Dynamic Output-Feedback Receding Horizon H_∞ Controller Design

Seung Cheol Jeong, Jeong Hye Moon, and PooGyeon Park*

Abstract: In this paper, we present a dynamic output-feedback receding horizon H_∞ controller for linear discrete-time systems with disturbance. The controller is obtained numerically from the finite horizon output-feedback H_∞ optimization problem, which is, in fact, hardly solved analytically. Under a matrix inequality condition on the terminal weighting matrix, the monotonic decreasing property of the cost is shown. This property guarantees both the closed-loop stability and the H_∞ norm bound. Then, we extend the proposed design method to a reference tracking problem and a problem for time-varying systems. Numerical examples are given to illustrate the performance of the proposed controller.

Keywords: Asymptotic stability, dynamic output-feedback controller, LMI, receding horizon H_∞ control.

1. INTRODUCTION

Receding horizon control (RHC), also known as model predictive control (MPC), has received much attention in control societies because of its good tracking performance and many applications to industrial processing systems such as distillation and paper processing [1-7]. The basic concept of the RHC is to solve an optimization problem over a fixed number of future time instants at the current time and to implement the first one among the solutions as the current control law. The procedure is then repeated at each subsequent instant. In particular, it is also a suitable control strategy for time-varying systems, reference tracking systems and constrained systems [8-13].

For systems with disturbance, there have been several results on the state-feedback receding horizon H_∞ control (RHHC), which not only guarantee the closed-loop stability but also provide the H_∞norm bound [14-17]. Alternatively, various results have been presented on the output-feedback RHC [18-23]. All these works consider the regulation problem for disturbance-free systems [18,20] or systems with bounded disturbance (the bounded disturbance is assumed owing to the input constraint). All these works take the observer-based approach where the controller is composed of a state observer and a static controller associated with observer states. To guarantee the closed-loop system stability in those control schemes, the state estimation errors should go to zero or remain as a bounded set as time goes on.

In many real plants, however, an abrupt disturbance with immense magnitude often enters the system. Moreover, since RHC also has merits in favor of reference tracking problems and problems for time-varying systems as well as constrained systems, there is still more to be studied to widen the RHC application area. Up to now, however, there has been little result on the output-feedback receding horizon H_∞ controller (ORHHC) design and tracking controller (ORHTC) design or on the output-feedback problem for time-varying systems.

In this paper, we design an ORHHC for linear discrete-time systems with disturbance. The proposed control scheme adopts a general form of dynamic controller rather than an observer-based controller. After solving the difficulty related to regulation, we extend the results to the tracking problem and to the problem for time-varying systems. The proposed controller is obtained numerically from the finite horizon output-feedback H_∞ optimization problem, which is, in fact, hardly solved analytically. Under a matrix inequality condition on the terminal weighting matrix, the closed-loop system stability and the H_∞ norm bound are guaranteed.

This paper is organized as follows. Section 2 describes the system, the dynamic controller form and the finite horizon LQ cost for the ORHHC. We also briefly address the ORHHC scheme using the *dynamic* controller. Section 3 presents an ORHHC.

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Section 4 indicates that the proposed controller stabilizes the closed-loop system and guarantees the H_∞ norm bound. Section 5 extends this control design method to a tracking problem and a problem for time-varying systems. Sections 6 and 7 provide numerical examples and concluding remarks, respectively.

2. PROBLEM FORMULATION

Let us consider the discrete-time linear system described by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Dw_k, \\ y_k &= Cx_k + Ew_k, \end{aligned} \tag{1}$$

where $x_k \in \Re^n, u_k \in \Re^m, y_k \in \Re^l$ and $w_k \in \Re^s$ denote state, input, output and disturbance vectors, respectively. It is assumed that the system described by (1) is stabilizable and detectable. In this paper, the notation $X \geq Y$ (respectively $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite).

The goal of this paper is to design an ORHHC, which stabilizes the system (1) and guarantees the H_∞ norm bound of the following form (so-called the dynamic output-feedback controller)

$$z_{k+1} = A_k^c z_k + B_k^c y_k, \tag{2}$$

$$u_k = C_k^c z_k + D_k^c y_k, \tag{3}$$

where $z_k \in \Re^n$ and A_k^c, B_k^c, C_k^c and D_k^c are design variables. We shall design the ORHHC based on the finite horizon output-feedback H_∞ optimal control, which can be solved through the dynamic game approach [24]. Thus, we first formulate the output-feedback H_∞ optimal control problem with the following cost

$$\begin{aligned} J_{k,k+N}(\eta_k) = \sum_{i=0}^{N-1} \{ & x_{k+i|k}^T Q x_{x+i|k} + u_{k+i|k}^T R u_{k+i|k} \\ & - \gamma^2 w_{k+i|k}^T w_{k+i|k} \} \\ & + \begin{bmatrix} x_{k+N|k} \\ z_{k+N|k} \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} x_{k+N|k} \\ z_{k+N|k} \end{bmatrix}, \end{aligned} \tag{4}$$

which is minimized by the control sequence $u_{k+i|k}$, $i = 0,\ldots,N-1$ and maximized by $w_{k+i|k}$, $i = 0,\ldots,N-1$. Here N is a finite positive integer, γ is a disturbance attenuation level and

$$\eta_k = \begin{bmatrix} x_k \\ z_k \end{bmatrix}, \quad 0 < Q \in \Re^{n \times n}, \quad 0 < R \in \Re^{m \times m},$$

$$0 < \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} = P_f \in \Re^{2n \times 2n}. \tag{5}$$

