Journal of the Korean Statistical Society (2004), **33**: *3, pp 323–337*

ESTIMATING VARIOUS MEASURES IN NORMAL POPULATION THROUGH A SINGLE CLASS OF ESTIMATORS

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ABSTRACT

This article coined a general class of estimators for various measures in normal population when some '*a priori*' or guessed value of standard deviation σ is available in addition to sample information. The class of estimators is primarily defined for a function of standard deviation. An unbiased estimator and the minimum mean squared error estimator are worked out and the suggested class of estimators is compared with these classical estimators. Numerical computations in terms of percent relative efficiency and absolute relative bias established the merits of the proposed class of estimators especially for small samples. Simulation study confirms the excellence of the proposed class of estimators. The beauty of this article lies in estimation of various measures like standard deviation, variance, Fisher information, precision of sample mean, process capability index C_p, fourth moment about mean, mean deviation about mean *etc.* as particular cases of the proposed class of estimators.

AMS 2000 subject classifications. Primary 62F10.
Keywords. Normal distribution, guessed value, mean squared error, absolute relative bias, percent relative efficiency.

1. INTRODUCTION

The normal distribution is probably the most important distribution in both the theory and application of statistics. We are always in a need to estimate various measures frequently encountered in practice like standard deviation, variance,

Received February 2004; accepted June 2004.
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Fisher information, precision of sample mean, process capability index C_p, fourth moment about mean, mean deviation about mean and many more. It is usual practice to estimate each of them separately. Can we think about a general estimation procedure that provides estimators for all these different measures? The answer is yes indeed. All the measures cited above are essentially the functions of standard deviation of the form $\kappa\sigma^r$; κ is any real number and r is a non-zero finite integer. Thus here we shall confine ourselves to estimate $\kappa\sigma^r = G$ (say). By putting appropriate values of κ and r one can obtain estimators for various measures. The approach is partially in accordance with Pandey and Singh (1978), Singh and Shukla (1999), and Saxena (2002) where they considered the problem of estimating the k^{th} exponent of scale parameter of exponential population.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a normal population $N(\mu, \sigma^2)$, the probability density function of which is given by

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}; -\infty < x < \infty, \quad -\infty < \mu < \infty, \ \sigma > 0 \quad (1.1)$$

where μ and σ^2 are the parameters known as population mean and population variance respectively.

2. Classical Estimation

An unbiased estimator of G is defined as

$$\widehat{G}_U = \kappa C_{(n,r)} s^r \quad (2.1)$$

where

$$C_{(n,r)} = \left(\frac{n-1}{2}\right)^{r/2} \frac{\Gamma((n-1)/2)}{\Gamma((n+r-1)/2)} \quad \text{and} \quad s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

The variance of $\widehat{G}_U$ is given by

$$\mathrm{Var}(\widehat{G}_U) = \kappa^2\sigma^{2r}(A_{(n,r)} - 1) \quad (2.2)$$

where

$$A_{(n,r)} = \frac{\Gamma((n-1)/2)\ \Gamma((n+2r-1)/2)}{\Gamma^2((n+r-1)/2)}.$$

The minimum mean squared error (MMSE) estimator of G is derived as

$$\widehat{G}_M = \kappa \left(\frac{n-1}{2}\right)^{r/2} \frac{\Gamma((n+r-1)/2)}{\Gamma((n+2r-1)/2)} s^r \tag{2.3}$$

having

$$\text{Bias}(\widehat{G}_M) = \kappa\sigma^r \left(A^{-1}_{(n,r)} - 1\right) \tag{2.4}$$

and

$$\text{MSE}(\widehat{G}_M) = \kappa^2\sigma^{2r} \left(1 - A^{-1}_{(n,r)}\right). \tag{2.5}$$

In addition to sample information, one kind of information is typically relevant. This non-sample information that is useful to consider is called prior information. This is information about the parameter under investigation arising from sources other than the statistical investigation and thus should also be incorporated in addition to random sample in any estimation procedure. Owing to considerable handling of the parameter in the past, one may have some prior information in the form of a point guessed value. Many authors including Thompson (1968), Mehta and Srinivasan (1971), Pandey and Singh (1977), Pandey (1979), Singh and Singh (1997), Singh *et al.* (1999), Saxena (2002), and Singh and Saxena (2003) have reported estimators of normal parameters by making use of prior point estimate. The next section provides a class of estimators for G when *a priori* or guessed value, say σ_0, of standard deviation σ is available. The properties of the proposed class of estimators are further discussed theoretically and empirically.

