Optimal Number of Failures before Group Replacement under Minimal Repair

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Abstract

In this paper, a group replacement policy based on a failure count is analysed. For a group of identical repairable units, a maintenance policy is performed with two phase considerations: a repair interval phase and a waiting interval phase. Each unit undergoes minimal repair at failure during the repair interval. Beyond the interval, no repair is made until a number of failures. The expected cost rate expressions under the policy is derived. A method to obtain the optimal values of decision variables are explored. Numerical examples are given to demonstrate the results.
Keywords: minimal repair, group replacement, maintenance policy

1. Introduction

As a decision rule for repair or replacement when an operating unit fails, Barlow and Hunter [2] first introduced a periodic replacement policy with minimal repair at failure. Under the policy, the unit is replaced at intervals and undergoes minimal repair at failure between the periodic replacements. A minimal repair involves only that amount of work that is necessary to restore the unit to the average condition for a working unit of its age. Thus the failure rate of a unit after a minimal repair is equal to the failure rate immediately before the failure.

Based on their maintenance model, many authors have developed minimal repair policies for single-unit operating environments. In particular, to alleviate a forced repair immediately prior to the scheduled replacement time, which is inherent in [2], several modifications have been given. A nice summary in this area is found in Valdez-Flores and Feldman [18] and Beichelt [4].

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In many practical situations, however, the unit is not singly operated but a group of identical units are put in service simultaneously. In such a situation, by replacing or overhauling all old units instead of individual replacement, fairly much cost reduction would be realized. This cost saving, known as the economy of scale, results mostly from the reduction of maintenance set-up cost per unit.

For a group of independent identical units, Okumoto and Elsayed [14] proposed a policy in which a group maintenance is undertaken when a prescribed time interval elapses. Gertsbakh [6] considered the same problem as in [14] under the exponential time-to-failure (TTF), and showed that the optimum policy is to carry out the group replacement when the number of failed units exceeds a prescribed number. Assaf and Shanthikumar [1] also examined the same maintenance problem under continuous and periodic inspections. Ritchken and Wilson [15] proposed a group maintenance policy with two decision variables: A group maintenance is performed either when a fixed time interval is expired or when a fixed number of units are failed, whichever comes first.

In these studies, repair is not included as a possible maintenance activity immediately after a failure. That is, if a unit fails, it is left idle until the next group maintenance set-up. However, if units are repairable upon failures, the early failures will not need a long wait to a group maintenance time. Yoo[19] presented a group replacement policy for a group of repairable units. Units are undergoes minimal repair for failures within a repair interval, and no repair is made during a waiting time interval.

In this paper, another maintenance policy for a group of identical units under minimal repair is presented. Starting operation at time 0 with N identical new units, the operating procedures of the policy is as follows: Each unit undergoes minimal repair at failure during $(0, \tau]$. Beyond τ, no repair is made until the kth failure, $1 \le k \le N$. All units are replaced on the kth failure beyond τ.

When N=1, the policy reduces to Makabe and Morimura [8], Morimura [10], or Muth [11]. When τ=0, the policy becomes the policy of Gertsbakh [6] or Assaf and Shanthikumar [1].

Notations

N	number of units to operate, all units are identical in the statistical sense;
c_f	average cost of a minimal repair;
c_r	average cost of a group replacement per unit;
c_d	downtime cost per unit time per unit;
$f(x), F(x)$	pdf, cdf of TTF, $F(x)$is differentiable;
μ, σ	mean, standard deviation of TTF;
$h(x)$	failure rate at age x;
$H(x)$	cumulative hazard, $H(x)=\int_0^x h(y)dy$
$R(x)$	random variable representing the residual life at age x;
$G_x(y)$	distribution function of $R(x), G_x(y)=1-\overline{F}(x+y)/\overline{F}(x)$ where $\overline{F}(\cdot)=1-F(\cdot)$
$\mu(x)$	mean residual life at x, $\mu(x)=\int_0^\infty y dG_x(y)$

Assumptions

(1) Minimal repairs do not disturb the failure rate of each unit.

(2) The term "replacement" means any maintenance activity such as physical replacement or overhaul which returns the age of the unit to zero.

