

## THE UNIFORM CONSISTENCY OF THE SAMPLE KERNEL QUANTILE PROCESS

JONGSIG BAE AND SUNGYEUN KIM

ABSTRACT. We obtain a kernel quantile process based on the kernel quantile estimator and prove the uniform consistency of the kernel quantile process by developing that of the usual sample quantile process. We apply our result to the classical kernel type processes.

### 1. Introduction and the results

Suppose that  $X_1, X_2, \dots, X_n$  are independent and identically distributed random variables with distribution function  $F$ . whose density function  $f : \mathbb{R} \rightarrow [0, \infty)$  with  $\int f(x)dx = 1$ . The unknown density  $f$  is often estimated by the classical density estimator

$$\hat{f}(x) := \hat{f}(x)(k, \alpha) = \frac{1}{n} \sum_{i=1}^n \frac{1}{\alpha} k \left( \frac{x - X_i}{\alpha} \right)$$

where  $k$  is a kernel which is bounded probability density function on  $\mathbb{R}$  with finite second moment, and is symmetric about the origin. Let  $\alpha := \alpha(n)$  be a sequence of bandwidths such that  $\alpha > 0$ ,  $\alpha \rightarrow 0$  as  $n \rightarrow \infty$ .

Observe that the density estimator  $\hat{f}$  depends on the kernel  $k$  and a sequence of bandwidths  $\alpha := \alpha(n)$ . We name  $(k, \alpha)$  as the *kernel bandwidth couple*.

Let  $p \in (0, 1)$  be fixed. Suppose  $F$  is smooth near the  $p$ -quantile  $F^{-1}(p)$  where  $F^{-1}(p) := \inf\{x : F(x) \geq p\}$ .

Observe that the kernel bandwidth couple  $(k, \alpha)$  depends on  $p$ . Hereafter, we write  $(k, \alpha)$  as  $(k_p, \alpha_p)$  in order to stress the dependency on  $p$ .

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For a fixed  $p \in (0, 1)$ , one often tries to infer the unknown quantity  $\hat{Q}(F, p)$  defined by

$$(1) \quad \hat{Q}(F, p) := \int_0^1 F^{-1}(x) \frac{1}{\alpha_p} k_p \left( \frac{p-x}{\alpha_p} \right) dx.$$

The quantity  $\hat{Q}(F, p)$  is known as the kernel type quantile. Observe that we have suppressed the  $n$  in  $\hat{Q}(F, p)$  as in case of  $\alpha$ . It has been estimated by the kernel type sample quantile  $\hat{Q}_n(F_n, p)$  given by

$$(2) \quad \hat{Q}_n(F_n, p) := \int_0^1 F_n^{-1}(x) \frac{1}{\alpha_p} k_p \left( \frac{p-x}{\alpha_p} \right) dx$$

where  $F_n^{-1}(p) := \inf\{x : F_n(x) \geq p\}$  denotes the usual sample  $p$  quantile based on  $X_1, \dots, X_n$ . See Falk [4] for the asymptotic normality of  $\hat{Q}_n(F_n, p)$ .

It is natural to ask

QUESTION 1. *What are the uniform behaviors of  $\{\hat{Q}_n(F_n, p) : p \in (0, 1)\}$  as a process indexed by  $p$ ?*

In this paper we prove the uniform consistency of the kernel quantile process by developing that of the usual sample quantile process.

We firstly get the following uniform consistency of the usual sample quantile process.

THEOREM 1. *Assume that  $F$  has the positive derivative on  $(0, 1)$ . Then, almost surely,*

$$\sup_{0 < p < 1} |F_n^{-1}(p) - F^{-1}(p)| \rightarrow 0.$$

REMARK.

1. In getting the result, we use a Glivenko-Cantelli theorem and a uniform analogue of the Bahadur representation.
2. Convergence in the mean can be obtained in a similar fashion by using a mean convergence version of Glivenko-Cantelli theorem. See, for example, Bae and Kim [1].

