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Reliability Equivalence Factors of a Bridge Network System

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Abstract. Improvements of a bridge network system are studied in this paper. Then equivalence between different improved designs of the bridge network system is discussed. Three different methods are used to get different better designs of the network in the sense of having higher reliability and mean time to failure. Then two different types of reliability equivalence factors of the system are derived. It is assumed here that the failure rates of the system's components are identical and constant. The reliability functions and mean time to failure of the original and improved designs of the network are derived. Comparison between the mean time to failures of the original system and improved designs of the system are presented. Numerical studies and conclusion are presented in order to explain how one can apply the the theoretical results obtained.

1. INTRODUCTION

A large number of real-world systems can be modelled using networks. These applications are as divers as numerous. Among the traditional applications, one may cite business, government, transportation, management systems and traditional telephony. Among the more recent ones are satellite and cellular telephony, information (internet) systems, and computer networks. Therefore, networks are mainstay of modern life, see Ahuja et al. [1].

Consequently, the reliability and mean time to failure of networks have become important issues and much development has been made in its evaluation. Some works have been made to estimate the reliability and mean time to failure of the networks such as, Lynn et al. [6], Colbourn [3] and Shier [13]. In this paper is we shall study how to improve the performance of an example of networks.

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One of the main problems related to networks is to improve their reliability and mean time to failure. Sometimes in reliability analysis, different system designs should be comparable based on a reliability characteristic such as the reliability function or mean time to failure in case of no repairs. Recently, the reliability equivalence factors of some certain systems are derived by applying the concept of comparing different designs, e.g. [7]-[11]. The reliability equivalence factors for two component parallel and two component series systems with independent and identical components are derived by Råde in [7], [8]. The concept of comparing different designs is followed by Sarhan [9] to derive the reliability equivalence factors of a series system consists of n independent and non-identical components. He used the reliability function as a characteristic measure to compare different system designs. Two types of reliability equivalence factors of a basic series/parallel system consists of three independent and non-identical components are derived by applying such concept, Sarhan [10]. He used both the reliability function and mean time to failure as characteristic measures to compare different system designs in obtaining these factors. Sarhan et al. [11], derived two types of reliability equivalence factors of a series-parallel system consists of four independent and identical components.

Generally, high reliability and mean time to failure of a system can be achieved in a variety of ways such as, Sarhan [9]:

(1) **Reduction method:** in which the system may improve by improving the quality of some components by reducing their failure rates by a factor ρ, $0 < \rho < 1$.

(2) **Hot duplication method:** in which the system may improve by assuming hot duplications of some of the system's components.

(3) **Cold duplication method:** in which the system may improve by assuming cold duplications of some of the system's components.

Using redundancy ways (cases 2-3) may not be the optimal solution in system in which the minimum size and weight are overriding considerations: for example, in satellites or other space applications, in well-logging equipment, and in pacemakers and similar biomedical applications, see Lewis [5]. In such applications space or weight limitations may dictate an increase in component reliability rather than redundancy. Then more emphasis must be placed on robust design, manufacturing quality control, an on controlling the operating environment. Therefore, the concept of reliability equivalence takes place. In such concept, the design of the system improved according to reduction method should equivalent with that design of the system improved according to one of the reset redundancy methods.

In order to derive the reliability equivalent factor of a system, we simply need the following definition.

Definition 1. [Sarhan [10]] A reliability equivalence factor is a factor by which a characteristic of components of a system design has to be multiplied in order to reach equality of a characteristic of this design and a different design.

The main aim of this paper we study the problem of reliability equivalence on a bridge network system. Using different characteristics as measures of equivalent between different designs, we derive two different types of reliability equivalence factors of the bridge network system.

Section 2 introduces the description of the system studied here. The structural importance and joint structural importance of the system are also calculated in Section 2. The reliability functions and mean time to failures of the designs of improved systems are presented in Section 3. Also, theoretical studies are established in this section to compare different methods used to improve the system design. In Section 4 we obtain two types of reliability equivalence factors of the system. Section 5 gives the $\alpha-$fractiles of the original design and improved designs. Numerical results and conclusion are listed in Section 6.

2. BRIDGE NETWORK SYSTEM

In what follows, we introduce the description of the system studied here. Also, we shall derive the survival function and mean time to failure of the system.

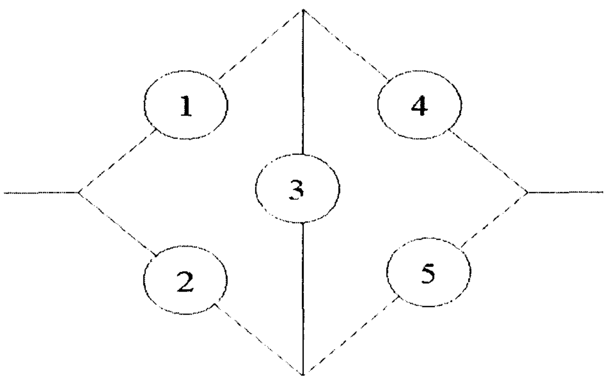

Figure 1. A bridge structure.

