq (o : G)$, then for all $x \in G$ and all $a \in R$, $x(I+a) = xa$ makes G into an R/I-group. Moreover, we get the following two conditions: If G_R is of type t, then $G_{R/I}$ is of type t. If G_R is faithful, then $G_{R/I}$ is also faithful.*
(2) *If G is R/I-group, then for all $x \in G$ and all $a \in R$, $xa = x(I+a)$ makes G into an R-group in a natural way.*

Observe that $(R/I)_0 = \{I + a \mid a \in R_0\} = R_0/I$.

The relation between G_R and G_{R_0} is particularly important as following.

PROPOSITION 3.3. *Let G be an R-group. Then we get the followings:*

(1) *G_R is simple if and only if G_{R_0} is simple.*

(2) *If G_{R_0} is monogenic by x, then G_R is monogenic by x.*
(3) *If G_{R_0} is strongly monogenic, then G_R is strongly monogenic or $o \neq \Omega \neq G$.*
(4) *If G is R_0-simple, then G is R-simple.*

PROOF. (1) From the Lemma 2.4 (1), since any R-ideal of G is an R_0-ideal of G, and also the converse is hold, this condition (1) is true.

(2) Suppose that G_{R_0} is monogenic by x. Then $G = xR_0 \subseteq xR \subseteq G$. Hence $G = xR$ and G_R is monogenic by x.

(3) Suppose that G_{R_0} is strongly monogenic. Then by the condition (2), G_R is monogenic. Let $x \in G$ and assume that $o \neq \Omega \neq G$ is not true, that is, $o = \Omega$ or $\Omega = G$.

i) In case $o = \Omega$, we see that

$$xR = x(R_0 + R_c) = xR_0 + xR_c = xR_0 + \Omega = xR_0$$

So this is o or G, because of G_{R_0} is strongly monogenic.

ii) In case $\Omega = G$, we get $xR = x(R_0 + R_c) = xR_0 + xR_c = xR_0 + \Omega = G$.

Consequently, from i) and ii), G_R is strongly monogenic.

(4) This condition is clear by the Lemma 2.4 (2). □

We remark that the converse of Proposition 3.3 (4) is not true: For example, let $R = M_c(Z_4) = \{0_c, 1_c, 2_c, 3_c\}$, where the lower subscript letter c of n_c means constant function into n. Then $R_0 = \{0_c\}$, and R-group Z_4 is R-simple but not R_0-simple, since $\{0, 2\}$ is R_0-subgroup of Z_4.

PROPOSITION 3.4. *Let G be an R-group and let $t \in \{0, 1, 2\}$.*
(1) *If G_R is of type t, then G_{R_0} is of type t or $GR_0 = o$.*
(2) *If G_{R_0} is of type t (for t=1 assume that $\Omega = o$ or $\Omega = G$ in G_R), then G_R is of type t.*

PROOF. (1) Already we know that G_R is simple if and only if G_{R_0} is simple by Proposition 3.3 (1). Let G_R is monogenic by x. At first, we see that R_0 is a right ideal of R, indeed for any $a \in R_0$, any $r,\ s \in R$, since $o\{(a + r)s - rs\} = (oa + or)s - ors = ors - ors = o$, $(a + r)s - rs \in R_0$. By Proposition 2.19 (1), For this right ideal R_0 of R, xR_0 is an R-ideal of G_R. Since G_R is simple, $xR_0 = o$ or $xR_0 = G$.

i) In case $xR_0 = G$, obviously, G_{R_0} is monogenic by x.

ii) In case $xR_0 = o$, then $G = xR = x(R_0 + R_c) = xR_0 + xR_c = o + \Omega = \Omega$. This implies that for any $y \in G$, $yR = G$. Again by Proposition 2.19

(1), for any $y \in G$, $yR_0 = o$ or $yR_0 = G$. Thus in either case, G_{R_0} is monogenic or $GR_0 = o$. Hence G_R is of type 0 implies that G_{R_0} is of type 0 or $GR_0 = o$.

Next, If G_R is of type 1, then G_R is strongly monogenic, so $\Omega = o$ or $\Omega = G$.

In case $\Omega = o$, then since for each $x \in G$,

$$xR = x(R_0 + R_c) = xR_0 + xR_c = xR_0 + \Omega = xR_0$$

G_{R_0} is again of type 1.

In case $\Omega = G$, then each $x \in G$ generates G_R by Lemma 2.3 (1) and (2). So again by Proposition 2.19 (1), for any $x \in G$, $xR_0 = o$ or $xR_0 = G$.

Consequently, in either case, G_{R_0} is either of type 1 or $GR_0 = o$. The assertion for $t = 2$ is trivial since every R_0-subgroup of G_{R_0} is R_0-subgroup of G_R.

(2) This condition is proved by using the Proposition 3.3. □

Any right ideal I of a near-ring R is called *modular* if there exists $e \in R$ such that $a - ea \in R$ for all $a \in R$. In this case, we also say that I is "modular by e" and that e is a left identity modulo I, since for each $a \in R$, $ea \equiv a(mod\ I)$.

In all what follows, t will be any number $\in \{0, 1, 2\}$ unless otherwise specified. A right ideal I of R is called *t-modular* if I is modular and R/I is an R-group (via $(I+a)r = I+ar$ for any $I+a \in R/I$ and $r \in R$) of type t.

