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INFERENCE FOR PEAKEDNESS ORDERING BETWEEN TWO DISTRIBUTIONS

MYONGSIK OH[1]

ABSTRACT

The concept of dispersion is intrinsic to the theory and practice of statistics. A formulation of the concept of dispersion can be obtained by comparing the probability of intervals centered about a location parameter. This is the peakedness ordering introduced first by Birnbaum (1948). We consider statistical inference concerning peakedness ordering between two arbitrary distributions. We propose nonparametric maximum likelihood estimators of two distributions under peakedness ordering and a likelihood ratio test for equality of dispersion in the sense of peakedness ordering.

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1. INTRODUCTION

The concept of dispersion is intrinsic to the theory and practice of statistics. A formulation of the concept of dispersion can be obtained by comparing the probability of intervals centered about a location point, typically the mean or median of the distribution. It seems natural to interpret dispersion in terms of the distance of a random variable X from a location parameter μ, that is, the magnitude of $|X - \mu|$. One might be interested in comparing such dispersions of two or more distributions.

Following Birnbaum (1948), a random variable X is said to be more peaked about $a \in \mathbb{R}^1$ than is Y about $b \in \mathbb{R}^1$ if, for all $x \geq 0$,

$$F(x + a) - F(-x + a) \geq G(x + b) - G(-x + b). \tag{1.1}$$

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[1]Department of Statistics, Pusan University of Foreign Studies, Busan 608-738, Korea (e-mail : moh@stat.pufs.ac.kr)

where F and G are the distribution functions of X and Y, respectively.

Proschan (1965), Karlin (1968), Bickel and Lehmann (1979), Shaked (1980, 1982), and Schweder (1982), among others, have considered properties and connections with other orderings. The statistical inference concerning peakedness ordering, however, has received little attention. We note that (1.1) is equivalent to

$$P(|X-a| \leq t) \geq P(|Y-b| \leq t) \quad \text{for every } t \in (0, \infty).$$

We see so-called stochastic ordering between two random variables, $|X-a|$ and $|Y-b|$. We say a random variable Y is more dispersed about ν than X about μ if $|Y-\nu|$ is stochastically larger than $|X-\mu|$. Based on this fact, El Barmi and Rojo (1996) considered the likelihood ratio tests for peakedness in multinomial populations. The usage of their result is, however, severely restricted because of the following reasons. First they provided the test for equality of two multinomial parameters against peakedness ordering rather than the equality in the sense of peakedness ordering. So their test can not be used for testing dispersive ordering. This will be discussed fully later in Section 3. Second, they considered only the case that the location parameter is assumed to be at center of the distribution, although peakedness ordering can be defined about at any location parameter. This seems to be based upon the fact that dispersion ordering can be explained by peakedness ordering about median. Finally they considered only multinomial populations. Even though their result can be extended to the case of arbitrary distribution functions, the extension to both estimation and testing problem requires some modifications, which are discussed later.

In this article we consider estimation of general distribution functions under peakedness ordering. The estimation procedure does not require that the location parameters should be at the center of the distributions. It is known that some estimator for one-sample problem may not satisfy consistency. See Rojo and Samaniego (1991). Here we only consider the two-sample problem, which is of particular interest in practice. We assume the location parameters to be known. In Section 2, maximum likelihood estimation of two general distribution functions under peakedness ordering is discussed. Though its estimation procedure is basically the same as given by El Barmi and Rojo (1996), it requires some modifications. In Section 3, the likelihood ratio test for equality in peakedness against peakedness ordering in discrete setting is discussed. In Section 4, real data set is analyzed for illustrative purpose.

2. ESTIMATION OF DISTRIBUTION FUNCTIONS

Let F and G be distribution functions with known median μ_x and μ_y. Without loss of generality we assume that $\mu_x = \mu_y = \mu$ and both random samples are observed at $-\infty < t_1 < t_2 < \cdots < t_k < +\infty$. Let δ_{1i} (δ_{2i}) be the number of observations from F (G) at t_i. The ordinary nonparametric maximum likelihood estimator can be obtained by finding F and G which maximize

$$\prod_{i=1}^{k}\{F(t_i) - F(t_i-)\}^{\delta_{1i}}\{G(t_i) - G(t_i-)\}^{\delta_{2i}}. \tag{2.1}$$

Now our problem is to find F and G subject to (1.1). This can be achieved by a reparametrization.