3. ESTIMATION WITH PRIOR INFORMATION

We propose a class of estimators $\tilde{G}_P$ for G in model (1.1), defined as

$$\tilde{G}_P = \kappa\sigma_0^r \left\{ v + W \left(\frac{s^r}{\sigma_0^r}\right)^u \right\} \tag{3.1}$$

where u and v are real numbers such that $u \neq 0$ and $v > 0$, W is a stochastic variable which may in particular be a scalar to be chosen such that the MSE of $\tilde{G}_P$ is minimum.

Assuming W as a scalar and using the result:

$$E(s^\alpha) = \left(\frac{2}{n-1}\right)^{\alpha/2} \frac{\Gamma((n+\alpha-1)/2)}{\Gamma((n-1)/2)} \sigma^\alpha,$$

the MSE of $\tilde{G}_P$ is given by

$$\begin{aligned} &\mathrm{MSE}(\tilde{G}_P) \\ &= \kappa^2\sigma^{2r}\left\{(v\Delta-1)^2 + W^2\Delta^{2(1-u)}\left(\frac{2}{n-1}\right)^{ru}\frac{\Gamma((n+2ru-1)/2)}{\Gamma((n-1)/2)}\right. \\ &\quad \left. +2\,(v\Delta-1)W\Delta^{(1-u)}\left(\frac{2}{n-1}\right)^{ru/2}\frac{\Gamma((n+ru-1)/2)}{\Gamma((n-1)/2)}\right\} \end{aligned} \tag{3.2}$$

where $\Delta = \sigma_0^r/\sigma^r$. Now minimizing (3.2) with respect to W and replacing σ^r by its unbiased estimator $\widehat{\sigma}_U^r = C_{(n,r)}s^r$, we get

$$\widehat{W} = \frac{-\left(v\sigma_0^r - C_{(n,r)}s^r\right)}{\sigma_0^{r(1-u)}\left(C_{(n,r)}s^r\right)^u}W_{(n,r,u)} \tag{3.3}$$

where

$$W_{(n,r,u)} = \left(\frac{n-1}{2}\right)^{ru/2}\frac{\Gamma((n+ru-1)/2)}{\Gamma((n+2ru-1)/2)}.$$

Substitution of (3.3) in (3.1) yields a class of estimators for $G = \kappa\sigma^r$ in a more feasible form as

$$\widehat{G}_P = \kappa v\sigma_0^r\left(1 - W'_{(n,r,u)}\right) + \kappa C_{(n,r)}s^r W'_{(n,r,u)} \tag{3.4}$$

where

$$W'_{(n,r,u)} = \frac{\Gamma((n+ru-1)/2)/\Gamma((n+2ru-1)/2)}{\{\Gamma((n-1)/2)/\Gamma((n+r-1)/2)\}^u}. \tag{3.5}$$

It is to be noted that if one changes the values of κ and r, the proposed class of estimators in (3.4) reduces to a class of estimators of a particular parametric function. Some of these are mentioned below:

Parametric Functions	*Values of* (κ, r)
Fisher Information ($1/\sigma^2$)	(1,−2)
Precision of Sample Mean (n/σ^2)	(n,−2)
Process Capability Index (C_p)	((USL−LSL)/6,−1)
Standard Deviation (σ)	(1,1)
Mean Devation about Mean ($\sigma\sqrt{2/\pi}$)	($\sqrt{2/\pi}$, 1)
Variance (σ^2)	(1,2)
Fourth Moment about Mean ($3\sigma^4$)	(3,4)