(3) The average cost of a minimal repair is less than the average cost of a group replacement per unit, i.e., $c_f < c_r$

(4) The presented policy is better than letting failed units idle indefinitely in terms of the expected cost rate. That is, the optimum(minimum) cost rate under the policy is less than c_d.

2. Expected Cost Rate under the Policy

To obtain the mean "waiting interval" under the policy, let $R_N^{(k)}(\tau)$ be the random variable representing the time to the kth failure from τ while the k-1 failed units are left idle, $1 \le k \le N$. Then $R_N^{(k)}(\tau)$ is the kth order statistic of N independent random samples from $G_\tau(x)$. From the theory of order statistics [9], the mean of $R_N^{(k)}(\tau)$, which is the mean waiting interval to the next group replacement is

$$\mu_N^{(k)}(\tau) = \sum_{i=0}^{k-1} \binom{N}{i} \int_0^\infty [G_t(x)]^i [\overline{G}_t(x)]^{N-i} dx \quad (1a)$$

$$= \overline{F}(t)^{-N} \sum_{i=0}^{k-1} \binom{N}{i} \int_t^\infty [\overline{F}(t) - \overline{F}(x)]^i [\overline{F}(x)]^{N-i} dx \quad (1b)$$

where $\overline{G}_t(\cdot) = 1 - G_t(\cdot)$

Under the policy, all units are renewed at intervals of τ plus the mean waiting interval beyond τ. Three cost factors are involved in the expected cost during this renewal cycle. The expected cost incurred for repairs and replacement is $c_r + c_f H(\tau)$per unit. The mean downtime per unit is, from (1a)

$$D_N^{(k)}(\tau) = \sum_{i=1}^{k-1} [\mu_N^{(k)}(\tau) - \mu_N^{(i)}(\tau)] \frac{1}{N}$$

$$= \sum_{i=1}^{k-1} \frac{i}{N} \binom{N}{i} \int_0^\infty [G_\tau(x)]^i [\overline{G}_\tau(x)]^{N-i} dx$$

$$= \overline{F}(\tau)^{-N} \sum_{i=1}^{k-1} \frac{i}{N} \binom{N}{i} \int_\tau^\infty [\overline{F}(\tau) - \overline{F}(x)]^i [\overline{F}(x)]^{N-i} dx \quad (2)$$

Therefore, the expected cost rate under the policy is, by the renewal reward theorem[16],

$$C(\tau, k) = \frac{c_r + c_f H(\tau) + c_d D_N^{(k)}(\tau)}{\tau + \mu_N^{(k)}(\tau)}. \quad (3)$$

3. Obtaining the Optimum Decision Variables

The optimum values of the policy variables, τ^* and k^*, are not obtainable in closed form. In this section, a method for obtaining these optimum values are explored when $h(x)$ is an IFR or a CFR.

Lemma 1. (a) Defining $\mu_N^{(k)}(\tau)\equiv 0$ for $N<1$ or $k<1$,

$$\frac{d\mu_N^{(k)}(\tau)}{d\tau} = Nh(\tau)[\mu_N^{(k)}(\tau)-\mu_{N-1}^{(k-1)}(\tau)]-1 \tag{4}$$

(b) If $h(x)$ is an IFR, the mean of order statistic of residual life at τ is decreasing in τ, that is, for k=1, ..., N,

$$\frac{d\mu_N^{(k)}(\tau)}{d\tau} < 0.$$

Proof. (a) Differentiating (1b) with respect to τ,

$$\frac{d\mu_N^{(k)}(\tau)}{d\tau} = N\overline{F}(\tau)^{-N-1}f(\tau)\sum_{i=0}^{k-1}\binom{N}{i}\int_\tau^\infty[\overline{F}(\tau)-\overline{F}(x)]^i[\overline{F}(x)]^{N-i}dx$$
$$+\overline{F}(\tau)^{-N}-f(\tau)\sum_{i=1}^{k-1}i\binom{N}{i}\int_\tau^\infty[\overline{F}(\tau)-\overline{F}(x)]^{i-1}[\overline{F}(x)]^{N-i}dx-\overline{F}(\tau)^N$$
$$=Nh(\tau)\mu_N^{(k)}(\tau)$$
$$-Nh(\tau)\overline{F}(\tau)^{-(N-1)}\sum_{i=0}^{k-2}\binom{N-1}{i}\int_\tau^\infty[\overline{F}(\tau)-\overline{F}(x)]^i[\overline{F}(x)]^{N-i-1}dx-1$$
$$=Nh(\tau)[\mu_N^{(k)}(\tau)-\mu_{N-1}^{(k-1)}(\tau)]-1$$

by the definition of $\mu_{N-1}^{(k-1)}(\tau)$ in 1(b).