We denote $x_{k+i|k}$, $u_{k+i|k}$ and $z_{k+i|k}$ (which will be defined below) as i th ahead predicted variables at time k with $x_{k|k} = x_k$, $z_{k|k} = z_k$. These variables satisfy

$$x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k}, \tag{6}$$

$$y_{k+i|k} = Cx_{k+i|k}, \tag{7}$$

$$z_{k+i+1|k} = A_{i,N}^c z_{k+i|k} + B_{i,N}^c y_{k+i|k}, \tag{8}$$

$$u_{k+i|k} = C_{i,N}^c z_{k+i|k} + D_{i,N}^c y_{k+i|k}. \tag{9}$$

Let $u_{k+i|k}^*$ and $w_{k+i|k}^*$ be unique saddle-point control and disturbance sequences of the dynamic game. Also let $x_{k+i|k}^*$, $\{A_{i,N}^{c*}, B_{i,N}^{c*}, C_{i,N}^{c*}, D_{i,N}^{c*}\}$ and $z_{k+i|k}^*$ be corresponding state trajectory, dynamics and state trajectory of the dynamic output-feedback controller, respectively, with $x_{k|k}^* = x_k$, $z_{k|k}^* = z_k$. Then the output-feedback receding horizon controller dynamics and input in (2) and (3) at time k are chosen as

$$\begin{bmatrix} A_k^c & B_k^c \\ C_k^c & D_k^c \end{bmatrix} = \begin{bmatrix} A_{0,N}^{c*} & B_{0,N}^{c*} \\ C_{0,N}^{c*} & D_{0,N}^{c*} \end{bmatrix} \text{ and } u_k = u_{k|k}^*. \tag{10}$$

In the following section, we shall derive the ORHHC based on the finite horizon output-feedback H_∞ optimal control problem.

3. DESIGN OF ORHHC

We start the derivation of the ORHHC with the augmentation of the system and controller equation. Combining (6) and (8) provides the following closed-loop system

$$\begin{aligned} \eta_{k+i+1|k} &= [A_0 + B_0 \Sigma_{i,N} C_0] \eta_{k+i|k} \\ &\quad + [D_0 + B_0 \Sigma_{i,N} E_0] w_{k+i|k}, \end{aligned} \tag{11}$$

where

$$\eta_{k+i|k} = \begin{bmatrix} x_{k+i|k} \\ z_{k+i|k} \end{bmatrix}, \Sigma_{i,N} = \begin{bmatrix} A_{i,N}^c & B_{i,N}^c \\ C_{i,N}^c & D_{i,N}^c \end{bmatrix}, A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, C_0 = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}, D_0 = \begin{bmatrix} D \\ 0 \end{bmatrix}, E_0 = \begin{bmatrix} 0 \\ E \end{bmatrix}.$$

Since it holds that

$$\begin{bmatrix} x_{k+i|k} \\ u_{k+i|k} \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2\Sigma_{i,N}C_0 \end{bmatrix}\eta_{k+i|k} + \begin{bmatrix} 0 \\ E_2\Sigma_{i,N}E_0 \end{bmatrix} w_{k+i|k},$$
$$E_1 = [I \quad 0], \ E_2 = [0 \quad I],$$

(4) can be represented as

$$J_{k,k+N}(\eta_k) = \sum_{i=0}^{N-1} S_{k+i|k}(\eta_{k+i|k}, \Sigma_{i,N}, w_{k+i|k}) + \eta_{k+N|k}^T P_f \eta_{k+N|k}, \tag{12}$$

where

$$\begin{aligned} S_{k+i|k}(\eta_{k+i|k}, \Sigma_{i,N}, w_{k+i|k}) &= \eta_{k+i|k}^T[E_1^T Q E_1 \\ &+ (E_2\Sigma_{i,N}C_0)^T R(E_2\Sigma_{i,N}C_0)]\eta_{k+i|k} \\ &+ \eta_{k+i|k}^T[(E_2\Sigma_{i,N}C_0)^T R(E_2\Sigma_{i,N}E_0)]w_{k+i|k} \\ &+ w_{k+i|k}^T[(E_2\Sigma_{i,N}E_0)^T R(E_2\Sigma_{i,N}C_0)]\eta_{k+i|k} \\ &+ w_{k+i|k}^T[(E_2\Sigma_{i,N}E_0)^T R(E_2\Sigma_{i,N}E_0)]w_{k+i|k} \\ &- \gamma^2 w_{k+i|k}^T w_{k+i|k}. \end{aligned} \tag{13}$$

Remark 1: We remark that the terminal weighting matrix plays a crucial role in guaranteeing the stability of the closed-loop system in the RHHC, which will be explained in Section 4. If the conventional RHHC where $Y = 0$ and $Z = 0$, and thus the terminal weighting matrix P_f is not full-rank, we shall have some difficulty in ensuring the closed-loop stability because the feasible region of the terminal weighting matrix in (25) of Section 4 becomes very narrow.