Consider imaginary data points $t_{i+k} = 2\mu - t_i$ with $\delta_{1,i+k} = \delta_{2,i+k} = 0$ for $i = 1, \ldots, k$. Let $-\infty < s_1 < s_2 < \cdots < s_l \leq s_{l+1} < \cdots < s_{2l} < \infty$ be ordered distinct values of t_i, $i = 1, \ldots, 2k$, except the case that μ is equal to one of t_i's so that $\mu = s_l = s_{l+1}$. We observe that $l \leq k$, $s_l < \mu < s_{l+1}$ (or possibly $s_l = \mu = s_{l+1}$) and $s_i = 2\mu - s_{2l-i+1}$, for $i = 1, \ldots, l$.

For $j = 1, \ldots, 2l$, let

$$d_{1j} = \sum_{i\in\{1,2,\ldots,2k\}:t_i=s_j} \delta_{1i} \quad \text{and} \quad d_{2j} = \sum_{i\in\{1,2,\ldots,2k\}:t_i=s_j} \delta_{2i}.$$

Then (2.1) can be rewritten as

$$\prod_{i=1}^{2l}\{F(s_i) - F(s_i-)\}^{d_{1i}}\{G(s_i) - G(s_i-)\}^{d_{2i}}. \tag{2.2}$$

The peakedness ordering (1.1) can be expressed as, for $j = 0, \ldots, l-1$,

$$F(s_{l+1+j}) - F(s_{l-j}-) \geq G(s_{l+1+j}) - G(s_{l-j}-). \tag{2.3}$$

Now we are going to find F and G which maximize (2.2) subject to (2.3).

Let $\theta_{11} = F(s_{l+1}) - F(s_l-)$, $\theta_{21} = G(s_{l+1}) - G(s_l-)$, and $\phi_{11} = (F(s_l) - F(s_l-))/(F(s_{l+1}) - F(s_l-))$, $\phi_{21} = (G(s_l) - G(s_l-))/(G(s_{l+1}) - G(s_l-))$, and for $j = 1, \ldots, l-1$,

$$\begin{aligned}
\theta_{1,j+1} &= F(s_{l-j}) - F(s_{l-j}-) + F(s_{l+j+1}) - F(s_{l+j+1}-),\\
\theta_{2,j+1} &= G(s_{l-j}) - G(s_{l-j}-) + G(s_{l+j+1}) - G(s_{l+j+1}-),\\
\phi_{1,j+1} &= \frac{F(s_{l-j}) - F(s_{l-j}-)}{\theta_{1,j+1}}, \quad \phi_{2,j+1} = \frac{G(s_{l-j}) - G(s_{l-j}-)}{\theta_{2,j+1}}.
\end{aligned}$$

Then (2.2) becomes

$$\prod_{i=1}^{2}\prod_{j=0}^{l-1}\left[\theta_{i,j+1}^{d_{i,l-j}+d_{i,l+j+1}}\,\phi_{i,j+1}^{d_{i,l-j}}(1-\phi_{i,j+1})^{d_{i,l+j+1}}\right] \tag{2.4}$$

with the convention $0^0 = 1$, and the restriction (2.3) is equivalent to

$$\sum_{j=0}^{\ell}\theta_{1,j+1} \geq \sum_{j=0}^{\ell}\theta_{2,j+1} \quad \text{for } \ell = 0,\ldots,l-1, \tag{2.5}$$

$$0 \leq \phi_{i,j+1} \leq 1 \quad \text{for } i = 1,2,\ j = 0,\ldots,l-1. \tag{2.6}$$

Noting that $\sum_{j=0}^{l-1}\theta_{1,j+1} = \sum_{j=0}^{l-1}\theta_{2,j+1} = 1$, we see a stochastic ordering between θ_{1j}'s and θ_{2j}'s. Moreover, restrictions (2.5) and (2.6) do not relate θ's and ϕ's. This means that maximization of (2.4) can be achieved by maximizing two parts (one involves θ's only and the other ϕ's only) separately.

