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FACTORIZATION AND DIVISIBILITY IN GENERALIZED REES RINGS

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Abstract. Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$ a generalized Rees ring, where t is an indeterminate. For suitable conditions, we show that R satisfies the ACCP (resp., is a BFD, an FFD, a (pre-)Schreier domain, a G-GCD domain, a PVMD, a v-domain) if and only if D satisfies the ACCP (resp., is a BFD, an FFD, a (pre-)Schreier domain, a G-GCD domain, a PVMD, a v-domain).

1. Introduction

Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$ a generalized Rees ring, where t is an indeterminate. In this paper, we study the various factorization properties and divisibility in the generalized Rees ring $R = D[It, t^{-1}]$. More precisely, for suitable conditions, we show that R satisfies the ACCP (resp., is a BFD, an FFD, a pre-Schreier domain, a Schreier domain, a G-GCD domain, a PVMD, a v-domain) if and only if D satisfies the ACCP (resp., is a BFD, an FFD, a pre-Schreier domain, a Schreier domain, a G-GCD domain, a PVMD, a v-domain).

General references for any undefined terminology or notation are [5, 8, 12, 13]. For an integral domain R, R^* is its set of nonzero elements and $U(R)$ is its group of units. Throughout this paper, $\mathbb{Z}$ denotes the set of integers. For two sets A and B, $A \subset B$ (or $B \supset A$) means that A is properly contained in B.

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2. Factorization properties

We first recall the various factorization properties which we will study in this section. Let R be an integral domain.

• [P. M. Cohn] R is *atomic* if each nonzero nonunit of R is a product of a finite number of irreducible elements (atoms) of R.

• R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R.

• [5, Anderson, Anderson, and Zafrullah] R is a *bounded factorization domain* (BFD) if R is atomic and for each nonzero nonunit of R there is a bound on the length of factorizations into products of irreducible elements.

• [5, Anderson, Anderson, and Zafrullah] R is a *finite factorization domain* (FFD) if each nonzero nonunit of R has only a finite number of nonassociate divisors (and hence, only a finite number of factorizations up to order and associates).

• [9, Anderson and Mullins] R is a *strong finite factorization domain* (SFFD) if each nonzero element of R has only finitely many divisors.

• [17, Zaks] R is a *half-factorial domain* (HFD) if R is atomic and whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$.

Let S be a grading monoid, i.e., a torsion-free cancellative monoid. We say that R is an *S-graded* integral domain if for each $s \in S$, there exists a subgroup R_s of the additive group of R such that

(1) $R = \bigoplus_{s \in S} R_s$ is the direct sum, as an abelian group, of the family $\{R_s\}$, and
(2) $R_s R_t \subseteq R_{s+t}$ for $s, t \in S$.

The next proposition shows that the factorization properties of a graded integral domain R contract to R_0.

Proposition 2.1. *Let $R = \bigoplus_{s \in S} R_s$ be a graded integral domain. Then R_0 satisfies the ACCP (resp., is a BFD, an FFD, an SFFD) if R satisfies ACCP(resp., a BFD, an FFD, an SFFD).*

Proof. Note that $U(R) \cap K_0 = U(R_0)$, where K_0 is the quotient field of R_0, and so $U(R) \cap R_0 = U(R_0)$. Thus if R satisfies the ACCP (resp., is a BFD, an FFD), then R_0 satisfies ACCP (resp., is a BFD, an FFD). If R is an SFFD, then R is an FFD and $U(R)$ is finite. Thus R_0 is an FFD and $U(R_0)$ is finite. Hence R_0 is an SFFD. $\square$

PROPOSITION 2.2. *Let $R = \bigoplus_{s \in S} R_s$ be a graded integral domain with $S \cap (-S) = \{0\}$. If R is atomic (resp., an HFD, a UFD), then R_0 is atomic (resp., an HFD, a UFD).*