From now on, we derive the finite horizon output-feedback H_∞ optimal controller for the augmented system (11) and the cost (12). Let us denote

$$J_{k+i+1,k+N}^*(\eta_{k+i+1|k}) = \eta_{k+i+1|k}^T P_{i+1,N}\eta_{k+i+1|k} \tag{14}$$

as the optimal value of $J_{k+i+1,k+N}(\eta_{k+i+1|k})$. Then, the application of the dynamic programming to this problem leads to the recurrence equation; for $i = 0,\ldots,N-1$,

$$J_{k+i,k+N}(\eta_{k+i|k}) = S_{k+i|k}(\eta_{k+i|k}, \Sigma_{i,N}, w_{k+i|k}) + J_{k+i+1,k+N}^*(\eta_{k+i+1|k}) \tag{15}$$

with initial value

$$J_{k+N,k+N}^*(\eta_{k+N|k}) = \eta_{k+N|k}^T P_{N,N}\eta_{k+N|k}, \quad P_{N,N} \equiv P_f. \tag{16}$$

The saddle point disturbance vector $w_{k+i|k}^*$ can be found through the partial differentiation of $S_{k+i|k}(\eta_{k+i|k}, \Sigma_{i,N}, w_{k+i|k})$ with respect to $w_{k+i|k}$;

$$\begin{aligned} w_{k+1|k}^* &= [\gamma^2 I - (E_2\Sigma_{i,N}E_0)^T R(E_2\Sigma_{i,N}E_0) \\ &\quad - (D_0 + B_0\Sigma_{i,N}C_0)^T P_{i+1,N} \\ &\quad \times (D_0 + B_0\Sigma_{i,N}C_0)]^{-1}[(E_2\Sigma_{i,N}E_0)^T \\ &\quad \times R(E_2\Sigma_{i,N}C_0) + (D_0 + B_0\Sigma_{i,N}C_0) \\ &\quad \times P_{i+1,N}\ (A_0 + B_0\Sigma_{i,N}C_0)]\eta_{k+i|k}, \end{aligned} \tag{17}$$

which maximizes $J_{k+i,k+N}(\eta_{k+i|k})$. We remark that (17) can exist uniquely if and only if

$$0 < \gamma^2 I - (E_2\Sigma_{i,N}E_0)^T R(E_2\Sigma_{i,N}E_0) - (D_0 + B_0\Sigma_{i,N}C_0)P_{i+1,N}\ (D_0 + B_0\Sigma_{i,N}C_0), \tag{18}$$

which shall be called 'the existence condition' of the unique saddle point solution of the dynamic game in this paper. Here, we apply $w_{k+i|k}^*$ to $J_{k+i,k+N}(\eta_{k+i|k})$ to consider the worst-case situation. The resulting $J_{k+i,k+N}(\eta_{k+i|k})$, *say* $J_{k+i,k+N}^w(\eta_{k+i|k})$, becomes

$$\begin{aligned} &J_{k+i,k+N}^w(\eta_{k+i|k}) \\ &= \eta_{k+i|k}^T \left[\begin{bmatrix} E_2\Sigma_{i,N}C_0 \\ A_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix}^T \left\{ \begin{bmatrix} R & 0 \\ 0 & P_{i+1,N} \end{bmatrix}^{-1} \right.\right. \\ &\left. - \gamma^{-2} \begin{bmatrix} E_2\Sigma_{i,N}E_0 \\ D_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix} \begin{bmatrix} E_2\Sigma_{i,N}E_0 \\ D_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix}^T \right\}^{-1} \\ &\left. \begin{bmatrix} E_2\Sigma_{i,N}C_0 \\ A_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix} + E_1^T Q E_1 \right] \eta_{k+i|k}. \end{aligned} \tag{19}$$

Then the output-feedback worst-case (H_∞) optimal controller $\Sigma_{i,N}^*$ that minimizes $J_{k+i,k+N}^w$ can be obtained from the following linear programming (in fact, $\Sigma_{i,N}^*$ is hardly found analytically);

Minimize the trace of $P_{i,N}$ subject to (20)

$$P_{i,N} \geq \left[\begin{bmatrix} E_2\Sigma_{i,N}C_0 \\ A_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix}^T \left\{ \begin{bmatrix} R & 0 \\ 0 & P_{i+1,N} \end{bmatrix}^{-1} \right. \right.$$
$$\left. -\gamma^{-2} \begin{bmatrix} E_2\Sigma_{i,N}E_0 \\ D_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix} \begin{bmatrix} E_2\Sigma_{i,N}E_0 \\ D_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix}^T \right\}^{-1} \tag{21}$$
$$\times \begin{bmatrix} E_2\Sigma_{i,N}C_0 \\ A_0 + B_0\Sigma_{i,N}C_0 \end{bmatrix} + E_1^T QE$$

with the given $P_{i+1,N}$.

Using Schur's complement, (21) can be converted into the following inequality

$$\begin{bmatrix} -R^{-1} & E_2\Sigma_{i,N}C_0 & 0 & E_2\Sigma_{i,N}E_0 \\ (E_2\Sigma_{i,N}C_0)^T & E_1^T QE_1 - P_{i,N} & (A_0 + B_0\Sigma_{i,N}C_0)^T & 0 \\ 0 & A_0 + B_0\Sigma_{i,N}C_0 & -P_{i+1,N}^{-1} & D_0 + B_0\Sigma_{i,N}E_0 \\ (E_2\Sigma_{i,N}E_0)^T & 0 & (D_0 + B_0\Sigma_{i,N}E_0)^T & -\gamma^2 I \end{bmatrix} \leq 0. \tag{22}$$

We summarize the output-feedback RHHC design procedure without proof in the following theorem.