First consider estimation of the ϕ's, which is just a binomial problem. Let * denote the estimate under peakedness ordering restriction. The ML estimate, $\phi^*_{i,j+1}$, for $i = 1,2$, $j = 0,\ldots,l-1$, of ϕ_{ij}, is given by $d_{i,l-j}/(d_{i,l-j}+d_{i,l+j+1})$ provided that $d_{i,l-j}+d_{i,l+j+1} > 0$. Next, let $E_{\mathbf{w}}(\mathbf{x}|A)$ denote the projection of $\mathbf{x}$ onto A provided it exists and is unique. See Robertson *et al.* (1988) for details of projection theory. Let $\mathbf{d}_1 = (d_{1l}+d_{1,l+1}, d_{1,l-1}+d_{1,l+2},\ldots,d_{11}+d_{1,2l})$, $\mathbf{d}_2 = (d_{2l}+d_{2,l+1}, d_{2,l-1}+d_{2,l+2},\ldots,d_{21}+d_{2,2l})$, $m = \sum_{j=0}^{l-1}(d_{1,l-j}+d_{1,l+j+1})$, and $n = \sum_{j=0}^{l-1}(d_{2,l-j}+d_{2,l+j+1})$. Robertson and Wright (1981) gave the ML estimate of two-sample multinomial parameters under stochastic ordering using Fenchel duality. See also Barlow and Brunk (1972). Using their notation the ML estimates of θ's under (2.5) are given by, for $j = 0,\ldots,l-1$,

$$\theta^*_{1,j+1} = \frac{d_{1,l-j}+d_{1,l+j+1}}{m}E_{\mathbf{d}_1}\left(\frac{\mathbf{d}_1+\mathbf{d}_2}{((m+n)/m)\mathbf{d}_1}|\mathcal{A}\right)_j,$$

$$\theta^*_{2,j+1} = \frac{d_{2,l-j}+d_{2,l+j+1}}{n}E_{\mathbf{d}_2}\left(\frac{\mathbf{d}_1+\mathbf{d}_2}{((m+n)/n)\mathbf{d}_2}|\mathcal{I}\right)_j,$$

where $\mathbf{x}+\mathbf{y} = (x_1+y_1,\ldots,x_k+y_k)$, $\mathbf{x}/\mathbf{y} = (x_1/y_1,\ldots,x_k/y_k)$, $\mathcal{I} = \{\mathbf{x} \in \mathbb{R}^k : x_1 \leq x_2 \leq \cdots \leq x_k\}$, $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^k : -\mathbf{x} \in \mathcal{I}\}$, and $E(\cdot)_j$ denotes the j^{th} component of $E(\cdot)$. Since we use imaginary data points it is likely to have missing components in $\mathbf{d}_1$ and $\mathbf{d}_2$. For the estimation procedure when missing values are present, see Lee (1987).

Now for $j = 0, \ldots, l-1$

$$\begin{aligned} F^*(s_{l-j}) - F^*(s_{l-j}-) &= \theta^*_{1,j+1}\phi^*_{1,j+1}, \\ F^*(s_{l+j+1}) - F^*(s_{l+j+1}-) &= \theta^*_{1,j+1}(1-\phi^*_{1,j+1}), \\ G^*(s_{l-j}) - G^*(s_{l-j}-) &= \theta^*_{2,j+1}\phi^*_{2,j+1}, \\ G^*(s_{l+j+1}) - G^*(s_{l+j+1}-) &= \theta^*_{2,j+1}(1-\phi^*_{2,j+1}). \end{aligned}$$

It follows immediately from Robertson and Wright (1981) that under the hypothesis of stochastic ordering the ML estimators of θ's are strongly consistent. Since ϕ's are also strongly consistent, F^* and G^* are strongly consistent, too.