It is apparent that (3.4) becomes the convex combination of $\kappa v\sigma_0^r$ and $\kappa C_{(n,r)}s^r$ if $W'_{(n,r,u)} > 0$ and $1 - W'_{(n,r,u)} > 0$. Since we are dealing with the problem of estimating G which cannot be negative unless κ is negative, obviously it is necessary that $\widehat{G}_P > 0$. Thus, subject to the condition that κ is positive and irrespective of the values of v, s^r, $C_{(n,r)}$ and σ_0^r this immediately leads to impose a constraint:

$$0 < W'_{(n,r,u)} < 1. \tag{3.6}$$

Therefore, accepatable range of values of u for all n and r is given by

$$\{u \mid 0 < W'_{(n,r,u)} < 1\}.$$

If $W'_{(n,r,u)} = 1$, the proposed class of estimators turns into the unbiased estimator $\widehat{G}_U$, otherwise it is biased with

$$\text{Bias}(\widehat{G}_P) = \kappa\sigma^r(v\Delta - 1)\left(1 - W'_{(n,r,u)}\right). \tag{3.7}$$

The mean squared error of $\widehat{G}_P$ is given by

$$\text{MSE}(\widehat{G}_P) = \kappa^2\sigma^{2r}\left\{(v\Delta - 1)^2\left(1 - W'_{(n,r,u)}\right)^2 + W'^2_{(n,r,u)}\left(A_{(n,r)} - 1\right)\right\}. \tag{3.8}$$

It is quite evident in expressions (3.7) and (3.8) that if $\Delta = v^{-1}$, the proposed class of estimators $\widehat{G}_P$ becomes unbiased and possesses minimum MSE, which is given by

$$\min \text{MSE}(\widehat{G}_P) = \kappa^2\sigma^{2r}W'^2_{(n,r,u)}\left(A_{(n,r)} - 1\right). \tag{3.9}$$

The quantity $\Delta = \sigma_0^r/\sigma^r$ represents the departure of natural origin σ_0^r from the true value σ^r. But in practical situations it is hardly possible to get an idea about Δ. Consequently, a consistent estimator of Δ is proposed as a guideline to know in practice whether Δ is within its acceptable range of dominance or not. It is defined as

$$\widehat{\Delta} = \left(\frac{n-1}{2}\right)^{r/2}\frac{\Gamma((n-1)/2)}{\Gamma((n+r-1)/2)}\left(\frac{\sigma_0}{s}\right)^r. \tag{3.10}$$

As it is already being pointed out that for better results v should lies in the close proximity of Δ. Thus the above estimate of Δ in (3.10) may help us in choosing v as $v = \widehat{\Delta}^{-1}$.

4. COMPARISON OF ESTIMATORS

Entire exercise in this paper consists in formulating efficient estimators. James and Stein (1961) promulgated that minimum MSE is a highly enviable characteristic and it is therefore used as a criterion to compare different estimators with each other. The conditions under which the proposed class of estimators is more efficient than the conventional estimators are discussed below.

MSE($\widehat{G}_P$) does not exceed Var($\widehat{G}_U$), if

$$(1-\sqrt{\xi})v^{-1} < \Delta < (1+\sqrt{\xi})v^{-1} \tag{4.1}$$

where

$$\xi = \left(A_{(n,r)} - 1\right)\left(\frac{1+W'_{(n,r,u)}}{1-W'_{(n,r,u)}}\right).$$

In a similar fashion, $\widehat{G}_P$ has smaller MSE than $\widehat{G}_M$, if

$$(1-\sqrt{\psi})v^{-1} < \Delta < (1+\sqrt{\psi})v^{-1} \tag{4.2}$$

where

$$\psi = \frac{\left(A_{(n,r)} - 1\right)\left(A^{-1}_{(n,r)} - W'^2_{(n,r,u)}\right)}{\left(1-W'_{(n,r,u)}\right)^2}.$$

Besides minimum MSE criterion, minimum bias is also important and therefore should be considered under study. Thus, ARB of $\widehat{G}_P$ is less than ARB of $\widehat{G}_M$, if

$$\left(1+\frac{A^{-1}_{(n,r)}-1}{1-W'_{(n,r,u)}}\right)v^{-1} < \Delta < \left(1-\frac{A^{-1}_{(n,r)}-1}{1-W'_{(n,r,u)}}\right)v^{-1}. \tag{4.3}$$

5. Empirical Study and Discussion

An exact analytical study of the performance of the proposed class of estimators is not possible because the expression of the estimator and its MSE appear to be too complex to obtain in nice compact forms. Therefore, we are left with no other better choice than empirical study. We have computed the Percent Relative Efficiency (PRE) of proposed class of estimators $\widehat{G}_P$ with respect to the MMSE estimator $\widehat{G}_M$ as

$$\text{PRE}(\widehat{G}_P, \widehat{G}_M) = \frac{\left(1-A^{-1}_{(n,r)}\right)\times 100}{(v\Delta-1)^2\left(1-W'_{(n,r,u)}\right)^2 + W'^2_{(n,r,u)}\left(A_{(n,r)}-1\right)}. \tag{5.1}$$

It should be noted that the above formula is independent of κ and therefore to see the performance of the suggested class of estimators in estimating different parametric functions the value of r is relevant no matter what is the value of κ.