Theorem 1: (Dynamic output-feedback RHHC) Given P_f, if the finite horizon output-feedback optimal controller dynamics $\Sigma_{i,N}^*$, $i = 0,\ldots,\ N-1$, which minimize (19), are obtained recursively by the linear programming (20), then the resulting saddle-point value, *say* $J_{k,k+N}^*(\eta_k)$, of the cost (12) with $\Sigma_{i,N}^*$ $i = 0,\ldots,N-1$ is given by

$$J_{k,k+N}^*(\eta_k) = \eta_k^T P_{0,N}\eta_k, \tag{23}$$

and the dynamic output-feedback receding horizon H_∞ controller u_k is defined as $u_k = u_{k|k}^*$.

Remark 2: In the linear programming (20), we used the well-known trick often employed in optimization problems;

$$\min_{\Sigma} trace(M(\Sigma)) \Leftrightarrow \min_{\Sigma} trace(N) \ \ s.t. \ \ N \geq M(\Sigma). \tag{24}$$

Remark 3: If the second column and raw are removed from the matrix in (22), the resulting inequality, say 'L1', is equivalent to (18). Moreover, if (22) is satisfied, 'L1' is always satisfied. Therefore (18) is always satisfied if (22) is feasible.

Remark 4: For given $P_{i+1,N}$, the minimal $P_{i,N}$ in the trace norm sense is unique. Even in this case, however, there may exist a lot of $\Sigma_{i,N}$ satisfying (20).

Remark 5: Let us think about the trajectories of $P_{i,N}^1$ and $P_{i,N}^2$ (for $i = N,\ldots 0$) subject to (20). Because $P_{i+1,N}$ is inserted as an inverse form $P_{i+1,N}^{-1}$ in (22), the smaller $P_{i+1,N}$ is, the bigger the resulting $P_{i,N}$ is. Therefore, if the initial values of the two trajectories, i.e., $P_{N,N}^1$ and $P_{N,N}^2$ have the relation of $P_{N,N}^1 \leq P_{N,N}^2$, then it holds that $P_{N-1,N}^1 \leq P_{N-1,N}^2$. In the same manner, if $P_{i+1,N}^1 \leq P_{i+1,N}^2$, then it holds that $P_{i,N}^1 \leq P_{i,N}^2$ for all subsequent times $i = N-2,\ldots,0$.

The RHHC strategy is to solve the finite horizon output-feedback H_∞ optimal control problem at current time k and to implement its first solution, i.e $u_{k|k}^*$, then to repeat the procedure at the next time $k+1$.

4. STABILITY OF ORHHC

This section indicates that the proposed controller stabilizes the closed-loop system and guarantees the H_∞ norm bound. In the following theorem, an inequality condition on the terminal weighting matrix P_f is suggested, which plays a crucial role in guaranteeing the stability of the RHHC.

Theorem 2: (Cost monotonicity) Assume that the terminal weighting matrix P_f satisfies the following inequality

$$P_f \geq \left[\begin{bmatrix} E_2\Sigma_{N,N+1}C_0 \\ A_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix}^T \left\{ \begin{bmatrix} R & 0 \\ 0 & P_f \end{bmatrix}^{-1} -\gamma^{-2} \right. \right.$$
$$\left. \times \begin{bmatrix} E_2\Sigma_{N,N+1}E_0 \\ D_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix} \begin{bmatrix} E_2\Sigma_{N,N+1}E_0 \\ D_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix}^T \right\}^{-1} \tag{25}$$
$$\times \begin{bmatrix} E_2\Sigma_{N,N+1}C_0 \\ A_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix} + E_1^T QE_1.$$

Then $J_{k.k+N}^*(\eta_k)$ satisfies the following relation

$$J^*_{k,k+N}(\eta_k) \geq J^*_{k,k+N+1}(\eta_k). \tag{26}$$

Proof: The difference of the costs $J^*_{k,k+N+1}(\eta_k)$ $-J^*_{k,k+N}(\eta_k)$ can be written as follows:

$$\begin{aligned} &J^*_{k,k+N+1}(\eta_k) - J^*_{k,k+N}(\eta_k) \\ &\quad = J^*_{k,k+N+1,1}(\eta_k) + J^*_{k,k+N+1,2}(\eta^1_{k+N|k}) \\ &\qquad - J^*_{k,k+N,3}(\eta_k) + J^*_{k,k+N,4}(\eta^2_{k+N|k}), \end{aligned} \tag{27}$$

where

$$J^*_{k,k+N+1,1}(\eta_k) = \sum_{i=0}^{N-1} S_{k+i|k}(\eta^1_{k+i|k}, \Sigma^1_{i,N+1}, w^1_{k+i|k}), \tag{28}$$

$$\begin{aligned} J^*_{k,k+N+1,2}(\eta^1_{k+N|k}) = {} & S_{k+N|k}(\eta^1_{k+N|k}, \Sigma^1_{N,N+1} w^1_{k+N|k}) \\ & + \eta^{1T}_{k+N+1|k} P_f \eta^1_{k+N+1|k}, \end{aligned} \tag{29}$$