Proof. Let x be a nonzero nonunit of R_0. Then $x = x_1 \cdots x_n$ with each x_i irreducible in R. Since $S \cap (-S) = \{0\}$, each $x_i \in R_0$. Note that $r \in R_0$ is irreducible in R_0 if and only if r is irreducible in R. Thus each x_i is irreducible in R_0. Thus R_0 is atomic. If R is an HFD, then there is a length function l_R on R. Define $l_{R_0} : R_0^* \longrightarrow \mathbb{Z}$ by $l_{R_0}(x) = l_R(x)$ for $x \in R_0^*$, i.e., $l_{R_0} = l_R|_{R_0}$. Then it is easy to show that l_{R_0} is a length function on R_0. Thus R_0 is an HFD. The UFD case appeared in [10, Proposition 6.3]. □

REMARK 1. In [10, p.96], D. F. Anderson gave the following example of $\mathbb{Z}$-graded integral domain R such that R is a UFD, but R_0 is not a UFD. Let K be a field and let $R = K[X, Y, Z, W]$, graded by *deg* X, $Y = 1$ and *deg* Z, $W = -1$. Then $R = \bigoplus_{n \in \mathbb{Z}} R_n$ with $R_0 = K[XZ, XW, YZ, YW]$. Thus R is a UFD, but R_0 is not a UFD since $Cl(R_0) = \mathbb{Z}$ [12, p.66].

Let D be an integral domain with quotient field K and let I be a proper ideal of D. If t is transcendental over D, let $R = D[It, t^{-1}]$ be the (generalized) Rees ring of D with respect to I. In general, the (generalized) Rees ring R is a $\mathbb{Z}$-graded ring with quotient field $K(t)$. Moreover, the element $u = t^{-1}$ is an irreducible element of R. On the other hand, since the ring R/uR is isomorphic to $G_I(D) = \sum_{n=0}^{\infty} I^n/I^{n+1}$, the associated graded ring with respect to I, it is clear that u is a prime element of R if and only if $G_I(D)$ is an integral domain.

In [16], Whitman showed that R is a UFD if and only if D is a UFD and u is a prime element of R. In [15, Proposition 3], J. Mott showed that R is a GCD-domain (respectively, UFD, pseudo-principal (every v-ideal is principal)) if and only if D is a GCD-domain (respectively, UFD, pseudo-principal) and u is a prime element of R. In [4], D. D. Anderson and D. F. Anderson investigated various factorization properties in the case when I is principal. To get the similar results, we need some definitions.

DEFINITION 2.3. Let R be a graded integral domain.

(1) R is said to be *graded atomic* if each nonzero nonunit homogeneous element of R is a product of a finite number of (homogeneous) irreducible elements of R.
(2) R is called a *graded BFD* if R is graded atomic and for each nonzero nonunit homogeneous element of R, there is a bound on the length

of factorizations into product of (homogeneous) irreducible elements.

(3) R is called a *graded FFD* if each nonzero nonunit homogeneous element of R has at most a finite number of nonassociate (homogeneous) irreducible divisors.

Recall from [8] that a saturated multiplicatively closed subset of an integral domain R is said to be a *splitting set* if for each $0 \neq d \in R$, we can write $d = sa$ for some $s \in S$ and $a \in R$ with $s'R \cap aR = s'aR$ for all $s' \in R$. A splitting set S is said to be an *lcm splitting set* if for each $s \in S$ and $d \in R$, $sR \cap dR$ is principal. Several characterizations of splitting set and lcm splitting sets are given in [8, Theorem 2.2 and Proposition 2.4].

PROPOSITION 2.4. *Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$. Assume that $u := t^{-1}$ is prime in R. Then we have the following results.*

(1) *u generates a splitting multiplicative set of R if and only if $\bigcap u^n R = \{0\}$.*
(2) *$\bigcap I^n = \{0\}$ if and only if $\bigcap u^n R = \{0\}$.*
(3) *If R is atomic, then $\bigcap u^n R = \{0\}$.*

Proof. (1) See [7, Proposition 1.6].

(2) Note that $u^n R \cap D = I^n$ for all $n \geq 0$, and so $(\bigcap u^n R) \cap D = \bigcap I^n$. Thus if $\bigcap u^n R = \{0\}$, then $\bigcap I^n = \{0\}$. Conversely, suppose that $\bigcap u^n R \neq \{0\}$. Then the intersection, being a homogeneous ideal, contains a nonzero homogeneous element, and hence a nonzero homogeneous element of degree 0. But then $(\bigcap u^n R) \cap D = \bigcap I^n \neq \{0\}$, a contradiction.