Consider a class  $\{(k_p, \alpha_p) : p \in (0, 1)\}$  of kernel bandwidth couples. We name  $\{\hat{Q}_n(F_n, p) : p \in (0, 1)\}$ , given in (2), as the kernel quantile process.

We secondly get the following uniform consistency for the sample kernel quantile process.

**THEOREM 2.** *Assume that  $F$  has the positive derivative on  $(0, 1)$ . Suppose that the class  $\{k_p : p \in (0, 1)\}$  of kernels has an envelope  $K$  with  $\int K(x)dx < \infty$  and  $0 < \alpha_p := \alpha_p(n) \rightarrow 0$  uniformly in  $p$  as  $n \rightarrow \infty$ . Then*

$$\sup_{0 < p < 1} |\hat{Q}_n(F_n, p) - \hat{Q}(F, p)| \rightarrow 0 \text{ almost surely.}$$

**REMARK.**

1. Under the smoothness assumption of  $F^{-1}$ ,  $\hat{Q}(F, p)$  may be replaced by  $F^{-1}(p)$ . See Remark 1.4 in Falk [4].
2. We may allow the bandwidth  $\alpha$  to be random. See for example Sheather and Marron [7].

**2. Proofs**

We begin by the following lemmas. Let  $\xi_n(p) := F_n^{-1}(p)$  and  $\xi(p) := F^{-1}(p)$ .

**LEMMA 1.** *If  $\xi(p) = 0$  for a fixed  $p \in (0, 1)$  and  $\alpha < 1/2$  then  $n^\alpha \xi_n(p) \rightarrow 0$  almost surely.*

*Proof.* See Lemma 4.6.2 in Fabian and Hannan [3]. □

**LEMMA 2.** *Suppose that  $\xi(p) = 0$ . Let  $\alpha$  and  $\gamma$  be numbers with  $0 < \alpha < \gamma < (\alpha + 1)/2$ ,*

$$I_n = (-n^{-\alpha}, n^{-\alpha}),$$

$$G_n(x) = F_n(x) - F_n(0) - (F(x) - F(0))$$

and

$$H_n = \sup \{|G_n(x)| : x \in I_n\}.$$

Then

$$P \{n^\gamma H_n \rightarrow 0\} = 1.$$

*Proof.* See Lemma 4.6.3 in Fabian and Hannan [3]. □

We have the following representation of sample quantile. We write  $d(p) := F'(\xi(p))$ , the derivative of  $F$  at  $\xi(p)$ .

LEMMA 3. Assume that  $F$  has the positive derivative  $d$ . Then

$$(3) \quad \xi_n(p) - \xi(p) = d(p)^{-1}[p - F_n(\xi(p))] + R_n(p)$$

with

$$\sup_{0 < p < 1} |R_n(p)| \rightarrow 0 \text{ almost surely.}$$

REMARK. The classical Bahadur representation states that for a fixed  $p \in (0, 1)$  the equation (3) is valid with  $n^{1/2}|R_n(p)| \rightarrow 0$  in probability. See Theorem 4.6.4 in Fabian and Hannan [3]. See also Bahadur [2].

In order to prove Lemma 3, we will modify the proof of Theorem 4.6.4 in Fabian and Hannan [3].

*Proof.* Let  $p \in (0, 1)$  be fixed. Let  $I_n, G_n$  be as in Lemma 2. By definition of  $\xi_n(p)$ , we have

$$-n^{-3/4} + F_n(\xi_n(p)-) \leq p \leq F_n(\xi_n(p)) + n^{-3/4}.$$

From this we get

$$(4) \quad F(\xi_n(p)) - F(\xi(p)) + F_n(\xi(p)) - p \geq -G_n(\xi_n(p)) - n^{-3/4}.$$

By Lemma 1,  $n^\alpha \xi_n(p) \rightarrow 0$  almost surely and  $P\{\xi_n(p) \notin I_n \text{ infinitely often}\} = 0$ . By Lemma 2, almost surely

$$(5) \quad n^{1/2}(G_n(\xi_n(p)) + n^{-3/4}) \rightarrow 0 \text{ uniformly in } p.$$