$$J^*_{k,k+N,3}(\eta_k) = \sum_{i=0}^{N-1} S_{k+i|k}(\eta^2_{k+i|k}, \Sigma^2_{i,N}, w^2_{k+i|k}), \tag{30}$$

$$J^*_{k,k+N,4}(\eta^2_{k+N|k}) = \eta^{2T}_{k+N|k} P_f \eta^2_{k+N|k}, \tag{31}$$

where $\Sigma^1_{i,N+1}, w^1_{k+i|k}, \eta^1_{k+i|k}$ (respectively, $\Sigma^2_{i,N+1}$, $w^2_{k+i|k}$, $\eta^2_{k+i|k}$) are minimizing controller dynamics, maximizing disturbance and corresponding state of $J^*_{k,k+N+1}(\eta_k)$ (respectively, $J^*_{k,k+N}(\eta_k)$). Let us replace $\Sigma^1_{i,N+1}$, $i = 0,\ldots,N-1$ (respectively, $\Sigma^1_{N,N+1}$, $w^2_{k+i|k}$) by $\Sigma^2_{i,N}$, $i = 0,\ldots,N-1$ (respectively, $\Sigma_{N,N+1}$, $w^1_{k+i|k}$) and denote the resulting costs as follows:

$$J^*_{k,k+N+1,1}(\eta_k) \Rightarrow J_{k,k+N+1,1}(\eta_k), \tag{32}$$

$$J^*_{k,k+N+1,2}(\eta^1_{k+N|k}) \Rightarrow J_{k,k+N+1,2}(\eta_{k+N|k}), \tag{33}$$

$$J^*_{k,k+N,3}(\eta_k) \Rightarrow J_{k,k+N,3}(\eta_k), \tag{34}$$

$$J^*_{k,k+N,4}(\eta^2_{k+N|k}) \Rightarrow J_{k,k+N,4}(\eta_{k+N|k}), \tag{35}$$

where $\eta_{k+N|k}$ is the trajectory constructed from η_k, $\Sigma^2_{i,N}$ and $w^1_{k+i|k}$ $i = 0,\ldots,N-1$. Therefore, the following inequalities are satisfied

$$\begin{aligned} &J^*_{k,k+N+1,1}(\eta_k) + J^*_{k,k+N+1,2}(\eta^1_{k+N|k}) \leq \\ &\qquad J_{k,k+N+1,1}(\eta_k) + J_{k,k+N+1,2}(\eta_{k+N|k}), \end{aligned} \tag{36}$$

$$\begin{aligned} &J^*_{k,k+N,3}(\eta_k) + J^*_{k,k+N,4}(\eta^2_{k+N|k}) \geq \\ &\qquad J_{k,k+N,3}(\eta_k) + J_{k,k+N,4}(\eta_{k+N|k}), \end{aligned} \tag{37}$$

$$J_{k,k+N+1,1}(\eta_k) = J_{k,k+N,3}(\eta_k). \tag{38}$$

Therefore, by (25), we have

$$\begin{aligned} &J^*_{kk+N+1}(\eta_k) - J^*_{kk+N}(\eta_k) \\ &\leq J_{kk+N+1}(\eta_k) + J_{kk+N+1,2}(\eta_{k+N|k}) \\ &\quad - J_{kk+N1,3}(\eta_k) - J_{kk+N,4}(\eta_{k+N|k}) \\ &= J_{kk+N+1,2}(\eta_{k+N|k}) - J_{kk+N,4}(\eta_{k+N|k}) \\ &= S_{k+N|k}(\eta_{k+N|k}, \Sigma_{N,N+1}, w_{k+N|k}) \\ &\quad + \eta^T_{k+N+1|k} P_f \eta_{k+N+1|k} - \eta^T_{k+N|k} P_f \eta_{k+N|k} \\ &= \eta^T_{k+N+1|k} \left[\begin{bmatrix} E_2 \Sigma_{N,N+1} C_0 \\ A_0 + B_0 \Sigma_{N,N+1} C_0 \end{bmatrix} \left\{ \begin{Bmatrix} R & 0 \\ 0 & P_f \end{Bmatrix}^{-1} - \gamma^{-2} \right.\right. \\ &\quad \left. \times \begin{bmatrix} E_2 \Sigma_{N,N+1} C_0 \\ D_0 + B_0 \Sigma_{N,N+1} C_0 \end{bmatrix} \begin{bmatrix} E_2 \Sigma_{N,N+1} E_0 \\ D_0 + B_0 \Sigma_{N,N+1} C_0 \end{bmatrix}^T \right\}^{-1} \\ &\quad \left. \times \begin{bmatrix} E_2 \Sigma_{N,N+1} C_0 \\ A_0 + B_0 \Sigma_{N,N+1} C_0 \end{bmatrix} + E_1^T Q E_1 - P_f \right] \eta_{k+N|k} \\ &\leq 0, \end{aligned}$$

where $w_{k+N|k}$ is eliminated by applying the maximizing value of it, i.e. $w^*_{k+N|k}$. □

Remark 6: Theorem 2 says that if it holds that, $J^*_{k+N,k+N+1} \leq J^*_{k+N,k+N}$, then $J^*_{k+i,k+N+1} \leq J^*_{k+i,k+N}$. We can generalize this result as follows. Assume that

$$J^*_{k+k',k+k_f+m} \leq J^*_{k+k',k+k_f}, \quad m \geq 1, \tag{39}$$

for some k'. Then, for any k'' ($0 \leq k'' \leq k'$)

$$J^*_{k+k'',k+k_f+m} \leq J^*_{k+k'',k+k_f}. \tag{40}$$

The proof of (40) is similar to that of Theorem 2. This result says that when the monotonicity of the optimal cost holds once, it holds for all subsequent times. Using this property, one can develop the block-shift RHHC.