(3) By [7, Corollary 1.7] and (1), (3) holds. □

REMARK 2. Let I be a finitely generated ideal of an integral domain R with *rank* $I \leq 1$. Then $\bigcap I^n = \{0\}$.

PROPOSITION 2.5. *Let D be an integral domain, I an ideal of D, and $R = D[It, t^{-1}]$. Assume that R is an atomic domain and $u = t^{-1}$ is a prime element of R. Then the following conditions are equivalent.*

(1) *R satisfies the ACCP (respectively, is a BFD, an FFD).*
(2) *D satisfies the ACCP (respectively, is a BFD, an FFD).*
(3) *R satisfies the ACC on homogeneous principal ideals (respectively, is a graded BFD, graded FFD).*

Proof. (1) $\Rightarrow$ (2): Note that $U(R) \cap K = U(D)$, where K is the quotient field of D. Thus this follows from [5, p. 16].

(1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1): If D satisfies the ACCP (respectively, is a BFD, an FFD), then $D[t]$ satisfies the ACCP (respectively, is a BFD, an FFD). Thus, since the saturated multiplicatively closed set generated by the prime element t in $D[t]$ is a splitting multiplicative set, $D[t]_t = D[t, u]$ satisfies the ACCP (respectively, is a BFD, an FFD). Also note that the saturated multiplicatively closed set generated by the prime element u in R is a splitting multiplicative set [7, Corollary 1.7]. Hence, by [6, Theorem 3.1], R satisfies the ACCP (respectively, is a BFD, an FFD) since $D[t, u] = R_u$. □

3. Divisibility

Let R be an integral domain with the quotient field K and with the group of units $U = U(R)$. Let $K^* = K \setminus \{0\}$ and let $G(R) = K^*/U$. Then the group $G(R)$, called a *group of divisibility* of R, may be considered to be a directed partially ordered group with the order relation: $xU \leq yU$ if and only if there exists $r \in R$ such that $y = xr$, i.e., $x \mid y$ in R.

PROPOSITION 3.1. *Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$. Assume that t^{-1} is prime in R and $\cap I^n = \{0\}$. Then $G(R)$ is order-isomorphic to $G(D[t])$.*

Proof. Since $\cap I^n = \{0\}$, the saturated multiplicatively closed set generated by t^{-1} in R is a splitting multiplicative set with $R_{t^{-1}} = D[t, t^{-1}]$. Hence $G(R)$ is order-isomorphic to $G(D[t, t^{-1}]) \oplus_C \mathbb{Z}$. Since $\{t^n\}_{n \geq 0}$ is a splitting multiplicative set generated by the prime t in $D[t]$, we also have that $G(D[t])$ is order-isomorphic to $G(D[t, t^{-1}]) \oplus_C \mathbb{Z}$. Hence $G(R)$ is order-isomorphic to $G(D[t])$. □

Several classes of integral domains are characterized by their groups of divisibility. In particular, an integral domain R is a GCD-domain (resp., pseudo-principal domain, i.e., every v-ideal is principal.) if and only if $G(R)$ is lattice-ordered (resp., complete lattice-ordered), and R is a UFD if and only if $G(R)$ is a cardinal sum of copies of $\mathbb{Z}$. It is well-known that R is a pseudo-principal domain (resp., UFD, GCD-domain) if and only if $R[X]$ is a pseudo-principal domain (resp., UFD, GCD-domain). Thus we recover Mott's results as follows.

COROLLARY 3.2. *Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$. Assume that $\cap I^n = \{0\}$. Then R is a pseudo-principal domain (resp., UFD, GCD-domain) if and only if D is a pseudo-principal domain (resp., UFD, GCD-domain) and t^{-1} is prime in R.*

PROPOSITION 3.3. *Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$. Assume that $u := t^{-1}$ is prime in R.*

1. *R is integrally closed if and only if D is integrally closed.*
2. *Assume further that $\cap I^n = \{0\}$. Then R is completely integrally closed if and only if D is completely integrally closed.*

Proof. (1) This follows from $R_u = D[t, u] = D[t]_t$ and [6, Proposition 4.2].