The bridge system consists of five components connected neither in series nor in parallel as shown in Scheme 1. The system components are assumed to be independent and identical each with reliability function given by (for $i = 1, 2, \ldots, 5$)

$$R_i(t) = e^{-\lambda t}, \;\; \lambda > 0, \; t \geq 0. \tag{2.1}$$

Let $X_i(t)$ be binary random variable denoting the state of component i at time t. That is,

$$X_i(t) = \begin{cases} 1 & \text{if component } i \text{ is functioning at time } t, \\ 0 & \text{if component } i \text{ is failed at time } t. \end{cases}$$

Let also $X(t)$ be binary random variable denoting the state of the system at time t. Namely,

$$Xt) = \begin{cases} 1 & \text{if the system is functioning at time } t, \\ 0 & \text{if system is failed at time } t. \end{cases}$$

The system has the following structure function, Leemis [4],

$$\Phi(\underline{X}) = X_1X_4 + X_2X_5 + X_2X_3X_4 + X_1X_3X_5 - \sum_{i=1}^{5} \prod_{j \in N_i} X_i + 2 \prod_{j \in N} X_j. \tag{2.2}$$

where $\underline{X} = (X_1, X_2, \ldots, X_5)$, $N = \{1, 2, 3, 4, 5\}$ and $N_i = N \setminus \{i\}$, $i \in N$.
The structure importance of the system components can be obtained and given by

$$I_\Phi(3) = \frac{1}{8} \text{ and } I_\Phi(i) = \frac{3}{8}, \; i \in N_3.$$

That is component 3 is less important than all the rest of the system components by virtue of its position in the system.

The following definition gives the joint structural importance of two components of a system.

Definition 3. [12] The joint structural importance of components i, j in a coherent system of n components is

$$JSI_\Phi(i,j) = \frac{1}{2^{n-2}} \sum_{\{\mathbf{x} \mid x_i=1, x_j=1\}} [\Phi(1_i, 1_j, \mathbf{x}) + \Phi(0_i, 0_j, \mathbf{x}) - \Phi(1_i, 0_j, \mathbf{x}) - \Phi(0_i, 1_j, \mathbf{x})] \tag{2.3}$$

for $i, j = 1, 2, \ldots, n$.

One of the components of i, j becomes more (less) important when the other is functioning if $JSI_\Phi(i,j) > 0$ $\big(JSI_\Phi(i,j) < 0\big)$, see Sarhan and Abouammoh [12]. For the given system, we have

$$JSI_\Phi(1,4) = JSI_\Phi(2,5) = \frac{1}{2},$$

$$JSI_\Phi(1,2) = JSI_\Phi(4,5) = -\frac{1}{2},$$

and

$$JSI_\Phi(1,3) = JSI_\Phi(2,3) = JSI_\Phi(3,4) = JSI_\Phi(3,5) = JSI_\Phi(1,5) = JSI_\Phi(2,4) = 0.$$

Therefore, one of the components in one of the pairs (1, 4), (2, 5) becomes more important when the other is functioning. While one of the components in one of the pairs (1, 2), (4, 5) becomes less important when the other is functioning. Finally, the importance of one of the components in any of the pairs (1,3), (2,3), (3,4), (3,5), (1,5) and (2,4) is unchanged by functioning the other.

In this paper we shall improve the system by improving one or two components according to:

1. Reduction method,
2. Hot duplication method,
3. Cold duplication method.

The selection of a single component to be improved is based on its structural importance. While selection of two components to be improved is based on their joint structural importance.

Let $R(t)$ be the reliability function of the system. One can obtained $R(t)$ as follows. The reliability function of a coherent system is given by

$$R(t) = E[\Phi(\underline{X}(t))]. \tag{2.4}$$

Then using (2.2), we get

$$R(t) = E[X_1X_4]+E[X_2X_5]+E[X_2X_3X_4]+E[X_1X_3X_5]-\sum_{i=1}^{5} E\left[\prod_{j\in N_i} X_i\right]+2\left[\prod_{j\in N} X_j\right]. \tag{2.5}$$

Since $R_i = R_i(t)$ is the probability that component i is functioning at a certain time t. That is, at time t, $R_i = P\{X_i = 1\} = E[X_i]$. Therefore, using the property that the components are independent we get

$$R(t) = R_1R_4 + R_2R_5 + R_2R_3R_4 + R_1R_3R_5 - \sum_{i=1}^{5}\prod_{j\in N_i} R_i + 2\prod_{j\in N} R_j. \tag{2.6}$$

Under the assumption that the components are identical, that is $R_i(t) = e^{-\lambda t}$ for $i = 1, 2, \ldots, 5$, then $R(t)$ becomes

$$R(t) = e^{-2\lambda t}\left\{2 + 2e^{-\lambda t} - 5e^{-2\lambda t} + 2e^{-3\lambda t}\right\}. \tag{2.7}$$