In the following theorem, it is shown that the proposed ORHHC guarantees stability of the closed-loop system.

Theorem 3: (Stability) If the terminal weighting matrix P_f satisfies the inequality (25), then the ORHHC stabilizes the system (1) as k goes to infinity.

Proof: Since the stability of linear systems is not affected by the value of disturbance, we shall assume that $w\ w_i = 0, i \geq 0$ to prove the asymptotic stability of the closed-loop system. If P_f satisfies (25), $J^*_{k,k+N}(\eta_k) \geq J^*_{k,k+N+1}(\eta_k)$ by Theorem 2. Therefore,

$$\begin{aligned} J^*_{k,k+N}(\eta_k) &\geq J^*_{k,k+N+1}(\eta_k) \\ &= x_k^T Q x_k + u_k^{*T} R u_k + J^*_{k+1,k+N+1}(\eta_{k+1}) \\ &> J^*_{k+1,k+N+1}(\eta_{k+1}), \end{aligned}$$

which is the monotonic decreasing property of the cost function. Since the cost function $J_{k,k+N}(\eta_k)$ is strictly bounded by zero, we can conclude that $J^*_{k,k+N}(\eta_k)$ goes to zero as k goes to infinity. As a result, x_k goes to zero as k goes to infinity. The proof ends. □

If there exists such a P_f satisfying (25), it can be obtained by the linear programming. With some efforts, we can transform (25) into the following;

$$0 > W + U\Sigma_{N,N+1}V^T + V\Sigma_{N,N+1}^T U^T, \tag{41}$$

where

$$W = \begin{bmatrix} -R^{-1} & 0 & 0 & 0 \\ 0 & E_1^T Q E_1 - P_f & A_0^T P_f & 0 \\ 0 & P_f A_0 & -P_f & P_f D_0 \\ 0 & 0 & D_0^T P_f & -\gamma^2 I \end{bmatrix},$$

$$U = \begin{bmatrix} E_2 \\ 0 \\ P_f B_0 \\ 0 \end{bmatrix}, V = \begin{bmatrix} 0 \\ C_0^T \\ 0 \\ E_0^T \end{bmatrix}.$$

The variable elimination procedure [25] states that (41) is solvable for some $\Sigma_{N,N+1}$ if and only if

$$0 > U_\perp^T W U_\perp \text{ and } 0 > V_\perp^T W V_\perp, \tag{42}$$

where $U_\perp$ and $V_\perp$ are orthogonal complements of U and V, respectively.

In the following Corollary, a set of linear matrix inequalities, which is equivalent to (42), is presented without proof.

Corollary 4.1: Suppose that there exists positive definite matrices X and $\overline{X}$ subject to

$$\begin{bmatrix} -\overline{X} + \gamma^{-2}DD^T - BR^{-1}B^T + A\overline{X}A^T & A\overline{X} \\ \overline{X}A^T & -Q^{-1}\overline{X} \end{bmatrix} < 0, \tag{43}$$

$$\left(\begin{bmatrix} E^T \\ C^T \end{bmatrix}_\perp\right)^T \begin{bmatrix} -\gamma^2 I + D^T X D & D^T X A \\ A^T X D & Q - X + A^T X A \end{bmatrix} \begin{bmatrix} E^T \\ C^T \end{bmatrix} < 0 \tag{44}$$

$$\begin{bmatrix} X & I \\ I & \overline{X} \end{bmatrix} \geq 0, \tag{45}$$

where $([E\ \ C]^T)_\perp$ denotes an orthogonal complement of $[E\ \ C]^T$. Then, there exists a P_f satisfying (25). One of methods to construct P_f from X and $\overline{X}$ is as follows:

$$P_f = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}, (YZ^{-1}Y^T) = SVD(X - \overline{X}^{-1}), \tag{46}$$

where SVD denotes the singular value decomposition.