(2) This follows from $R_u = D[t, u] = D[t]_t$ and [6, Proposition 4.3 and Corollary 4.5], since $\{t^n\}_{n \geq 0}$ (resp., $\{u^n\}_{n \geq 0}$) is a splitting multiplicative set generated by the prime t (resp., u) in $D[t]$ (resp., R). □

Let D be an integral domain. An element $x \in D$ is said to be *primal* if $x|ab$ implies $x = rs$, where $r|a$ and $s|b$. We call x *completely primal* if every factor of x is primal. An integral domain D is said to be *pre-Schreier* if every nonzero element of D is primal. Integrally closed pre-Schreier domains, called *Schreier* domains, were introduced by P. M. Cohn. In [11], Cohn also proved the following analog of Nagata's UFD Theorem which we call Cohn's Theorem: Let D be an integral domain and let S be a subset of D which is multiplicatively generated by completely primal elements of D. If D_S is pre-Schreier, then so is D.

PROPOSITION 3.4. *Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$.*

1. *Assume that t^{-1} is a completely primal element in R. Then R is a pre-Schreier domain if and only if D is a pre-Schreier domain.*
2. *Assume that t^{-1} is prime in R. Then R is a Schreier domain if and only if D is a Schreier domain.*

Proof. (1) If D is a pre-Schreier domain, then $D[t]$ is a pre-Schreier domain. Thus $D[t]_t = D[t, t^{-1}]$ is a pre-Schreier domain. Note that $R_{t^{-1}} = D[t, t^{-1}]$. It follows from Cohn's theorem that R is a pre-Schreier domain. Conversely, suppose that R is a pre-Schreier domain. We show that each nonzero element d of D is primal. Let $d|bc$ in D. Then $d|bc$ in R. Since R is a (pre-)Schreier domain, we have $d = fg$ such that $f|b$ and $g|c$ in R for some $f, g \in R$. Thus $f = b_1 t^m$ and $g = c_1 t^n$ for some $b_1, c_1 \in D$ and $m, n \in \mathbb{Z}$. But $d = fg$ shows that $m + n = 0$, and so

$f = b_1 t^m$ and $g = c_1 t^{-m}$. Clearly $b_1 | b$ in D and $c_1 | c$ in D and $d = b_1 c_1$. Hence d is primal. Thus D is a pre-Schreier domain.

(2) This follows from (1) and Proposition 3.3, since a Schreier domain is an integrally closed pre-Schreier domain. □

Recall that an integral domain R is called a *GCD domain* (resp., *G-GCD domain*, *PVMD*, and *v-domain*) if every finite type v-ideal of R is principal (resp., invertible, t-invertible, and v-invertible). These classes of domains are investigated by many authors. The following result is well-known.

LEMMA 3.5. *Let S be a multiplicatively closed subset of an integral domain R. If I is a v-ideal of R of finite type such that I^{-1} is also of finite type, then I_S is a v-ideal of R_S of finite type.*

The following result shows that if I is a finite type v-ideal of an integral domain R, then I is invertible if and only if I is locally principal. Note that it is a well-known result in the case when I is finitely generated [14, Theorem 62].

LEMMA 3.6. *Let I be a finite type v-ideal of an integral domain R. Then I is invertible if and only if I_M is principal for each maximal ideal M of R.*

Proof. Assume that I is invertible. Let M be a maximal ideal of R. Then I_M is also invertible in R_M. Since R_M is quasi-local, I_M is principal. Conversely, assume that I_M is principal for each maximal ideal M of R. If I is not invertible, then $II^{-1} \subseteq M$ for some maximal ideal M of R. Since I_M is principal, we can choose $a \in I$ such that $I_M = aR_M$. Let $I = (a_1, \ldots, a_n)_v$. Then for each $i = 1, \ldots, n$, we have $s_i a_i \in aR$ for suitable elements $s_i \in R \backslash M$. Let $s = s_1 \cdots s_n \in R \backslash M$. Then $sa^{-1} a_i \in R$ for each $i = 1, \ldots, n$ and hence $sa^{-1} \in R : (a_1, \ldots, a_n) = R : I = I^{-1}$. Thus $s = aa^{-1} s \in II^{-1} \subseteq M$, a contradiction. □

Recall that an integral domain R is called a *v-coherent* domain if for each nonzero finitely generated ideal I of R there exists a finitely generated ideal J such that $I^{-1} = J_v$, equivalently, for each v-ideal I of finite type, I^{-1} is a v-ideal of finite type. Note that the class of v-coherent domains includes G-GCD domains and PVMD's.