(5) implies that almost sure convergence of  $G_n(\xi_n(p)) + n^{-3/4}$  to zero uniformly in  $p$ . Since  $F'(\xi(p)) = d(p)$ , we have  $F(\xi_n(p)) = F(\xi(p)) + d_n(p)\xi_n(p)$  with  $d_n(\cdot) \rightarrow d(\cdot)$  uniformly in  $p$  almost surely. Use this in (4) to obtain, on  $\{\inf_p d_n(p) > 0\}$ ,

$$\xi_n(p) + \frac{F_n(\xi(p)) - p}{d_n(p)} \geq -\frac{G_n(\xi_n(p)) + n^{-3/4}}{d_n(p)}$$

and

$$R_n(p) \geq \left( \frac{1}{d(p)} - \frac{1}{d_n(p)} \right) (F_n(\xi(p)) - p) - \frac{G_n(\xi_n(p)) + n^{-3/4}}{d_n(p)}.$$

We may assume that  $d$  is bounded above, and hence  $d_n$  is bounded above. Observe that

$$EF_n(\xi(p)) = P\{X_1 \leq \xi(p)\} = F(\xi(p)) = p.$$

Then, by Glivenko-Cantelli theorem,  $\sup_p |F_n(\xi(p)) - p|$  converges to zero almost surely. It follows that

$$(6) \quad \liminf_{n \rightarrow \infty} \inf_p R_n(p) \geq 0 \text{ almost surely.}$$

By symmetry (consider  $-X_i$ ) we obtain

$$(7) \quad \limsup_{n \rightarrow \infty} \sup_p R_n(p) \leq 0 \text{ almost surely.}$$

(6) and (7) imply that

$$\sup_p |R_n(p)| \rightarrow 0 \text{ almost surely.}$$

The proof of Lemma 3 is completed.  $\square$

Write  $S_n(p) = p - F_n(\xi(p))$ . Under the assumption that  $F' = d > 0$ , Lemma 3 allows us to write

$$\xi_n(p) - \xi(p) = S_n(p)/d(p) + R_n(p)$$

with

$$(8) \quad \sup_p |R_n(p)| \rightarrow 0 \text{ almost surely.}$$

Observe that

$$\begin{aligned} S_n(p) : &= p - F_n(\xi(p)) \\ &= -(F_n(\xi(p)) - EF_n(\xi(p))) \\ &= -n^{-1} \sum_{i=1}^n [\{X_i \leq \xi(p)\} - P\{X_1 \leq \xi(p)\}] \\ &= -(F_n - F)(\xi(p)). \end{aligned}$$

Notice that a  $(0, 1)$ -indexed integral process  $S_n(p)$  can be regarded as an  $\mathbb{R}$ -indexed empirical process  $-(F_n - F)(\xi(p))$  by a transformation from  $(0, 1)$  onto  $\mathbb{R}$ .

We now perform the proof of Theorem 1.

*Proof of Theorem 1.* Observe that, by (8), we get  $\sup_p |R_n(p)| \rightarrow 0$  almost surely. In order to finish the proof for Theorem 1, it remains to prove that  $\sup_p |S_n(p)/d(p)| \rightarrow 0$  almost surely. Glivenko-Cantelli theorem, together with the uniform boundedness of  $d$ , gives that  $\sup_{\xi(p) \in \mathbb{R}} |(F_n - F)(\xi(p))/d(p)| \rightarrow 0$  almost surely. The proof is completed.  $\square$

*Proof of Theorem 2.* Observe, by a change of variables, that

$$\begin{aligned} \sup_{0 < p < 1} \int_0^1 \frac{1}{\alpha_p} k_p \left( \frac{p-x}{\alpha_p} \right) dx &= \sup_{0 < p < 1} \int_{\frac{p-1}{\alpha_p}}^{\frac{p}{\alpha_p}} k_p(y) dy \\ &\leq \int K(y) dy. \end{aligned}$$