Various combinations of quantities involved in $\widehat{G}_P$ have been examined and the proposed class of estimators appears to give better results than $\widehat{G}_M$ for a number of combinations. An extort from an extensive computer printout is presented in Tables 5.1 to 5.5 where the value of r is considered as -2, -1, 1, 2 and 4 depending upon the problem of estimation of parametric function. Several values of $n = 5(5)25$, $\Delta = 0.5(0.5)4.0$, $u = -2(1)2$ and $v = 0.5$ have been considered in each of these tables. The range of dominance of Δ is also computed for each set of (u, v).

It has been observed from Tables 5.1 to 5.5 that the gain in relative efficiency

(i) decreases as sample size n increases,

(ii) increases up to $\Delta = v^{-1}$, attains its maximum at this point and then decreases symmetrically in magnitude as Δ increases,

subject to the condition that other quantities are fixed.

The absolute relative bias (ARB) of $\widehat{G}_P$ and $\widehat{G}_M$ have also been compared with the help of a computer program. To meet the constraint of brevity we have not presented the results. However, it is being perceived that for fixed n, u, v, r the ARBs of the proposed class of estimators decreases up to $\Delta = v^{-1}$, the estimator becomes unbiased at this point and then ARBs increases symmetrically in magnitude as Δ increases in its range of dominance.

6. SIMULATION

In previous section we have discussed empirical findings about the performance of the suggested class of estimators. However, despite of the fact that empirical study can, and does, help to analyze a wide variety of problems, there are many situations that are too complex to be handled this way. For instance, in the present case the value of Δ is not known and the entire empirical study is more or less based on it. In these circumstances, it is often possible to simulate the given system and study its behaviour. Thus a random sample of 30 observations is generated from a normal population (1.1) with $\mu = 20$ and $\sigma = 5$ (see, Sinha and Kale, 1980).

If a prior point estimate of standard deviation σ is available from some similar study in the past as 5.5, $\widehat{\Delta}$ can be calculated for different values of r. Table 6.1 summarizes the findings along with the true values of Δ.

In practice the true value of Δ is not known, therefore for choosing v we used the criterion $v = \widehat{\Delta}^{-1}$ as $\widehat{\Delta}$ can be easily evaluated during experimentation. The

TABLE 5.1 *PRE of* $\widehat{G}_P$ *with respect to* $\widehat{G}_M$ *for* $r = -2$
(applicable for estimation of Fisher information and Precision of sample mean)

v	u	$\Delta \downarrow$ $n \rightarrow$	5	10	15	20	25
0.5	−2	0.50	115.76	83.67	71.07	68.33	74.06
		1.00	258.16	172.36	136.61	123.39	120.84
		1.50	985.88	473.41	305.81	238.84	194.58
		2.00	16326.52	1133.17	520.83	347.10	244.27
		2.50	985.88	473.41	305.81	238.84	194.58
		3.00	258.16	172.36	136.61	123.39	120.84
		3.50	115.76	83.67	71.07	68.33	74.06
		4.00	65.32	48.64	42.52	42.06	48.03
		Range of Δ	(0.39, 3.61)	(0.64, 3.36)	(0.78, 3.22)	(0.83, 3.17)	(0.81, 3.19)
	−1	0.50	143.96	120.87	112.87	109.15	107.07
		1.00	217.41	146.58	130.08	121.98	116.56
		1.50	313.33	168.01	143.18	131.24	123.10
		2.00	367.35	176.63	148.15	134.65	125.45
		2.50	313.33	168.01	143.18	131.24	123.10
		3.00	217.41	146.58	130.08	121.98	116.56
		3.50	143.96	120.87	112.87	109.15	107.07
		4.00	97.73	97.04	95.24	95.14	96.11
		Range of Δ	(0.03, 3.97)	(0.07, 3.93)	(0.14, 3.86)	(0.17, 3.83)	(0.17, 3.83)
	1	0.50	134.94	114.24	107.87	105.43	104.14
		1.00	179.82	127.18	114.29	109.68	107.32
		1.50	224.64	136.45	118.52	112.40	109.32
		2.00	245.00	139.85	120.00	113.33	110.00
		2.50	224.64	136.45	118.52	112.40	109.32
		3.00	179.82	127.18	114.29	109.68	107.32
		3.50	134.94	114.24	107.87	105.43	104.14
		4.00	100.00	100.00	100.00	100.00	100.00
		Range of Δ	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)
	2	0.50	117.57	59.07	61.22	66.77	71.22
		1.00	261.29	132.11	125.00	121.98	118.98
		1.50	980.27	511.97	333.33	242.10	199.11
		2.00	11838.44	12335.65	750.00	360.39	256.75
		2.50	980.27	511.97	333.33	242.10	199.11
		3.00	261.29	132.11	125.00	121.98	118.98
		3.50	117.57	59.07	61.22	66.77	71.22
		4.00	66.42	33.29	35.71	40.87	45.59
		Range of Δ	(0.37, 3.63)	(0.85, 3.15)	(0.86, 3.14)	(0.85, 3.15)	(0.84, 3.16)