Theorem 4: (H_∞ norm bound) With zero initial condition $\eta_0 = 0$, if we apply the dynamic ORHHC to the system (1), then the H_∞ norm bound of the closed-loop system is guaranteed, i.e.,

$$\frac{\sum_{k=0}^{\infty} x_k^T Q x_k + u_k^{*T} R u_k^*}{\sum_{k=0}^{\infty} w_k^{*T} w_k^*} \leq \gamma^2. \tag{47}$$

Proof: Along the state trajectory of (1), we obtain

$$\begin{aligned} &-\eta_0^T P_{0,N} \eta_0 \\ &= \sum_{k=0}^{\infty} \{J^*_{k+1,k+N+1}(\eta_{k+1}) - J^*_{k,k+N}(\eta_k) \\ &\leq \sum_{k=0}^{\infty} \{J^*_{k+1,k+N}(\eta_{k+1}) - J^*_{k,k+N}(\eta_k)\} \\ &= \sum_{k=0}^{\infty} \{ -x_k^T Q x_k - u_k^{*T} R u_k^* + \gamma^2 w_k^{*T} w_k^* \}. \end{aligned}$$

Therefore, we obtain (47) with $\eta_0 = 0$. □

5. EXTENSIONS

5.1. Reference trajectory tracking problem

In this section, it is shown that the dynamic output feedback RHHC can be extended to the reference trajectory tracking problem such that the output of the plant should follow arbitrary reference trajectories. It is assumed that the arbitrary reference trajectories are given for finite future horizons.

For the tracking problem, the dynamic output-feedback tracking controllers have the following structure

$$z_{k+i+1|k} = A^c_{i,N} z_{k+i|k} + B^c_{i,N} y_{k+i|k} + F_{i,N}, \tag{51}$$

$$u_{k+i|k} = C^c_{i,N} z_{k+i|k} + D^c_{i,N} y_{k+i|k} + G_{i,N}, \tag{52}$$

and the cost function $J^{tr}_{k,k+N}$ is given as follows

$$\begin{aligned} J^{tr}_{k,k+N} &= \sum_{i=0}^{N-1} \{(y_{k+i|k} - y^r_{k+i|k})^T Q (y_{k+i|k} - y^r_{k+i|k}) \\ &+ u^T_{k+i|k} R\, u_{k+i|k} - \gamma^2 w^T_{k+i|k}\, w_{k+i|k}\} \\ &+ \begin{bmatrix} (y_{k+i|k} - y^r_{k+i|k}) \\ z_{k+N|k} \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} (y_{k+i|k} - y^r_{k+i|k}) \\ z_{k+N|k} \end{bmatrix}, \end{aligned}$$

where $y^r_{k+i|k}$, $i = 0$, ..., N are given reference trajectories. Using the state argumentation technique, we augment the reference trajectories into the state of (6) such that

$$\bar{x}_{k+1+i|k} = \bar{A}\bar{x}_{k+i|k} + \bar{B}u_{k+i|k} + \bar{D}w_{k+i|k}, \tag{53}$$

$$y_{k+i|k} = \bar{C}\bar{x}_{k+i|k} + Ew_{k+i|k}, \tag{54}$$

where

$$\bar{x}_{k+i|k} = \begin{bmatrix} x_{k+i|k} \\ 1 \end{bmatrix},\ \bar{A} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix},\ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix},\ \bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

Combining (51), (52), (53), and (54) provides a closed-loop system similar to (11), whose state consists of $x_{k+i|k}$, $z_{k+i|k}$ and 1. The rest of the work is similar to the procedure of the previous chapters. Owing to space limitations, we shall omit the details.

5.2. Problems for time-varying systems

We can also extend the result of this paper to regulation or tracking problems for *time-varying* systems. For time-varying systems, the development of the output feedback RHHC is also similar to the procedure of the previous chapters except for the stability issue. The stability condition for time-varying systems is as follows:

$$\begin{aligned} P_{k+N} \geq & \begin{bmatrix} E_2\Sigma_{N,N+1}C_0 \\ A_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix}^T \left\{ \begin{bmatrix} R & 0 \\ 0 & P_{k+N+1} \end{bmatrix}^{-1} \right. \\ & \left. -\gamma^{-2} \begin{bmatrix} E_2\Sigma_{N,N+1}E_0 \\ D_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix} \begin{bmatrix} E_2\Sigma_{N,N+1}E_0 \\ D_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix}^T \right\}^{-1} \\ & \times \begin{bmatrix} E_2\Sigma_{N,N+1}C_0 \\ A_0 + B_0\Sigma_{N,N+1}C_0 \end{bmatrix} + E_1^T Q E_1, \end{aligned} \tag{55}$$

which is the same as (25) except that we replace the constant terminal weighting matrix P_f with the time-varying terminal weighting matrices P_{k+N} and P_{k+N+1} corresponding to time k and $k+1$, respectively. Here we note that for certain time-varying systems, the terminal weighting matrix satisfying the condition (55) may not exist all the time k. In this case, we cannot always guarantee the stability of the closed-loop system. If we restrict target systems to periodically time-varying systems, we have more chance to find the terminal weighting matrix satisfying the condition (55) and to guarantee the stability of the closed-loop system. We shall show the tracking performance for the periodic system by an example in the following section.

6. SIMULATION RESULTS

6.1. Regulation problem for time-invariant systems

In this example, we consider the regulation problem of the proposed controller for the following system, which is a modified system of Kwakernaak and Sivan [26].

$$x_{k+1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.495 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.787 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} w_k, \tag{56}$$

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.1 \end{bmatrix} w_k. \tag{57}$$

The weighting matrices and initial values for the regulation problem are as follows; $Q = diag[1\ \ 1]$, $R = 0.01$, the disturbance attenuation level $\gamma = 1$, the control horizon $N = 4$, the order of dynamic controller $l = 2$, $u_0 = 0$ and $z_0 = [0\ \ 0]^T$. Here note that the existence of X and $\bar{X}$ satisfying (43)-(45) (and thus P_f) is connected with the values of Q, R and γ. Hence the design parameters must be selected to make the set of LMIs (43)-(45) feasible.