(b) Using the well-known identity

$$\binom{N}{i}=\binom{N-1}{i-1}+\binom{N-1}{i}$$

and noticing that $G_\tau(x)+\overline{G}_\tau(x)=1$

$$\mu_{N-1}^{(k-1)}(\tau)=\sum_{i=0}^{k-2}\binom{N-1}{i}\int_0^\infty[G_\tau(x)]^i[\overline{G}_\tau(x)]^{N-i-1}dx$$

$$= \sum_{i=0}^{k-2}\binom{N-1}{i}\int_0^\infty [G_\tau(x)]^i[\overline{G}_\tau(x)]^{N-i-1}[\overline{G}_\tau(x)+G_\tau(x)]dx$$

$$= \sum_{i=0}^{k-2}\binom{N-1}{i}+\sum_{i=1}^{k-1}\binom{N-1}{i-1}\int_0^\infty [G_\tau(x)]^i[\overline{G}_\tau(x)]^{N-i}dx$$

$$= \sum_{i=0}^{k-2}\binom{N}{i}\int_0^\infty [G_\tau(x)]^i[\overline{G}_\tau(x)]^{N-i}dx+\binom{N-1}{k-2}\int_0^\infty [G_\tau(x)]^{k-1}[\overline{G}_\tau(x)]^{N-k+1}dx$$

Therefore, the bracket of (4) becomes

$$[\mu_N^{(k)}(\tau)-\mu_{N-1}^{(k-1)}(\tau)]=[\binom{N}{k-1}-\binom{N-1}{k-2}]\int_0^\infty [G_\tau(x)]^{k-1}[\overline{G}_\tau(x)]^{N-k+1}dx$$

$$=\binom{N-1}{k-1}\int_0^\infty [G_\tau(x)]^{k-1}[\overline{G}_\tau(x)]^{N-k+1}dx$$

By the IFR assumption stating $h(\tau)<h(\tau+x)$ for $x>0$,

$$Nh(\tau)[\mu_N^{(k)}(\tau)-\mu_{N-1}^{(k-1)}(\tau)]= h(\tau)N\binom{N-1}{k-1}\int_0^\infty [G_\tau(x)]^{k-1}[\overline{G}_\tau(x)]^{N-k+1}dx$$

$$=h(\tau)k\binom{N}{k}\int_0^\infty [G_\tau(x)]^{k-1}[\overline{G}_\tau(x)]^{N-k}\overline{F}(\tau+x)/\overline{F}(\tau)dx$$

$$<\int_0^\infty k\binom{N}{k}[G_\tau(x)]^{k-1}[\overline{G}_\tau(x)]^{N-k}f(\tau+x)/\overline{F}(\tau)dx \qquad (5)$$

$$=1$$

since the integrand of (5) represents the probability density function of $R_N^{(k)}(\tau)$.

If N=1 (thus k=1), (4) reduces to the well-known identity on the mean residual life [7]:

$$\frac{d\mu(\tau)}{d\tau}=h(\tau)\mu(\tau)-1$$

Lemma 2. If $h(x)$ is an IFR, for k=1, ..., N,

$$\mu_N^{(k)}(\tau) \le \rho \equiv \mu + \frac{(N-1)\sigma}{\sqrt{2N-1}}$$

Proof. The chain of induction is

$$\mu_N^{(k)}(\tau) \le \mu_N^{(N)}(\tau) \le \mu_N^{(N)}(0) \le \rho.$$

The first and second inequality follows by the definition of order statistic and Lemma 1(b) respectively. The final inequality, where $\mu_N^{(N)}(0)$ represents the mean

of the largest order statistic in a random sample of size N from $F(x)$, is the inequality (4.2.6) of David[5].