The system mean time to failure, say MTTF, can be obtained by using (2.7) according to the following relation

$$\text{MTTF} = \int_0^{\infty} R_s(t)dt. \tag{2.8}$$

That is,

$$\text{MTTF} = \frac{2}{3\lambda} - \frac{1}{4\lambda} + \frac{2}{5\lambda} = \frac{49}{60\lambda}. \tag{2.9}$$

3. DESIGNS OF THE IMPROVED SYSTEMS

In this section, we present different designs of the system that may be obtained improving some of its components according to different improvement methods previously mentioned. Also we derive reliability function and mean time to failure of each design.

3.1 Reduction method

In this method we assume that the system can be improved by reducing the failure rates of some of its components by a factor ρ, $0 < \rho < 1$. Let A be a set of components that are improved according to the reduction method. Let $R_{A,\rho}(t)$ denote the reliability function of the system improved by improving the set A components according to the reduction method.
In what follows, we present $R_{A,\rho}(t)$ for some different A.

1. $A \in \mathcal{S}_1 = \Big\{\{3\}\Big\}$:

$$R_{A,\rho}(t) = e^{-2\lambda t}\left\{2 - e^{-2\lambda t} + 2e^{-\rho\lambda t}\left(1 - e^{-\lambda t}\right)^2\right\}. \tag{3.1}$$

2. $A \in \mathcal{S}_2 = \Big\{\{1\},\ \{2\},\ \{4\},\ \{5\}\Big\}$:

$$R_{A,\rho}(t) = e^{-(1+\rho)\lambda t}\left\{1 + e^{-\lambda t} - 4e^{-2\lambda t} + 2e^{-3\lambda t}\right\} + e^{-2\lambda t}\left\{1 + e^{-\lambda t} - e^{-2\lambda t}\right\}. \tag{3.2}$$

3. $A \in \mathcal{S}_3 = \Big\{\{1,3\},\ \{1,5\},\ \{2,3\},\ \{2,4\}, \{3,4\},\ \{3,5\}\Big\}$:

$$R_{A,\rho}(t) = e^{-2\lambda t} + e^{-(1+\rho)\lambda t}\left\{1 + e^{-\lambda t} - 2e^{-2\lambda t} + e^{-\rho\lambda t}\left(1 - 3e^{-\lambda t} + 2e^{-2\lambda t}\right)\right\}. \tag{3.3}$$

4. $A \in \mathcal{S}_4 = \Big\{\{1,2\},\ \{4,5\}\Big\}$:

$$R_{A,\rho}(t) = e^{-(1+\rho)\lambda t}\left\{2\left[1 + e^{-\lambda t} - e^{-2\lambda t}\right] + e^{-(\rho+1)\lambda t}\left(2e^{-\lambda t} - 3\right)\right\}. \tag{3.4}$$

5. $A \in \mathcal{S}_5 = \Big\{\{1,4\},\ \{2,5\}\Big\}$:

$$R_{A,\rho}(t) = e^{-2\lambda t} + 2e^{-(\rho+2)\lambda t}\left(1 - e^{-\lambda t}\right) + e^{-2\rho\lambda t}\left(1 - 3e^{-2\lambda t} + 2e^{-3\lambda t}\right). \quad (3.5)$$

Using the relations (3.2-3.5) one can derive the mean time to failures of the improved designs. In what follows, we present $\mathrm{MTTF}_{A,\rho}$ for some different A.

1. $A \in \mathcal{S}_1 = \Big\{\{3\}\Big\}$:

$$\mathrm{MTTF}_{A,\rho} = \frac{1}{\lambda}\left\{\frac{3}{4} + \frac{2}{2+\rho} + \frac{2}{4+\rho} - \frac{4}{3+\rho}\right\}. \quad (3.6)$$

2. $A \in \mathcal{S}_2 = \Big\{\{1\},\ \{2\},\ \{4\},\ \{5\}\Big\}$:

$$\mathrm{MTTF}_{A,\rho} = \frac{1}{\lambda}\left\{\frac{7}{12} + \frac{1}{1+\rho} + \frac{1}{2+\rho} + \frac{2}{4+\rho} - \frac{4}{3+\rho}\right\}. \quad (3.7)$$

3. $A \in \mathcal{S}_3 = \Big\{\{1,3\},\ \{1,5\},\ \{2,3\},\ \{2,4\}, \{3,4\},\ \{3,5\}\Big\}$:

$$\mathrm{MTTF}_{A,\rho} = \frac{1}{\lambda}\left\{\frac{1}{2} + \frac{1}{2+\rho} + \frac{1}{1+2\rho} + \frac{2}{3+2\rho} - \frac{2}{3+\rho} - \frac{1}{2(1+\rho)}\right\}. \quad (3.8)$$