Target systems (56) and (57) suffer from unbounded disturbance. The regulation and tracking performances of the proposed controller are compared with those of observer-based ORHHC, where the state observer is designed without considering

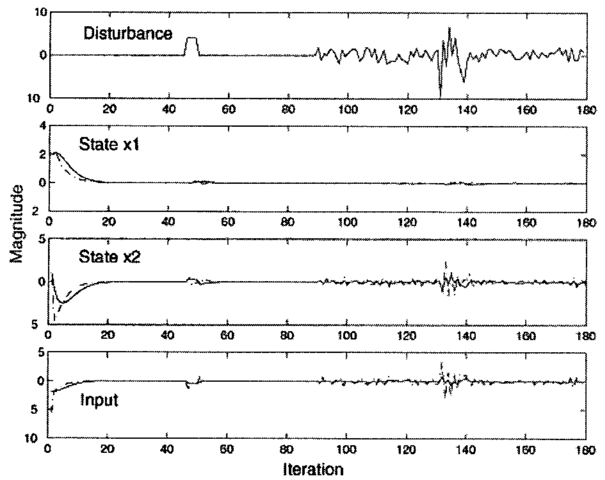

Fig. 1. Regulation performance of the proposed controller (solid-line) and the observer-based controller (dash-line).

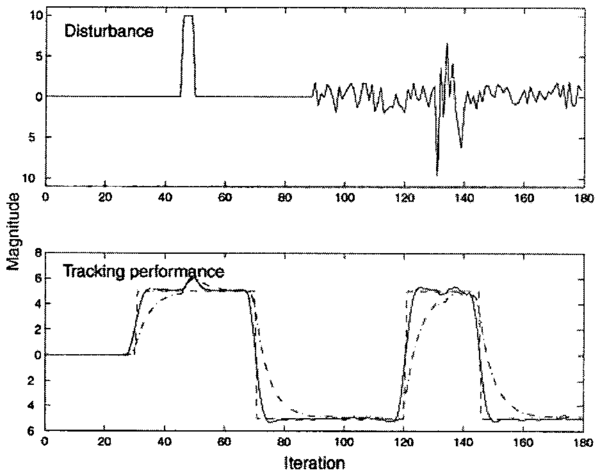

Fig. 2. Tracking performance for the periodic system: (dash-line) reference trajectory, (solid-line) the proposed controller, (dotted dash-line) the observer-based controller.

measurement noise. Simulation results demonstrate that the proposed controller yields better regulation performance than the observer-based controller, especially in the case of transient performance (Fig. 1).

6.2. Tracking problem for periodic systems

For simplicity, we demonstrate the tracking performance of the proposed controller for the following simple periodic system

$$x_{k+1}=\begin{bmatrix}0.1074+g_k & 0.0539 & 0.02\\ -0.1078 & 0.1591 & 0.12\\ 0.032 & -0.104 & 0.121\end{bmatrix}x_k+\begin{bmatrix}-0.0013\\ -0.0539\\ 0.0021\end{bmatrix}u_k, \tag{58}$$

$$y_k=\begin{bmatrix}0.2 & 0 & -1\end{bmatrix}x_k+0.1w_k, \tag{59}$$

where $g_k=0.4\sin(0.4\pi k)$. We chose the simulation parameters as follows: $Q=10\quad diag[1\ 1\ 1]$, $R=0.01$, $\gamma=1$, $N=4$, $u_0=0$, $l=3$ and $z_0=[0\ \ 0\ \ 0]^T$. The performance of the proposed controller is compared with that of observer-based receding horizon controller in Fig. 2. We chose the periodic observer gains through the general Lyapunov function method so that all the eigenvalues of the periodic observer systems are within the unit circle. There are four kinds of situations described in Fig. 2. For the first 90 iterations, there is no disturbance. Between the 45th and 50th iterations, the constant disturbance with magnitude 10 is excited. After the 90th iteration, the uniformly distributed random disturbances are excited. Between the 130th and 140th iterations, the magnitudes of disturbances become bigger due to certain reasons. We can easily know that the proposed controller yields better performance. We guess the reason as follows; some amount of estimation error that always exists in the estimated state causes tracking performance to be poor.

7. CONCLUDING REMARKS

In this paper, we introduced a dynamic ORHHC for linear discrete-time systems with unbounded disturbance, and extended it to tracking problems. By using a general form of the dynamic controller, the proposed controller overcame some drawbacks in the existing observer-based receding horizon controller. Using the matrix inequality condition on the terminal weighting matrix, the closed-loop system stability and the H_∞ norm bound were guaranteed. It was also shown that the proposed controller could be easily extended to tracking problems. Numerical examples demonstrated how the proposed controller improved the regulation and tracking performance compared with the observer-based receding horizon controller. The results of this paper can be easily extended to constrained or uncertain systems, which will be our next topic of research.

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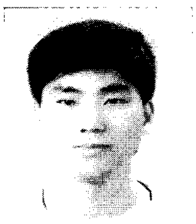

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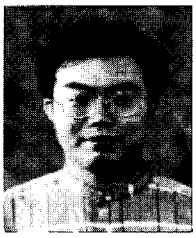

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