Theorem 1. (a) Suppose that $h(x)$ is an IFR and $c_f h(\infty) > c_d$. The optimum τ^* is in the bounded interval I, where

$$I = \left\{ \tau \mid A(\tau) \equiv \frac{c_r + c_f H(\tau)}{\tau + \rho} < c_d \right\}$$

The function $C(\tau, k)$ has exactly one local minimum in k for a fixed τ.

(b) If $h(x)$ is a CFR, $\tau^* \to \infty$.

Proof. (a) Since $C^* < c_d$ by Assumption 4, the optimum τ^* is in the set of points τ satisfying $C(\tau, k) < c_d$. Deleting the downtime cost component in the

numerator of $C(\tau, k)$ in (3) and noting Lemma 2, it follows that $A(\tau) \le C(\tau, k)$ for any k. Thus, the optimum τ^* is in the set of points τ satisfying $A(\tau) < c_d$, which is the set I. Note that the set I is not empty since $A(\tau^*) \le C(\tau^*, k^*) < c_d$ Further, it is easy to show that $A(\tau)$ is unimodal in τ with $A(\infty) = c_f h(\infty) > c_d$ Therefore the set I is a bounded interval.

Forming the inequality $C(\tau, k) < C(\tau, k+1)$ implies

$$J(k) \equiv k[\tau + \mu_N^{(k)}(\tau)]/N - D_N^{(k)}(\tau) > [c_r + c_f H(\tau)]/c_d, \; 1 \le k \le N-1$$

Since

$$J(k+1) - J(k) = [\tau + \mu_N^{(k)}(\tau)]/N > 0$$

$C(\tau, k)$ has exactly one (local) minimum in k for a fixed τ.

(b) If $F(x) = 1 - e^{-\lambda x}, \lambda > 0$ then $G_\tau(x) = F(x)$ From Corollary 2.7 of Barlow and Proschan [3],

$$\mu_N^{(k)}(\tau) = \sum_{i=0}^{k-1} \frac{1}{(N-i)\lambda} \tag{6}$$

In a way similar to the derivation of (2),

$$D_N^{(k)}(\tau) = \sum_{i=1}^{k-1} \frac{i}{N(N-i)\lambda} \tag{7}$$

Inserting (6) and (7) into (3), and differentiating it with respect to τ, it is easily shown that (3) is strictly decreasing function in τ for all k. Thus $C(\tau, k)$ is minimized at $\tau^* \to \infty$.

Using Theorem 1, a method for obtaining the finite optimum values τ^* and k^* is summarized as follows: <1> Find the interval I, then choose points τ_i, $i=1,2,\ldots,m$, equally dividing I. <2> For a fixed $\tau = \tau_i$ find the corresponding $k = k_i$ for the local minimum $C(\tau_i, k_i)$. <3> Repeat <2> for all i to find the global minimum $C^* = \min_i C(\tau_i, k_i)$ Slicing up the interval I, the higher numerical accuracy can be attained.

4. Numerical Examples

Suppose that a unit's TTF follows a Weibull failure distribution, $F(x) = 1 - \exp[-(\lambda x)^\beta]$ where the scale parameter is $\lambda = 1$/year and the shape parameter is $\beta = 2$ so that $\mu = 0.886$ and $\sigma = 0.463$ Its failure rate is $h(x) = 2x$ Since $h(x)$ is an IFR with $h(\infty) = \infty$ the interval I is bounded by Theorem 1.

Table 1 summarizes the optimum cost rates of the presented policy for a fleet size N=5 with several values of cost parameters. The table shows that (1)the interval I is bounded (Theorem 1), (2)the interval I becomes broaden as the downtime cost c_d increases, (3)the optimum cost rate and optimum repair interval τ^* increase as c_d increases.

< Table 1 > The optimum decision variables and cost rates($c_f = 25$, $c_r = 100$

c_d	I	τ^*	k^*	C^*
200	[0, 8.90)	1.80	3	**95.02**
300	[0, 13.07)	1.85	2	**96.32**
400	[0, 17.16)	1.88	2	**96.95**
500	[0, 21.22)	1.90	1	**97.49**

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