Next we consider the estimation of distribution when two distributions are equal in sense of peakedness ordering, which means that, for $j = 0, \ldots, l-1$,

$$\begin{aligned} &F(s_{l-j}) - F(s_{l-j}-) + F(s_{l+j+1}) - F(s_{l+j+1}-) \\ &= G(s_{l-j}) - G(s_{l-j}-) + G(s_{l+j+1}) - G(s_{l+j+1}-). \end{aligned} \tag{2.7}$$

We use the same reparametrization scheme. Then the equality in peakedness ordering restriction is equivalent to

$$\theta_{1,j+1} = \theta_{2,j+1} \text{ for } j = 0, \ldots, l-1, \tag{2.8}$$

with (2.6). And the likelihood becomes

$$\prod_{j=0}^{l-1} \theta_{1,j+1}^{d_{1,l-j}+d_{1,l+j+1}+d_{2,l-j}+d_{2,l+j+1}} \prod_{i=1}^{2}\prod_{j=0}^{l-1} \phi_{i,j+1}^{d_{i,l-j}}(1-\phi_{i,j+1})^{d_{i,l+j+1}}. \tag{2.9}$$

The likelihood consists of two parts; one is multinomial likelihood and the other is a product of binomials. The ML estimate, $\theta^\circ_{1,j+1}$, for $j = 0, \ldots, l-1$, of $\theta_{1,j+1}$ is given by

$$\theta^\circ_{1,j+1} = \theta^\circ_{2,j+1} = \frac{d_{1,l-j} + d_{1,l+j+1} + d_{2,l-j} + d_{2,l+j+1}}{m+n},$$

where $\circ$ denotes estimate under H_0. Note that $\phi^\circ_{i,j+1} = \phi^*_{i,j+1}$, for $j = 0, \ldots, l-1$. Hence we have, for $j = 0, \ldots, l-1$,

$$\begin{aligned} F^\circ(s_{l-j}) - F^\circ(s_{l-j}-) &= \theta^\circ_{1,j+1}\phi^\circ_{1,j+1}, \\ F^\circ(s_{l+j+1}) - F^\circ(s_{l+j+1}-) &= \theta^\circ_{1,j+1}(1-\phi^\circ_{1,j+1}), \\ G^\circ(s_{l-j}) - G^\circ(s_{l-j}-) &= \theta^\circ_{1,j+1}\phi^\circ_{2,j+1}, \\ G^\circ(s_{l+j+1}) - G^\circ(s_{l+j+1}-) &= \theta^\circ_{1,j+1}(1-\phi^\circ_{2,j+1}). \end{aligned}$$

The ML estimator under the equality assumption in the sense of peakedness ordering is also strongly consistent.

3. HYPOTHESES TESTING

Assume that F and G have support on the fixed set $(t_1, \ldots, t_k)$ and that each point has positive probability, so that we are concerned with discrete distributions with common support. In this section we consider the likelihood ratio test for equality in a sense of peakedness ordering of two distributions against peakedness ordering. The hypotheses are, for all $x > 0$ and given a and b,

$$H_0 : F(x+a) - F(-x+a) = G(x+b) - G(-x+b),$$
$$H_1 : F(x+a) - F(-x+a) \geq G(x+b) - G(-x+b)$$

with strict inequality for at least one x. After reparametrization given random samples, H_0 is related to (2.8) and H_1 to (2.5). The test rejects H_0 for large values of test statistic, which is given by

$$T = -2m \sum_{j=0}^{l-1} \hat{\theta}_{1,j+1}(\log \theta^*_{1,j+1} - \log \theta^\circ_{1,j+1}) - 2n \sum_{j=0}^{l-1} \hat{\theta}_{2,j+1}(\log \theta^*_{2,j+1} - \log \theta^\circ_{2,j+1}),$$

where

$$\hat{\theta}_{1,j+1} = F_m(s_{l-j}) - F_m(s_{l-j}-) + F_m(s_{l+j+1}) - F_m(s_{l+j+1}-),$$
$$\hat{\theta}_{2,j+1} = G_n(s_{l-j}) - G_n(s_{l-j}-) + G_n(s_{l+j+1}) - G_n(s_{l+j+1}-),$$

and F_m and G_n are the empirical distributions of F and G.