PROPOSITION 3.7. *The following conditions are equivalent for an integral domain R.*

(1) *R is a G-GCD domain.*

(2) *R_P is a GCD domain for each prime ideal P of R and R is a v-coherent domain.*
(3) *R_M is a GCD domain for each maximal ideal M of R and R is a v-coherent domain.*

Proof. (1) $\Rightarrow$ (2): This follows from [1, Corollary 1] and above remarks. (2) $\Rightarrow$ (3): Obvious. (3) $\Rightarrow$ (1): Let I be a v-ideal of finite type. Then we show that I is invertible. Let M be a maximal ideal of R. Since R is a v-coherent domain, I^{-1} is a v-ideal of finite type and hence I_M is also a v-ideal of R_M of finite type by Lemma 3.5. Since R_M is a GCD domain, I_M is principal. Therefore I is invertible by Lemma 3.6. □

Recall that an integral domain R is a π-domain if every nonzero principal ideal of R is a (finite) product of prime ideals, equivalently, every t-ideal is invertible. An integral domain R is called a *Mori* domain if it satisfies the ACC on v-ideals. Note that if R is a Mori domain, then every v-ideal is of finite type and hence every t-ideal of R is a v-ideal (the converse is always true).

COROLLARY 3.8. *The following conditions are equivalent for an integral domain R.*
(1) *R is a π-domain.*
(2) *R_M is a UFD for each maximal ideal M of R and R is a Mori domain.*
(3) *R_M is a π-domain for each maximal ideal M of R and R is a Mori domain.*

Proof. (1) $\Rightarrow$ (2): This follows from [13, Theorem 46.7]. (2) $\Rightarrow$ (3): This follows from [13, Theorem 46.5 and Theorem 46.7]. (3) $\Rightarrow$ (1): This follows from Proposition 3.7 and above remark. □

We denote by $\mathscr{D}_f(R)$ the set of all finite type v-ideals of an integral domain R. The other notations below follow from [8].

PROPOSITION 3.9. *Let S be an lcm splitting multiplicative set for an integral domain R. Then R is a G-GCD domain (resp., a v-domain) if and only if R_S is a G-GCD domain (resp., a v-domain).*

Proof. Let S be an lcm splitting multiplicative set with the m-complement S'. By [8, Theorem 3.7] we have an isomorphism $\mathscr{D}_f(R) \to \mathscr{D}_f(R_S) \oplus_C \mathscr{D}_f(R_{S'})$ that takes $Inv(R)$ (resp., $Inv_v(R)$) to $Inv(R_S) \oplus_C Inv(R_{S'})$ (resp., $Inv_v(R_S) \oplus_C Inv_v(R_{S'})$). It follows from [8, Theorem 4.1] that $R_{S'}$ is a GCD-domain. Thus $\mathscr{D}_f(R_{S'})=Inv(R_{S'})=Inv_v(R_{S'}) = P(R_{S'})$. Thus R is a G-GCD domain if and only if $\mathscr{D}_f(R_S)=Inv(R_S)$ if and only if R_S is a G-GCD domain. Also, R is a v-domain if and only if $\mathscr{D}_f(R_S)=Inv_v(R_S)$ if and only if R_S is a v-domain. □

COROLLARY 3.10. *Let D be an integral domain, I a proper ideal of D, and $R = D[It, t^{-1}]$. Assume that t^{-1} is prime in R and $\cap I^n = \{0\}$. Then R is a G-GCD domain (resp., a PVMD, a v-domain) if and only if D is a G-GCD domain (resp., a PVMD, a v-domain).*

Proof. Let S be the saturated multiplicative set generated by the prime t^{-1} in R. Then S is an lcm splitting multiplicative set. Thus the assertions follow immediately from Proposition 3.9. □

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