TABLE 5.2 *PRE of $\widehat{G}_P$ with respect to $\widehat{G}_M$ for $r = -1$ (applicable for estimation of Process capability index C_p)*

v	u	$\Delta \downarrow\ n \rightarrow$	5	10	15	20	25
0.5	−2	0.50	86.01	75.68	77.30	79.81	83.72
		1.00	156.68	115.50	107.64	104.76	103.97
		1.50	309.06	168.78	140.80	128.96	121.62
		2.00	457.31	199.45	156.91	139.72	128.92
		2.50	309.06	168.78	140.80	128.96	121.62
		3.00	156.68	115.50	107.64	104.76	103.97
		3.50	86.01	75.68	77.30	79.81	83.72
		4.00	52.72	51.04	55.43	59.85	65.78
		Range of Δ	(0.64, 3.36)	(0.83, 3.17)	(0.88, 3.12)	(0.91, 3.09)	(0.90, 3.10)
	−1	0.50	114.17	105.33	103.17	102.24	101.74
		1.00	129.18	111.94	107.51	105.46	104.21
		1.50	140.24	116.33	110.29	107.50	105.75
		2.00	144.36	117.86	111.25	108.19	106.27
		2.50	140.24	116.33	110.29	107.50	105.75
		3.00	129.18	111.94	107.51	105.46	104.21
		3.50	114.17	105.33	103.17	102.24	101.74
		4.00	98.19	97.28	97.65	98.05	98.47
		Range of Δ	(0.06, 3.91)	(0.16, 3.84)	(0.20, 3.80)	(0.22, 3.78)	(0.22, 3.78)
	1	0.50	109.70	103.10	101.82	101.28	100.99
		1.00	117.86	105.44	103.15	102.22	101.71
		1.50	123.36	106.90	103.97	102.79	102.15
		2.00	125.32	107.39	104.25	102.98	102.29
		2.50	123.36	106.90	103.97	102.79	102.15
		3.00	117.86	105.44	103.15	102.22	101.71
		3.50	109.70	103.10	101.82	101.28	100.99
		4.00	100.00	100.00	100.00	100.00	100.00
		Range of Δ	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)
	2	0.50	71.48	72.00	81.86	86.64	89.39
		1.00	143.76	113.41	109.53	107.18	105.71
		1.50	365.49	173.15	137.39	124.96	118.70
		2.00	752.21	210.03	150.12	132.28	123.77
		2.50	365.49	173.15	137.39	124.96	118.70
		3.00	143.76	113.41	109.53	107.18	105.71
		3.50	71.48	72.00	81.86	86.64	89.39
		4.00	41.95	47.65	60.48	68.30	73.51 2
		Range of Δ	(0.76, 3.24)	(0.86, 3.14)	(0.84, 3.16)	(0.83, 3.17)	(0.82, 3.18)