4. $A \in \mathcal{S}_4 = \Big\{\{1,2\},\ \{4,5\}\Big\}$:

$$\mathrm{MTTF}_{A,\rho} = \frac{1}{\lambda}\left\{\frac{1}{2(1+\rho)} + \frac{2}{2+\rho} + \frac{2}{3+2\rho} - \frac{2}{3+\rho}\right\}. \quad (3.9)$$

5. $A \in \mathcal{S}_5 = \Big\{\{1,4\},\ \{2,5\}\Big\}$:

$$\mathrm{MTTF}_{A,\rho} = \frac{1}{\lambda}\left\{\frac{1}{2} + \frac{1}{2\rho} + \frac{2}{2+\rho} + \frac{2}{3+2\rho} - \frac{3}{2(1+\rho)} - \frac{2}{3+\rho}\right\}. \quad (3.10)$$

Figure 1 shows the the behavior of $\lambda \times \mathrm{MTTF}_{A,\rho}$ with ρ for different A's in the presence of MTTF. In this figure, the symbol S-i means that $A \in \mathcal{S}_i$, $i = 1, 2, \ldots, 5$.

One can conclude from Fig. 1 that follows:

1. As it was expected, for any A the $\mathrm{MTTF}_{A,\rho}$ is decreasing with ρ.

2. The following statements are fulfilled:

$$\mathrm{MTTF} < \mathrm{MTTF}_{A_1,\rho} < \mathrm{MTTF}_{A_2,\rho}, \quad (3.11)$$

$$\mathrm{MTTF}_{A_2,\rho} < \mathrm{MTTF}_{A_3,\rho} < \mathrm{MTTF}_{A_4,\rho} < \mathrm{MTTF}_{A_5,\rho}. \quad (3.12)$$

where $A_i \in \mathcal{S}_i$, $i = 1, 2, \ldots, 5$.

Based on the above two results, one can say that:

1. Reducing the failure rate of the most structural importance component produces a better design than that obtained by reducing the failure rate of the less structural importance component by the the same factor ρ, in the sense of having greater MTTF.

2. Reducing the failure rates of any two components, whatever their joint structural importance is positive, zero or negative, gives a better design than that obtained by reducing the failure rate of a single component even that of the most structural importance one by the the same factor ρ, in the sense of having greater MTTF.

3. Reducing the failure rates of two components with positive joint structural importance produces a better design than that produced by reducing the failure rates of any other two components with either zero or negative joint structural importance by the the same factor ρ, in the sense of having greater MTTF.

4. Reducing the failure rates of two components with zero joint structural importance produces a better design than that produced by reducing the failure rates of any other two components with negative joint structural importance by the the same factor ρ, in the sense of having greater MTTF.

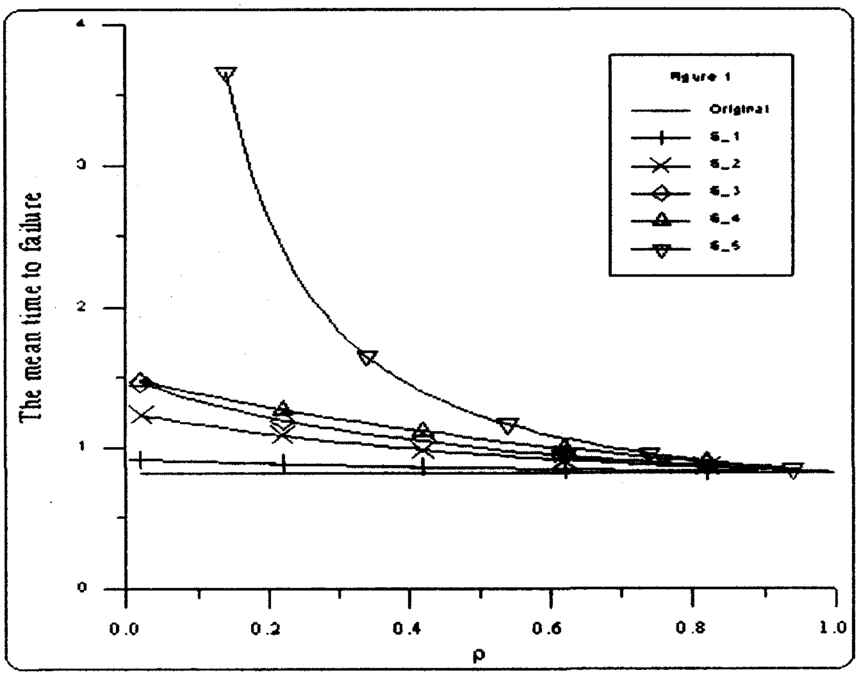

Figure 1. $\text{MTTF}_{A,\rho}$ against ρ for different subsets A.