LEMMA 3.5. *For any right ideal I of a near-ring R, the following statements are equivalent:*

(1) *I is modular.*
(2) *There exists a monogenic R-group G with generator x such that $I = (o : x)$.*

PROOF. (1) $\Longrightarrow$ (2) Suppose I is modular by e. Then $G := R/I$ is monogenic by $I + e =: x$. Indeed, since I is modular by e, $a - ea \in I$, that is, $I + a = I + ea$, we deduce that

$$(I + e)R = \{(I + e)a \mid a \in R\} = \{I + a \mid a \in R\} = R/I$$

Moreover,$a \in (o : x) = (o : I + e)$ if and only if $I + a = I + ea = I + o$ if and only if $a \in I$. Thus $I = (o : x)$.

(2) $\Longrightarrow$ (1) Suppose that G is a monogenic R-group with generator x such that $I = (o : x)$. Then there exists an element $e \in R$ such that $x = xe$. On the other hand, for any $a \in R$, $xa = xea$, this implies that $x(a - ea) = o$, that is, $(a - ea) \in (o : x) = I$. Therefore I is modular by e. $\square$

Applying Proposition 2.15 (2) and Lemma 3.5, we get the following first statement:

PROPOSITION 3.6. *Let R be a near-ring and I be a right ideal of R.*

(1) *If I is modular, then $(I : R) \subseteq I$.*
(2) *If I is modular by e, then $(I : R) = (I : eR)$ which is the largest ideal of R contained in I.*

PROOF. (1) By Lemma 3.5, we take a monogenic R-group $G := R/I$ with generator x such that $I = (o : x)$. Then $(I : R) = (o : R/I) = (o : G) \subseteq (o : x) = I$.

(2) First, from $eR \subseteq R$ we have $(I : R) \subseteq (I : eR)$. For converse inclusion, let $r \in (I : eR)$. Then for any $a \in R$, $ear \subseteq I$. On the other hand, since I is modular by e, $ar - ear$ I for $ar \in R$. Hence for all $a \in R$, $ar \in I$, that is, $Rr \subseteq I$. Thus $r \in (I : R)$. So we proved that $(I : R) = (I : eR)$.

Next, from the Lemma 2.10 (1) and (3), $(I : R)$ is a two-sided ideal of R, and from the condition (1), $(I : R) \subseteq I$. If J is a two-sided ideal of R with $J \subseteq I$, then since J is a left ideal of R,

$$RJ \subseteq J \subseteq I$$

This implies that $J \subseteq (I : R)$. Therefore $(I : R) = (I : eR)$ is the largest ideal of R contained in I. $\square$

DEFINITION 3.7. Let R be a near-ring and I be an ideal of R.

(1) R is called *t-primitive on G_R* if G is faithful and of type t;
(2) R is called *t-primitive* if there exists an R-group G such that R is t-primitive on G_R;
(3) I is called *t-primitive* if R/I is a t-primitive near-ring.

PROPOSITION 3.8. *Let I be an ideal of R. Then the following statements are equivalent:*

(1) *I is t-primitive.*

(2) *There exists an R/I-group G such that G is faithful and of type t.*
(3) *There exists an R-group G such that $I = (o : G)$ and G is of type t.*
(4) *There exists a right ideal J of R such that $I = (J : R)$ and J is t-modular.*

PROOF. (1) $\iff$ (2) By the definition of t-primitive ideal, I is t-primitive if and only if R/I is a t-primitive near-ring if and only if there exists an R/I-group G such that R/I is t-primitive on $G_{R/I}$ if and only if there exists an R/I-group G such that G is faithful and of type t.

(2) $\iff$ (3) We can prove this condition by defining $x(I + a) = xa$ and the fact G is faithful if and only if $I = (o : G)$ using the Lemma 3.2 (1) and (2).

(3) $\Longrightarrow$ (4) Consider a nonzero monogenic R-group $G = xR$ and $J = (o : x)$. Then by Corollary 2.11 (1), J is a right ideal of R, moreover this J is modular by Lemma 3.5. Again by Proposition 2.15 (2),

$$R/J = R/(o : x) \cong xR = G$$

Hence J is t-modular. Finally, $I = (o : G) = (o : R/J) = (J : R)$.

(4) $\Longrightarrow$ (3) Assume the condition (4). If we take $R/J =: G$, since J is t-primitive, J is modular and G is an R-group of type t and as above $I = (J : R) = (o : R/J) = (o : G)$. □

COROLLARY 3.9. *For any near-ring R, the following statements are equivalent:*

(1) *R is t-primitive.*
(2) *$\{o\}$ is a t-primitive ideal of R.*
(3) *There exists an R-group G such that $o = (o : G)$ and G is of type t.*
(4) *There exists a right ideal J of R such that $o = (J : R)$ and J is t-modular.*

PROPOSITION 3.10. *Let G be an R-group. If R is simple and G is of type t, then R is t-primitive on G.*

PROOF. The Corollary 2.11 (2) implies that $(o : G)$ is an ideal of R. Since R is simple, so $(o : G) = 0$ or $(o : G) = R$.

i) In case $(o : G) = 0$, then G is faithful. So we are done.

ii) The case $(o : G) = R$ does not arise, for if the case $(o : G) = R$ arise, then $GR = o$. Since G is of type t, $G \neq o$ and monogenic with some generator $x \in G$, thus $o \neq G = xR \subsetneq GR = o$. This is a contradiction. Consequently, R is t-primitive on G. □

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