Notice that

$$\begin{aligned} &\sup_{0 < p < 1} |\hat{Q}_n(F_n, p) - \hat{Q}(F, p)| \\ &= \sup_{0 < p < 1} \left| \int_0^1 (F_n^{-1}(x) - F^{-1}(x)) \frac{1}{\alpha_p} k_p \left( \frac{p-x}{\alpha_p} \right) dx \right| \\ &= \sup_{0 < x < 1} |F_n^{-1}(x) - F^{-1}(x)| \sup_{0 < p < 1} \int_0^1 \frac{1}{\alpha_p} k_p \left( \frac{p-x}{\alpha_p} \right) dx \\ &\leq \int K(y) dy \cdot \sup_{0 < x < 1} |F_n^{-1}(x) - F^{-1}(x)|. \end{aligned}$$

The proof is finished by applying Theorem 1.  $\square$

### 3. Applications

Suppose that  $X_1, X_2, \dots, X_n$  are independent and identically distributed random variables with absolutely continuous distribution function  $F$ . Let  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  be the order statistics of  $X_i$ 's.

Throughout this section we suppose that  $K$  be a fixed kernel and  $\alpha := \alpha(n)$  be a fixed sequence of bandwidths. Let

$$K_\alpha(\cdot) = \frac{1}{\alpha} K \left( \frac{\cdot}{\alpha} \right).$$

We list assumptions needed to derive the uniform consistency. See Yang [8].

1.  $F$  has a pdf  $f$ , which is continuous and positive on  $(0, 1)$ .
2.  $f'$  exists and is continuous on  $(0, 1)$ .
3.  $K$  is a pdf with finite support.
4.  $K$  is bounded.
5.  $\lim_{x \rightarrow \infty} x^a [F(-x) + 1 - F(x)] = 0$  for some  $a > 0$ .

The kernel type quantile function  $\hat{Q}(F, p)$ , given in (1), is boiled down to

$$\hat{Q}_K(F, p) := \int_0^1 F^{-1}(t)K_\alpha(t - p)dt.$$

As the first example, we consider the kernel quantile process  $\{\hat{Q}_K(F_n, p) : p \in (0, 1)\}$  defined by

$$\hat{Q}_K(F_n, p) := \int_0^1 F_n^{-1}(t)K_\alpha(t - p)dt.$$

Trace back the process to Parzen [6].

**COROLLARY 1.** *Suppose the assumptions 1 – 4 are satisfied. Then,*

$$\sup_{0 < p < 1} |\hat{Q}_K(F_n, p) - \hat{Q}_K(F, p)| \rightarrow 0 \text{ almost surely.}$$

*Proof.* Observe that the kernel  $K$  is itself envelope with  $\int K(x)dx = 1$ . Theorem 2 completes the proof.  $\square$

Observe that

$$\begin{aligned} \hat{Q}_K(F_n, p) &= \int_0^1 F_n^{-1}(t)K_\alpha(t - p)dt \\ &= \sum_{i=1}^n X_{(i)} \int_{(i-1)/n}^{i/n} K_\alpha(t - p)dt \end{aligned}$$

which is an  $L$  process.

As the second example, we consider the  $L$  process  $\{S_n(F_n, p) : p \in (0, 1)\}$  defined by

$$S_n(F_n, p) := \sum_{i=1}^n X_{(i)}n^{-1}K_\alpha\left(\frac{i}{n} - p\right).$$

This process is proposed by Yang [8] as a modification of

$$\{\hat{Q}_K(F_n, \cdot)\}.$$

The following lemma is appeared in Yang [8].

**LEMMA 4.** *Suppose the assumptions 3 and 5 are satisfied and  $K$  satisfies a Lipsuitz condition. Then*

$$E \sup_{0 < p < 1} |S_n(F_n, p) - \hat{Q}_K(F_n, p)|^2 = o\left[\frac{1}{n\alpha^2(n)a_n^2}\right]$$

where  $a_n \rightarrow \infty$  and  $a_n/(n\alpha(n)) \rightarrow 0$  as  $n \rightarrow \infty$ .