3.2 Hot duplication method

Let us now assume that the system is improved by improving a set B of its components according to hot duplication method. Also, let $R_B^H(t)$ denote the reliability function of the system improved by improving the set B components according to the hot duplication method.
Let $R_i^H(t)$ be the reliability function of component i when it is improved according to hot duplication method, as diagramed in Figure 2. That is, $R_i^H(t) = e^{-\lambda t}\left(2 - e^{-\lambda t}\right)$.

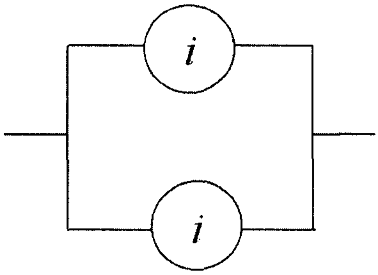

Figure 2. Hot duplication of component i.

For a given B, the function $R_B^H(t)$ can be deduced from (2.6) when $R_i^H(t)$ replaced R_i for all $i \in B$. In that follows, we present $R_B^H(t)$ for some B.

1. $B \in S_1$:

$$R_B^H(t) = e^{-2\lambda t}\left\{2 + 4e^{-\lambda t} - 11e^{-2\lambda t} + 8e^{-3\lambda t} - 2e^{-4\lambda t}\right\}. \tag{3.13}$$

2. $B \in S_2$:

$$R_B^H(t) = e^{-2\lambda t}\left\{3 + 2e^{-\lambda t} - 10e^{-2\lambda t} + 8e^{-3\lambda t} - 2e^{-4\lambda t}\right\}. \tag{3.14}$$

3. $B \in S_3$:

$$R_B^H(t) = e^{-2\lambda t}\left\{3 + 5e^{-\lambda t} - 21e^{-2\lambda t} + 23e^{-3\lambda t} - 11e^{-4\lambda t} + 2e^{-5\lambda t}\right\}. \tag{3.15}$$

4. $B \in S_4$:

$$R_B^H(t) = e^{-2\lambda t}\left\{4 + 2e^{-\lambda t} - 18e^{-2\lambda t} + 22e^{-3\lambda t} - 11e^{-4\lambda t} + 2e^{-5\lambda t}\right\}. \tag{3.16}$$

5. $B \in S_2$:

$$R_B^H(t) = e^{-2\lambda t}\left\{5 - 17e^{-2\lambda t} + 22e^{-3\lambda t} - 11e^{-4\lambda t} + 2e^{-5\lambda t}\right\}. \tag{3.17}$$

The system mean time to failure of the system improved by improving the set B components according to hot duplication method, say MTTF_B^H, can be obtained by using the function $R_B^H(t)$. Table 1 shows MTTF_B^H for some different B.

3.3 Cold duplication method

Assume that the system can be improved by improving a set B of its components according to cold duplication method. Let $R_B^C(t)$ be the reliability function of the system improved when the set B components are improved according to the cold duplication method.
Let $R_i^C(t)$ denote the reliability function of component i when it is improved according to cold duplication method, as diagramed in Figure 3. That is, $R_i^C(t) = (1 + \lambda_i t)\, e^{-\lambda_i t}$, see [2].

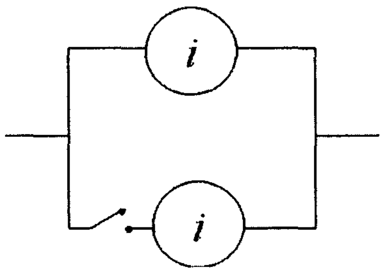

Figure 3. Cold duplication of component i.

The function $R_B^C(t)$ can be derived from (2.6), for a specified B, when $R_i^C(t)$ replaced R_i for all $i \in B$. In that follows, we present $R_B^C(t)$ for some B.

1. $B \in \mathcal{S}_1$:

$$R_B^C(t) = e^{-2\lambda t}\left\{2 + 2e^{-\lambda t} - 5e^{-2\lambda t} + 2e^{-3\lambda t} + 2\lambda t\, e^{-\lambda t}\left[1 - 2\,e^{-\lambda t} + e^{-2\lambda t}\right]\right\}. \tag{3.18}$$

2. $B \in \mathcal{S}_2$:

$$R_B^C(t) = e^{-2\lambda t}\left\{2 + 2e^{-\lambda t} - 5e^{-2\lambda t} + 2e^{-3\lambda t} + \lambda t\left[1 + e^{-\lambda t} - 4e^{-2\lambda t} + 2\,e^{-3\lambda t}\right]\right\} \tag{3.19}$$

3. $B \in \mathcal{S}_3$:

$$\begin{aligned} R_B^C(t) &= e^{-2\lambda t}\left\{2 + 2\,e^{-\lambda t} - 5\,e^{-2\lambda t} + 2\,e^{-3\lambda t}\right. \\ &\quad + \lambda t\left[1 + 3\,e^{-\lambda t} - 8\,e^{-2\lambda t} + 4\,e^{-3\lambda t}\right] \\ &\quad \left. + \lambda^2 t^2 e^{-\lambda t}\left[1 - 3\,e^{-\lambda t} + 2\,e^{-2\lambda t}\right]\right\}. \end{aligned} \tag{3.20}$$