TABLE 5.3 *PRE of* $\widehat{G}_P$ *with respect to* $\widehat{G}_M$ *for* $r = 1$
(applicable for estimation of Standard deviation and Mean deviation about mean)

v	u	$\Delta \downarrow$ $n \rightarrow$	5	10	15	20	25
		0.50	34.71	46.76	60.58	68.85	73.50
		1.00	76.09	86.71	95.31	98.02	98.47
		1.50	267.34	177.89	145.31	131.42	123.68
		2.00	1648.33	273.88	176.10	148.27	135.22
	−2	2.50	267.34	177.89	145.31	131.42	123.68
		3.00	76.09	86.71	95.31	98.02	98.47
		3.50	34.71	46.76	60.58	68.85	73.50
		4.00	19.71	28.43	40.11	48.60	54.25
		Range of Δ	(1.13, 2.87)	(1.10, 2.90)	(1.05, 2.95)	(1.03, 2.97)	(1.03, 2.97)
		0.50	97.60	102.84	102.25	101.77	101.36
		1.00	135.09	113.43	108.08	105.77	104.44
		1.50	175.57	120.90	111.91	108.32	106.39
		2.00	195.05	123.61	113.25	109.20	107.05
	−1	2.50	175.57	120.90	111.91	108.32	106.39
		3.00	135.09	113.43	108.08	105.77	104.44
		3.50	97.60	102.84	102.25	101.77	101.36
		4.00	70.28	90.95	95.07	96.65	97.33
		Range of Δ	(0.54, 3.46)	(0.38, 3.62)	(0.34, 3.66)	(0.32, 3.68)	(0.32, 3.68)
0.5		0.50	105.44	102.42	101.56	101.15	100.88
		1.00	109.69	104.21	102.70	101.99	101.53
		1.50	112.42	105.32	103.40	102.49	101.91
		2.00	113.36	105.70	103.63	102.66	102.04
	1	2.50	112.42	105.32	103.40	102.49	101.91
		3.00	109.69	104.21	102.70	101.99	101.53
		3.50	105.44	102.42	101.56	101.15	100.88
		4.00	100.00	100.00	100.00	100.00	100.00
		Range of Δ	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)
		0.50	83.20	85.78	88.68	90.67	92.98
		1.00	132.98	114.94	109.69	107.17	105.65
		1.50	207.44	144.39	127.88	120.30	115.06
		2.00	255.05	157.88	135.36	125.42	118.58
	2	2.50	207.44	144.39	127.88	120.30	115.06
		3.00	132.98	114.94	109.69	107.17	105.65
		3.50	83.20	85.78	88.68	90.67	92.98
		4.00	54.59	63.30	69.92	74.59	79.61
		Range of Δ	(0.70, 3.30)	(0.76, 3.24)	(0.77, 3.23)	(0.78, 3.22)	(0.77, 3.23)

TABLE 5.4 *PRE of $\widehat{G}_P$ with respect to $\widehat{G}_M$ for $r = 2$ (applicable for estimation of Population variance)*

v	u	$\Delta \downarrow\ n \rightarrow$	5	10	15	20	25
		0.50	61.03	35.48	37.96	42.98	45.34
		1.00	137.27	79.73	82.72	88.47	88.44
		1.50	548.37	316.96	282.78	242.32	205.90
		2.00	309377.87	38602.49	1458.93	576.56	369.45
	−2	2.50	548.37	316.96	282.78	242.32	205.90
		3.00	137.27	79.73	82.72	88.47	88.44
		3.50	61.03	35.48	37.96	42.98	45.34
		4.00	34.33	19.96	21.60	24.99	26.95
		Range of Δ	(0.83, 3.17)	(1.11, 2.89)	(1.10, 2.90)	(1.07, 2.93)	(1.08, 2.92)
		0.50	88.38	101.24	105.21	105.20	103.75
		1.00	186.02	154.10	133.98	124.19	118.32
		1.50	551.72	224.39	160.28	139.28	129.20
		2.00	1600.67	264.64	171.50	145.16	133.29
	−1	2.50	551.72	224.39	160.28	139.28	129.20
		3.00	186.02	154.10	133.98	124.19	118.32
		3.50	88.38	101.24	105.21	105.20	103.75
		4.00	50.94	68.39	80.90	86.65	88.50
		Range of Δ	(0.60, 3.40)	(0.49, 3.51)	(0.40, 3.60)	(0.36, 3.64)	(0.38, 3.62)
0.5		0.50	117.07	108.64	105.79	104.35	103.27
		1.00	133.33	115.78	110.34	107.70	105.74
		1.50	145.45	120.54	113.27	109.81	107.28
		2.00	150.00	122.22	114.29	110.53	107.80
	1	2.50	145.45	120.54	113.27	109.81	107.28
		3.00	133.33	115.78	110.34	107.70	105.74
		3.50	117.07	108.64	105.79	104.35	103.27
		4.00	100.00	100.00	100.00	100.00	100.00
		Range of Δ	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)
		0.50	87.72	78.84	78.58	80.15	86.50
		1.00	185.19	146.86	132.66	125.30	120.76
		1.50	555.56	304.50	225.97	189.28	158.39
		2.00	1666.67	474.14	295.19	228.11	176.75
	2	2.50	555.56	304.50	225.97	189.28	158.39
		3.00	185.19	146.86	132.66	125.30	120.76
		3.50	87.72	78.84	78.58	80.15	86.50
		4.00	50.51	47.83	50.02	53.27	61.91
		Range of Δ	(0.60, 3.40)	(0.70, 3.30)	(0.74, 3.26)	(0.75, 3.25)	(0.71, 3.29)

