Journal of the Korean Statistical Society (2004), **33**: *3, pp 299–302*

A MARKOVIAN APPROACH TO THE FORWARD RECURRENCE TIME IN THE RENEWAL PROCESS†

JONGWOO KIM[1], EUI YONG LEE[2] AND GYOOCHEOL SHIM[3]

ABSTRACT

A Markovian approach is introduced to find the Laplace transform of the forward recurrence time in the renewal process at finite time $t > 0$. Until now, most works on the forward recurrence time have been done through renewal arguments.

AMS 2000 subject classifications. Primary 60K05; Secondary 60J25.
Keywords. Renewal process, Laplace transform, forward recurrence time, forward differential equation.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of renewal intervals which are independent and identically distributed with distribution function G and probability density function g. We assume that $\mu = E[X_i] < \infty$. Let $N(t)$ be the number of renewals up to time t, $M(t) = E[N(t)]$ the renewal function, and $m(t) = dM(t)/dt$ the renewal density function. Define $R(t)$ as the forward recurrence time at time $t > 0$, and denote $F(x,t)$ and $f(x,t)$ as the distribution function and density function of $R(t)$, respectively. Let $f^*(s,t) = \int_0^\infty e^{-sx} f(x,t)dx$ be the Laplace transform of $f(x,t)$.

The followings are well known results of $R(t)$ (see Cox, 1962):

(i) $f(x,t) = g(t+x) + \int_0^t m(t-u)g(u+x)du$.

Received April 2004; accepted April 2004.
†This research was supported by KOSEF through Statistical Research Center for Complex Systems at Seoul National University.
[1]Statistical Research Center for Complex Systems, Seoul National University, Seoul 151-747, Korea (e-mail : jwkim@stats.snu.ac.kr)
[2]Department of Statistics, Sookmyung Women's University, Seoul 140-742, Korea
[3]Financial Engineering Research Center, Graduate School of Management, Korea Advanced Institute of Science and Technology, Seoul 130-722, Korea

(ii) $R(t)$ has the limiting density function given by $f(x) = \bar{G}(x)/\mu$ as $t \to \infty$.

(iii) The Laplace transform of $f(x)$ is given by

$$f^*(s) = \frac{1 - g^*(s)}{s\mu},$$

where $g^*(s) = \int_0^\infty e^{-sx} g(x)dx$.

Coleman (1982) obtained the moments of $R(t)$. These works, however, are done by making use of renewal arguments.

In this paper, we obtain the formula for the Laplace transform of the forward recurrence time at finite time $t > 0$, by establishing a Kolmogorov's forward differential equation for $F(x,t)$. As a consequence, we obtain the moments of $R(t)$ by differentiation.

2. LAPLACE TRANSFORM OF THE FORWARD RECURRENCE TIME

First, note that the forward recurrence time $R(t)$ satisfies the Markovian property, that is, once $R(t_0)$ is given, $\{R(t),\ t > t_0\}$ and $\{R(t),\ t < t_0\}$ are conditionally independent (see Figure 2.1). This fact enables us to deduce a forward differential equation for $F(x,t) = Pr\{R(t) \le x\}$.

In a small interval $[t, t+\Delta t]$, $R(t+\Delta t)$ satisfies the followings:

$$R(t+\Delta t) = \begin{cases} R(t) - \Delta t, & \text{almost surely, if } R(t) > \Delta t, \\ X - \Delta t, & \text{almost surely, if } R(t) \le \Delta t, \end{cases}$$

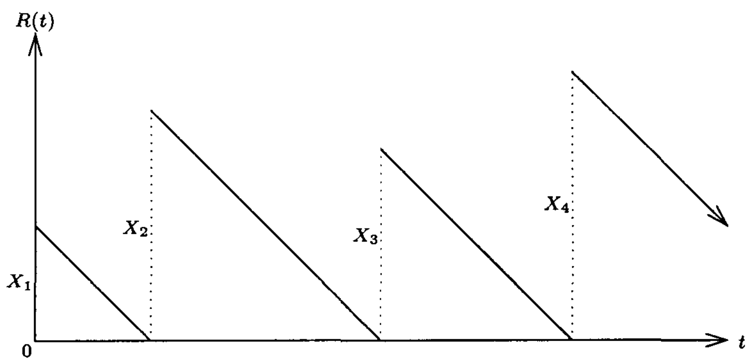

FIGURE 2.1 *A sample path of forward recurrence time $R(t)$*

where X is a random variable having distribution function G. These two events are mutually exclusive, and hence