4. $B \in \mathcal{S}_4$:

$$R_B^C(t) = e^{-2\lambda t}\left\{2 + 2\,e^{-\lambda t} - 5\,e^{-2\lambda t} + 2\,e^{-3\lambda t}\right.$$

$$+2\lambda t\left[1+e^{-\lambda t}-4\,e^{-2\lambda t}+2\,e^{-3\lambda t}\right]$$
$$+\lambda^2 t^2 e^{-2\lambda t}\left[2\,e^{-\lambda t}-3\right]\Big\}. \qquad (3.21)$$

5. $B \in \mathcal{S}_5$:

$$\begin{aligned} R^C_B(t) &= e^{-2\lambda t}\Big\{2+2\,e^{-\lambda t}-5\,e^{-2\lambda t}+2\,e^{-3\lambda t} \\ &+2\,\lambda t\left[1+e^{-\lambda t}-4\,e^{-2\lambda t}+2\,e^{-3\lambda t}\right] \\ &+\lambda^2 t^2\left[1-3\,e^{-2\,\lambda t}+2\,e^{-3\,\lambda t}\right]\Big\}. \end{aligned} \qquad (3.22)$$

The mean time to failure of the system improved by improving the system B components according to cold duplication method, say MTTF^C_B, can be obtained by using the function $R^C_B(t)$. Table 1 shows MTTF^C_B for some different B.

Table 1.

B	MTTF^H_B	MTTF^C_B
$B \in \mathcal{S}_1$	$0.850/\lambda$	$0.869/\lambda$
$B \in \mathcal{S}_2$	$0.933/\lambda$	$1.008/\lambda$
$B \in \mathcal{S}_3$	$0.969/\lambda$	$1.072/\lambda$
$B \in \mathcal{S}_4$	$1.019/\lambda$	$1.137/\lambda$
$B \in \mathcal{S}_5$	$1.102/\lambda$	$1.387/\lambda$

Based on the results given in Table 1, one can conclude that:

1. $MTTF^H_B < MTTF^C_B$ for any set B of components.
2. $MTTF^H_{B_1} = \min_B\left\{MTTF^H_B,\, MTTF^C_B\right\}$.
3. $MTTF^C_{B_5} = \max_B\left\{MTTF^H_B,\, MTTF^C_B\right\}$.
4. $MTTF^H_{B_1} < MTTF^C_{B_1} < MTTF^H_{B_2} < MTTF^H_{B_3} < MTTF^C_{B_2} < MTTF^H_{B_4} < MTTF^C_{B_3} < MTTF^H_{B_5} < MTTF^C_{B_4} < MTTF^C_{B_5}$.

Where $B_i \in \mathcal{S}_i$, $i = 1, 2, \ldots, 5$.

Figure 1 shows the reliability functions of the original system and the improved systems obtained by improving one component according hot and cold duplication methods. Figure 2 shows the reliability functions of the original system and the improved systems obtained by improving two components according to hot and cold duplication methods.

4. RELIABILITY EQUIVALENCE FACTORS

Generally there are two different reliability equivalence factors: survival reliability equivalence factor and mean reliability equivalence factor, Sarhan[?]. From now and henceforth, we use the abbreviations SREF and MREF instead of survival and mean reliability equivalence factors, respectively. The reliability equivalence factors can be derived by using different characteristics of the system.

Equating the characteristics for a design of the system improved according to reduction method with those for designs of the system improved according to redundancy methods permits us to derive the reliability equivalence factors of the system. In what follows we present the two different reliability equivalence factors.

4.1 SREF

In order to derive the SREFs, we need simply the following definition.

Definition 3. The hot (cold) SREF, say $\rho^D_{A,B}(\alpha)$, $D = H(C)$, is defined as that factor by which the failure rates of the set A components should be reduced in order to improve the system reliability to be as that reliability of system improved by assuming hot (cold) duplications of the set B components.

Based on this definition, the hot (cold) SREF $\rho^D_{A,B}(\alpha)$ can be obtained by solving the following system of two equations:

$$R^D_B(t) = \alpha, \quad R_{A,\rho}(t) = \alpha, \quad D = H\,(C). \tag{4.1}$$

In what follows, we present the different forms of hot (cold) SREFs of the bridge system that can be derived from the above system of equations (4.1). Using the second equation in (4.1) and for a specific set A, we have:

1. When $A \in \mathcal{S}_1$:

$$\rho^H_{A,B}(\alpha) = \frac{1}{\ln x} \ln\left[\frac{\alpha - x^2\,(2 - x^2)}{2x^2\,(1 - x)^2}\right]. \tag{4.2}$$