COROLLARY 2. Suppose the assumptions 1 – 5 are satisfied and  $K$  satisfies a Lipsuitz condition. Then,

$$\sup_{0 < p < 1} |S_n(F_n, p) - \hat{Q}_K(F, p)| \rightarrow 0 \text{ in the mean.}$$

*Proof.* Using Lemma 4, we have

$$(9) \quad \sup_{0 < p < 1} |S_n(F_n, p) - \hat{Q}_K(F_n, p)| \rightarrow 0 \text{ in the mean.}$$

Corollary 1 implies that

$$(10) \quad \sup_{0 < p < 1} |\hat{Q}_K(F_n, p) - \hat{Q}_K(F, p)| \rightarrow 0 \text{ in the mean.}$$

The proof is completed by combining (9) and (10).  $\square$

REMARK. For the choice of  $a_n$  and  $\alpha(n)$  satisfying

$$(11) \quad \sum_{n=1}^{\infty} \frac{1}{n\alpha^2(n)a_n^2} < \infty,$$

the result of Corollary 1 can be strengthened to almost sure convergence. To see this, observe that, for  $\epsilon > 0$ , we have

$$\begin{aligned} & \sum_{n=1}^{\infty} P \left( \sup_{0 < p < 1} |S_n(F_n, p) - \hat{Q}_K(F_n, p)| > \epsilon \right) \\ & \leq \epsilon^{-2} \sum_{n=1}^{\infty} E \sup_{0 < p < 1} |S_n(F_n, p) - \hat{Q}_K(F_n, p)|^2 \\ & < \infty \end{aligned}$$

by Lemma 4 and (11). This implies that

$$\sup_{0 < p < 1} |S_n(F_n, p) - \hat{Q}_K(F_n, p)| \rightarrow 0 \text{ almost surely.}$$

See for example Proposition 5.7 in Karr [5]. This together with Corollary 1 finishes the proof.

As the third example we consider  $\{KQ_2(F_n, p) : p \in (0, 1)\}$  defined by

$$KQ_2(F_n, p) := n^{-1} \sum_{i=1}^n X_{(i)} K_{\alpha} \left( \frac{i - \frac{1}{2}}{n} - p \right).$$

See Sheather and Marron [7].



**COROLLARY 3.** *Suppose the assumptions 1 – 5 are satisfied and  $K$  satisfies a Lipsuitz condition. Then,*

$$\sup_{0 < p < 1} |KQ_2(F_n, p) - \hat{Q}_K(F, p)| \rightarrow 0 \text{ almost surely.}$$

*Proof.* Let  $M$  be the Lipsuitz constant, then

$$\begin{aligned} & \sup_{0 < p < 1} |S_n(F_n, p) - KQ_2(F_n, p)| \\ & \leq n^{-1} \sum_{i=1}^n X_{(i)} \sup_{0 < p < 1} \left| K_\alpha \left( \frac{i}{n} - p \right) - K_\alpha \left( \frac{i - \frac{1}{2}}{n} - p \right) \right| \\ & \leq M(2n)^{-1} n^{-1} \sum_{i=1}^n X_{(i)} \rightarrow 0 \text{ almost surely.} \end{aligned}$$

This together with Corollary 2 finishes the proof. □

**REMARK.** Consider the kernel type processes  $KQ_3$  and  $KQ_4$  given by

$$KQ_3(F_n, p) := n^{-1} \sum_{i=1}^n X_{(i)} K_\alpha \left( \frac{i}{n+1} - p \right)$$

and

$$KQ_4(F_n, p) := \sum_{i=1}^n K_\alpha \left( \frac{i - \frac{1}{2}}{n} - p \right) X_{(i)} / \sum_{j=1}^n K_\alpha \left( \frac{j - \frac{1}{2}}{n} - p \right).$$

See Sheather and Marron [7]. Then, one can employ the similar reasoning to establish uniform asymptotic equivalences between  $S_n(F_n, p)$  and  $KQ_i(F_n, p)$  for  $i = 3, 4$ .

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DEPARTMENT OF MATHEMATICS AND INSTITUTE OF BASIC SCIENCE, SUNG-KYUNKWAN  
UNIVERSITY, SUWON 440-746, KOREA

*E-mail*: jsbae@skku.edu

sykim@math.skku.ac.kr