Now we need to find the asymptotic null distribution of test statistic T to find a critical value. It follows immediately from Robertson and Wright (1981) that

$$\lim_{m,n\to\infty} P[T \geq t] = \sum_{i=1}^{l} P_s(i, l; \boldsymbol{\theta}) P[\chi^2_{l-i} \geq t] \tag{3.1}$$

provided that $m/(m+n) \to a \in (0,1)$. We note that $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_l)$, and that $P(i, l; \boldsymbol{\theta})$ is the probability that $E_{\boldsymbol{\theta}}(W|\mathcal{A})$ has exactly i distinct levels, where $\mathbf{W} = (W_1, \ldots, W_l)$ and $W_1, \ldots, W_l$ are independent normal variables with zero means and variances $\theta_1^{-1}, \ldots, \theta_l^{-1}$, respectively. To find a critical value we need to know the value of $\boldsymbol{\theta}$, which is unknown because the null distribution does not specify the two distributions. Moreover the level probabilities are intractable if l is large. One might use the plug-in method using an estimate of $\boldsymbol{\theta}$ such as $\boldsymbol{\theta}^\circ$ provided that l is not so big. This approximation is generally known to be very adequate.

Robertson and Wright (1983) showed that the equal-weights null distribution of chi-bar-square test statistics provide reasonable approximation for the case of unequal sample sizes if the sample sizes are not too different for the simple ordering. We recommend the use of equal-weight level probabilities for finding a critical value or a p-value. The equal weight level probability can be obtained by the following recursive relationship,

$$\begin{aligned} P(1,l) &= \frac{1}{l}, \quad P(l,l) = \frac{1}{l!}, \\ P(i,l) &= \frac{1}{l}P(i-1,l-1) + \frac{l-1}{l}P(i,l) \quad \text{for } i = 2,\ldots,l-1. \end{aligned}$$

Finally one might use the least favorable distribution for a conservative test. The least favorable test is given by

$$\sup_{H_0} \lim_{m,n\to\infty} P[T \geq t] = \frac{1}{2}(P[\chi_l^2 \geq t] + P[\chi_{l-1}^2 \geq t]).$$

One might be interested in testing the equality of two distribution against peakedness ordering. El Barmi and Rojo (1996) provided only the test of equality of two multinomial parameter against peakedness ordering. This is, however, not common in testing equality of dispersion of two distribution. On the other hand, their test statistic, T', say, consists of two parts; one has chi-bar-square distribution and the other chi-square distribution. This means that the asymptotic null distribution does not contain degenerate distribution. Specifically,

$$\lim_{m,n\to\infty} P[T' \geq t] = \sum_{i=1}^{l} P_s(i,l;\boldsymbol{\theta})P[\chi_{2l-i}^2 \geq t].$$

For most order restricted statistical problems the asymptotic or exact distribution does involve a degenerate distribution which is a chi-square distribution with zero degrees of freedom. However the above asymptotic null distribution does not involve any degenerate distributions. Moreover the chi-square variables included in the null distribution have bigger degrees of freedom. This guarantees higher power.