2. When $A \in \mathcal{S}_2$:

$$\rho^H_{A,B}(\alpha) = \frac{1}{\ln x} \ln\left[\frac{\alpha - x^2\,(1 + x - x^2)}{x\,(1 + x - 4x^2 + 2x^3)}\right]. \tag{4.3}$$

3. When $A \in \mathcal{S}_3$:

$$\rho^H_{A,B}(\alpha) = \frac{1}{\ln x} \ln\left[\frac{-C_2 + \sqrt{C_2^2 - 4\,C_1\,C_3}}{2\,C_1}\right] \tag{4.4}$$

where

$$C_1 = x(1 - 3x + 2x^2),\; C_2 = x(1 + x - 2x^2),\; C_3 = x^2 - \alpha\,.$$

4. When $A \in \mathcal{S}_4$:

$$\rho^H_{A,B}(\alpha) = \frac{1}{\ln x} \ln \left[\frac{-C_2 + \sqrt{C_2^2 + 4\alpha\, C_1}}{2\, C_1} \right] \tag{4.5}$$

where

$$C_1 = x^2(2x-3),\; C_2 = 2x(1+x-x^2)\,.$$

5. When $A \in \mathcal{S}_5$:

$$\rho^H_{A,B}(\alpha) = \frac{1}{\ln x} \ln \left[\frac{-C_2 + \sqrt{C_2^2 - 4\, C_1\, C_3}}{2\, C_1} \right] \tag{4.6}$$

where

$$C_1 = 1 - 3x + 2x^2,\; C_2 = 2x^2(1-x),\; C_3 = x^2 - \alpha\,.$$

where x in the above relations is the solution of that equation which can be obtained from the first equation of (4.1) for a specified B and assuming the improving method. In what follows, we present different forms of this equation for different possible subsets B and following hot and cold duplication methods.

1. Hot duplication method, $D = H$:

1.1 For $B \in \mathcal{S}_1$: x is the solution of the following equation

$$x^2\left(2x^4 - 8x^3 + 11x^2 - 4x - 2\right) + \alpha = 0. \tag{4.7}$$

1.2 For $B \in \mathcal{S}_2$: x is the solution of the following equation

$$x^2\left(2x^4 - 8x^3 + 10x^2 - 2x - 3\right) + \alpha = 0. \tag{4.8}$$

1.3 For $B \in \mathcal{S}_3$: x is the solution of the following equation

$$x^2\left(2x^5 - 11x^4 + 23x^3 + 21x^2 + 5x + 3\right) - \alpha = 0. \tag{4.9}$$

1.4 For $B \in \mathcal{S}_4$: x is the solution of the following equation

$$x^2\left(2x^5 - 11x^4 + 22x^3 - 18x^2 + 2x + 4\right) - \alpha = 0. \tag{4.10}$$

1.5 For $B \in \mathcal{S}_5$: x is the solution of the following equation

$$x^2\left(2x^5 - 11x^4 + 22x^3 - 17x^2 + 5\right) - \alpha = 0. \tag{4.11}$$

2. Cold duplication method, $D = C$:

2.1 For $B \in S_1$: x is the solution of the following equation

$$x^2 \left\{2x^3 - 5x^2 + 2x + 2 - 2x \ln x \left(x^2 - 2x + 1\right)\right\} - \alpha = 0. \tag{4.12}$$

2.2 For $B \in S_2$: x is the solution of the following equation

$$x^2 \left\{2x^3 - 5x^2 + 2x + 2 - \ln x \left(2x^3 - 4x^2 + x + 1\right)\right\} - \alpha = 0. \tag{4.13}$$

2.3 For $B \in S_3$: x is the solution of the following equation

$$x^2 \left\{2+2x-5x^2+2x^3-\ln x \left(1+3x-8x^2+4x^3\right)+x \ln^2 x \left(1-3x+2x^2\right)\right\}-\alpha = 0. \tag{4.14}$$

2.4 For $B \in S_4$: x is the solution of the following equation

$$x^2 \left\{2x^3 - 5x^2 + 2x + 2 - 2\left(2x^3 - 4x^2 + x + 1\right) \ln x + (2x-3)\, x^2 \ln^2 x\right\} - \alpha = 0. \tag{4.15}$$

2.5 For $B \in S_5$: x is the solution of the following equation

$$x^2 \left\{2x^3 -5x^2 +2x+2-2\left(2x^3 -4x^2 +x+1\right) \ln x+\left(1-3x^2+2x^3\right) \ln^2 x\right\}-\alpha = 0. \tag{4.16}$$

4.1 MREF

In order to derive the MREFs, we need simply the following definition.

Definition 4. The hot (cold) MREF, say $\xi^D_{A,B}(\alpha)$, $D = H(C)$, is defined as that factor by which the failure rates of the set A components should be reduced in order to improve the system MTTF to be as that MTTF of system improved by assuming hot (cold) duplications of the set B components.