TABLE 5.5 *PRE of $\widehat{G}_P$ with respect to $\widehat{G}_M$ for $r = 4$*
(applicable for estimation of Fourth moment about mean)

v	u	$\Delta \downarrow$ $n \rightarrow$	5	10	15	20	25
		0.50	124.46	87.52	67.17	54.67	41.79
		1.00	280.04	196.92	151.13	123.00	93.95
		1.50	1120.15	787.68	604.52	491.90	374.16
		2.00	i. l.	i. l.	i. l.	i. l.	63709.78
	−2	2.50	1120.15	787.68	604.52	491.90	374.16
		3.00	280.04	196.92	151.13	123.00	93.95
		3.50	124.46	87.52	67.17	54.67	41.79
		4.00	70.01	49.23	37.78	30.75	23.51
		Range of Δ	(0.33, 3.67)	(0.60, 3.40)	(0.77, 3.23)	(0.89, 3.11)	(1.03, 2.97)
		0.50	126.86	94.26	100.70	105.66	98.80
		1.00	285.30	211.38	207.32	191.14	165.69
		1.50	1138.10	830.79	568.40	371.50	279.02
		2.00	313245.18	35777.20	1355.06	541.95	361.44
	−1	2.50	1138.10	830.79	568.40	371.50	279.02
		3.00	285.30	211.38	207.32	191.14	165.69
		3.50	126.86	94.26	100.70	105.66	98.80
		4.00	71.37	53.08	58.55	64.97	63.12
		Range of Δ	(0.31, 3.69)	(0.54, 3.46)	(0.49, 3.51)	(0.45, 3.55)	(0.51, 3.49)
0.5		0.50	144.14	127.45	119.80	115.33	110.60
		1.00	210.53	158.53	139.53	129.52	119.66
		1.50	290.91	185.71	154.84	139.84	125.84
		2.00	333.33	196.96	160.71	143.66	128.05
	1	2.50	290.91	185.71	154.84	139.84	125.84
		3.00	210.53	158.53	139.53	129.52	119.66
		3.50	144.14	127.45	119.80	115.33	110.60
		4.00	100.00	100.00	100.00	100.00	100.00
		Range of Δ	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)	(0.00, 4.00)
		0.50	127.38	98.88	87.27	86.40	93.35
		1.00	286.40	220.37	190.38	178.77	164.58
		1.50	1140.97	838.36	653.97	498.60	303.59
		2.00	211676.47	12855.76	3472.31	1235.22	422.55
	2	2.50	1140.97	838.36	653.97	498.60	303.59
		3.00	286.40	220.37	190.38	178.77	164.58
		3.50	127.38	98.88	87.27	86.40	93.35
		4.00	71.67	55.81	49.64	50.13	58.13
		Range of Δ	(0.31, 3.69)	(0.51, 3.49)	(0.60, 3.40)	(0.61, 3.39)	(0.57, 3.43)

NOTE : *i. l. stands for indefinitely large.*

Simulated data from $N(20, 25)$

22.320	24.530	17.590	11.065	10.975	20.995
20.300	25.895	13.120	19.475	14.070	20.795
27.430	12.495	14.950	13.305	23.290	31.365
25.110	16.550	19.975	25.205	17.805	20.205
26.970	26.860	26.965	21.395	13.005	14.340