We close this section with a brief discussion of the likelihood ratio test for peakedness ordering against all alternatives. The test rejects the null hypothesis for the large value of T'', which is given by

$$-2m\sum_{j=0}^{l-1}\hat{\theta}_{1,j+1}(\log\theta_{1,j+1}^* - \log\hat{\theta}_{1,j+1}) - 2n\sum_{j=0}^{l-1}\hat{\theta}_{2,j+1}(\log\theta_{2,j+1}^* - \log\hat{\theta}_{2,j+1}).$$

The least favorable distribution is given by

$$\begin{aligned}\sup_{H_0} \lim_{m,n\to\infty} P[T'' \geq t] &= \sup_{F=G} \lim_{m,n\to\infty} P[T'' \geq t] \\ &= \sum_{i=1}^{l} \binom{l-1}{i-1} 2^{-l+1} P[\chi^2_{i-1} \geq t].\end{aligned}$$

4. An Example

In this section, a real data set, which has been used in El Barmi and Rojo (1996), is analyzed to illustrate the proposed method. The data have been abstracted from the data set presented in O'Neill *et al.* (1985). Table 1 in El Barmi and Rojo (1996) shows lung cancer mortality in South Australia in 1981 classified according to gender and age. There are a total of 425 deaths of which 338 are males and 87 females. Let p_i (q_i) be the proportion of deaths which are males (females) and belong to i^{th} age group. Apparently, a peak is seen at age interval 65–69 (6^{th} age group) for males and at age 70–74 (7^{th} age group) for females. We are going to compare peakedness about at age interval 65–69. Under H_0, the MLE for parameters are

$$\begin{aligned}\mathbf{p}^\circ &= (0.0141, 0.0282, 0.0762, 0.1205, 0.1750, 0.2024, 0.1873, 0.1289, 0.0674), \\ \mathbf{q}^\circ &= (0.0141, 0.0282, 0.0957, 0.1174, 0.1510, 0.2024, 0.2114, 0.1320, 0.0478),\end{aligned}$$

and under H_1,

$$\begin{aligned}\mathbf{p}^* &= (0.0086, 0.0288, 0.0750, 0.1240, 0.1721, 0.2084, 0.1842, 0.1326, 0.0663), \\ \mathbf{q}^* &= (0.0384, 0.0256, 0.1024, 0.1024, 0.1598, 0.1811, 0.2238, 0.1152, 0.0512).\end{aligned}$$

We have $T = 3.979238$ with $l = 6$. We use the equal-weights level probability, which can be found in Robertson *et al.* (1988). The computed p-value is 0.3471 and hence we can conclude that there is no solid evidence that male is more peaked about age 65–69 than female.

Next we are going to compare peakedness about the interval 60–64 as El Barmi and Rojo (1996) did. We have

$$\begin{aligned}\mathbf{p}^\circ &= (0.0090, 0.0282, 0.0816, 0.1238, 0.1694, 0.1986, 0.1914, 0.1295, 0.0687), \\ \mathbf{q}^\circ &= (0.0333, 0.0287, 0.0753, 0.1032, 0.1694, 0.2192, 0.1976, 0.1290, 0.0444), \\ \mathbf{p}^* &= (0.0086, 0.0287, 0.0777, 0.1284, 0.1703, 0.2061, 0.1822, 0.1320, 0.0660), \\ \mathbf{q}^* &= (0.0392, 0.0261, 0.0887, 0.0887, 0.1662, 0.1884, 0.2327, 0.1176, 0.0523),\end{aligned}$$

and $T = 2.9465$ with $l = 5$. We note that the $\mathbf{p}^*$ and $\mathbf{q}^*$ in El Barmi and Rojo (1996) does not maximize the likelihood and hence they are not the MLE under H_1. The computed p-value is 0.4975 when the equal-weight approximation is used for level probability. No evidence that the male is more peaked than the female is shown. We have the opposite result compare to the example in El Barmi and Rojo (1996). This is due to that their null hypothesis postulates the equality of two parameter rather than equality in peakedness.

5. CONCLUDING REMARKS

The likelihood ratio test concerning stochastic ordering for general distributions has been studied by several authors. Dykstra *et al.* (1983), Franck (1984) and Wang (1996) are among others. Their results can be applied with some modifications if needed. We will not pursue this subject in this paper.

In this paper we did not consider the case of unknown location parameter, such as median, which is more frequently encountered situation in practice. We may also impose symmetry assumption, which is quite common in nonparametric setting. These problems are, however, nontrivial extension of the procedure given in this article.

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