$$\begin{aligned} &Pr\{R(t+\Delta t) \le x\} \\ &= Pr\{R(t)-\Delta t \le x,\ R(t) > \Delta t\} + Pr\{X-\Delta t \le x,\ R(t) \le \Delta t\} \\ &= Pr\{R(t)-\Delta t \le x\} - Pr\{R(t)-\Delta t \le x,\ R(t) \le \Delta t\} \\ &\quad + Pr\{X-\Delta t \le x,\ R(t) \le \Delta t\}. \end{aligned} \tag{2.1}$$

Applying the power series expansion to $Pr\{R(t)-\Delta t \le x\}$ around x gives

$$Pr\{R(t)-\Delta t \le x\} = Pr\{R(t) \le x\} + \frac{\partial}{\partial x} Pr\{R(t) \le x\}\Delta t + o(\Delta t).$$

Substituting this into equation (2.1), we have

$$\begin{aligned} &Pr\{R(t+\Delta t) \le x\} - Pr\{R(t) \le x\} \\ &= \frac{\partial}{\partial x} Pr\{R(t) \le x\}\Delta t - Pr\{X > x+\Delta t\} Pr\{R(t) < \Delta t\} + o(\Delta t), \end{aligned} \tag{2.2}$$

where we use the fact that X is independent of $R(t)$. Divide both sides by Δt and let $\Delta t \to 0$ in equation (2.2), we obtain the following differential equation:

$$\frac{\partial}{\partial t} F(x,t) = f(x,t) - \bar{G}(x) f(0,t), \tag{2.3}$$

where $f(x,t) = \partial F(x,t)/\partial x$ and $f(0,t) = \lim_{\Delta t \to 0} Pr\{R(t) \le \Delta t\}/\Delta t$. By taking the Laplace transform in equation (2.3), we have

$$\frac{\partial}{\partial t} f^*(s,t) - s f^*(s,t) = -s f(0,t) \left(\frac{1}{s} - \frac{1}{s} g^*(s) \right). \tag{2.4}$$

By solving equation (2.4) for $f^*(s,t)$ and using the boundary condition $f^*(s,0) = g^*(s)$, we have

$$f^*(s,t) = e^{st} \left[g^*(s) + (g^*(s)-1) \int_0^t e^{-su} f(0,u)\,du \right].$$

By a simple renewal argument, $f(0,t) = m(t)$. Thus,

$$f^*(s,t) = e^{st} \left[g^*(s) + (g^*(s)-1) \int_0^t e^{-su} m(u)\,du \right]. \tag{2.5}$$

Now differentiate equation (2.5) with respect to s. This yields the moments of $R(t)$. For example,

$$\begin{aligned} E[R(t)] &= -t + \mu[1 + M(t)], \\ E[R(t)^2] &= t^2 - 2\mu \left[t + \int_0^t M(u)du\right] + \mu_2[1 + M(t)], \\ E[R(t)^3] &= -t^3 + 3\mu \left[t^2 + 2t \int_0^t M(u)du - 2\int_0^t uM(u)du\right] \\ &\quad -3\mu_2 \left[t + \int_0^t M(u)du\right] + \mu_3[1 + M(t)], \end{aligned}$$

where $\mu_2 = E[X^2]$ and $\mu_3 = E[X^3]$. It can be seen that the moments of $R(t)$ are consistent with those of Coleman (1982).

REMARK. It turns out that it is possible to derive equation (2.5) from the density of the forward recurrence time after a long calculation. As far as we know, equation (2.5) is the first simple and explicit formula in the literature for the Laplace transform of the forward recurrence time, especially, by making use of the Kolmogorov's forward differential equation.

REFERENCES

COLEMAN, R. (1982). "The moments of forward recurrence time", *European Journal of Operational Research*, **9**, 181–183.

COX, D. R. (1962). *Renewal Theory*, John Wiley & Sons, New York.