TABLE 6.1 *Values of* Δ, $\widehat{\Delta}$, v *and* u

r	Δ (*True Value*)	$\widehat{\Delta}$ (*Consistent Estimate*)	$v = \widehat{\Delta}^{-1}$	u
−2	0.8264	0.9773	1.0232	2.00
−1	0.9090	0.9963	1.0037	2.50
1	1.1000	0.9900	1.0101	2.75
2	1.2100	0.9684	1.0326	2.00
4	1.4641	0.9000	1.1111	1.25

suitable values of u have also been shown in Table 6.1.

Table 6.2 provides a perfect delineation about the deed of the conventional and proposed estimators of numerous parametric functions. To know the actual prospective, true values of Δ are used in ARB and RMSE computations. Undoubtedly, one can say that the proposed estimators perform tremendously better than the conventional estimators both in terms of bias and efficiency.

7. CONCLUDING REMARKS

It seems really stunning that a single class of estimators estimates numerous important parameters. The suggested class of estimators has substantial gain in efficiency for a number of choices of scalars comprehend in it, particularly for small sample sizes. Even for large sample sizes, so far as the proper selection of scalars (u, v) is concerned, many estimators of the class are found more efficient than the MMSE estimator.

It is interesting to note that at $v = \Delta^{-1}$ the proposed class of estimators is unbiased with maximum gain in efficiency and hence in the vicinity of $v = \Delta^{-1}$ also, the suggested class of estimators not only renders massive gain in efficiency but it marginally biased in comparison of MMSE estimator. Thus in order to have considerable gain in efficiency for fixed Δ one should choose v in the close proximity of $v = \Delta^{-1}$. The advantage and attraction of this criterion of choosing

TABLE 6.2 *Simulation results*

Parametric Function	*True Value*	*Estimators*	*Estimate*	*ARB*	*RV or RMSE*	*PRE of proposed estimator w.r.t. others*
Fisher Information	0.0400	Unbiased	0.0287	0.0000	0.0713	202.56
		MMSE	0.0268	0.0665	0.0665	188.92
		Proposed	0.0303	0.0499	0.0352	100.00
Precision of Sample Mean	1.2000	Unbiased	0.8605	0.0000	0.0713	202.56
		MMSE	0.8033	0.0665	0.0665	188.92
		Proposed	0.9103	0.0499	0.0352	100.00
**Process Capability Index (C_p)	0.1000	Unbiased	0.8053	0.0000	0.0157	131.93
		MMSE	0.0840	0.0154	0.0154	129.41
		Proposed	0.0861	0.0116	0.0119	100.00
Standard Deviation	5.0000	Unbiased	5.7787	0.0000	0.0121	123.47
		MMSE	5.7099	0.0119	0.0119	121.43
		Proposed	5.7548	0.0119	0.0098	100.00
Mean Deviation about Mean	3.9894	Unbiased	4.6098	0.0000	0.0121	123.47
		MMSE	4.5549	0.0119	0.0119	121.43
		Proposed	4.5908	0.0119	0.0098	100.00
Variance	25.0000	Unbiased	32.9957	0.0000	0.0421	131.97
		MMSE	31.6642	0.0404	0.0404	126.64
		Proposed	32.7352	0.0369	0.0319	100.00
Fourth Moment about Mean	1875.0000	Unbiased	3134.3466	0.0000	0.1264	127.80
		MMSE	2782.5402	0.1122	0.1122	113.45
		Proposed	3120.5229	0.1029	0.0989	100.00

NOTE : ** *Specification limits are set as* 20 ± 1.5 *so that* $USL=21.5$ *and* $LSL=18.5$.

v lies in the fact that if one selects smaller values of v then massive gain in efficiency is observed for larger values of Δ and *vice-versa.* This insinuates that for smaller values of v, the suggested family of shrinkage estimators permits to select the guessed value much away from the true value. Consequently, even if the experimenter has weak believe in the guessed value the risk of estimation using the proposed class of estimators is quite low.

ACKNOWLEDGEMENTS

The authors are grateful to the two referees for their constructive comments that led to improvement of the earlier version of the paper.

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