According to the above definition, the hot (cold) MREF $\xi^D_{A,B}(\alpha)$ can be obtained by solving the following equation with respect to $\rho = \xi$:

$$MTTF_{A,\rho} = MTTF^D_B,\ D = H,\ C. \tag{4.17}$$

In what follows we present the equations that should be solved to get such factors for different A and B:

1. $A \in \mathcal{S}_1 = \Big\{\{3\}\Big\}$:

$$\frac{3}{4} + \frac{2}{2+\xi^D} + \frac{2}{4+\xi^D} - \frac{4}{3+\xi^D} = m_B^D\,. \tag{4.18}$$

2. $A \in \mathcal{S}_2 = \Big\{\{1\},\,\{2\},\,\{4\},\,\{5\}\Big\}$:

$$\frac{7}{12} + \frac{1}{1+\xi^D} + \frac{1}{2+\xi^D} - \frac{4}{3+\xi^D} + \frac{2}{4+\xi^D} = m_B^D\,. \tag{4.19}$$

3. $A \in \mathcal{S}_3 = \Big\{\{1,3\},\,\{1,5\},\,\{2,3\},\,\{2,4\},\{3,4\},\,\{3,5\}\Big\}$:

$$\frac{1}{2} - \frac{1}{2(1+\xi^D)} + \frac{1}{2+\xi^D} - \frac{2}{3+\xi^D} + \frac{1}{1+2\xi^D} + \frac{2}{3+2\xi^D} = m_B^D\,. \tag{4.20}$$

4. $A \in \mathcal{S}_4 = \Big\{\{1,2\},\,\{4,5\}\Big\}$:

$$\frac{2}{2+\xi^D} + \frac{2}{3+2\xi^D} + \frac{1}{2(1+\xi^D)} - \frac{2}{3+\xi^D} = m_B^D\,. \tag{4.21}$$

5. $A \in \mathcal{S}_5 = \Big\{\{1,4\},\,\{2,5\}\Big\}$:

$$\frac{1}{2} + \frac{1}{2\xi^D} + \frac{2}{2+\xi^D} + \frac{2}{3+2\xi^D} - \frac{3}{2(1+\xi^D)} - \frac{2}{3+\xi^D} = m_B^D\,. \tag{4.22}$$

where $m_B^D = \lambda\,\mathrm{MTTF}_B^D$, $D = H,\,C$, and MTTF_B^D are given in Table 1.

5. The α–FRACTILES

In this section we present the α–fractiles of the original and improved systems. Firstly, let us assume that $L(\alpha)$ denotes to the α–fractile of the original system. Also, let $L_B^D(\alpha)$ denote to the α–fractile of the improved designs produced by improving the set B of the system's components according to hot and cold duplication methods, for D=H, D, respectively.

Now we explain how one can derive $L(\alpha)$ and $L_B^D(\alpha)$. Generally, $L(\alpha)$ can be derived by solving the following equation with respect to L

$$R(L/\lambda) = \alpha\,. \tag{5.1}$$

Using (2.7) and (5.1), one can get that

$$e^{-2L}\Big\{2 + 2e^{-L} - 5e^{-2L} + 2\,e^{-3L}\Big\} - \alpha = 0. \tag{5.2}$$

Let $x = e^{-L}$, then

$$L(\alpha) = -\ln x, \tag{5.3}$$

where x is a solution of the following equation

$$2\,x^5 - 5x^4 + 2x^3 + 2\,x^2 - \alpha = 0. \tag{5.4}$$

Numerical technique method can be used to derive a solution of (5.4) which can be used together with (5.3) to calculate $L(\alpha)$ for a specified value of α.

The $\alpha-$fractile $L_B^D(\alpha)$, $D = H,\ C$, is defined as the solution of the following equation with respect to L

$$R_B^D(L/\lambda) = \alpha\,. \tag{5.5}$$

Similar to the approach used in deriving $L(\alpha)$ of the original system, one can derive $L_B^D(\alpha)$. After very short calculations by using (5.5) and the forms of the reliability functions $R_B^D(t)$ previously obtained, one can get $L_B^D(\alpha)$ as

$$L_B^D(\alpha) = -\ln x, \tag{5.6}$$

where x is a solution of the equation obtained by specifying the method of improvement used (hot or cold), the set B and α. That is, for $B \in \mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_5$, x is a solution of equation

1. (4.7), (4.8), (4.9), (4.10) and (4.11), respectively, when $D = H$.
2. (4.12), (4.13), (4.14), (4.15) and (4.16), respectively, when $D = C$.

Numerical results of $L(\alpha)$ and $L_B^D(\alpha)$ are given in the next section.

6. NUMERICAL RESULTS AND ANALYSIS

Numerical studies are given in this section to explain how one can utilize the theoretical results obtained in the context. The values of $L(\alpha)$ and $L_B^D(\alpha)$ are calculated for different sets A and B, when $\alpha = 0.1,\ 0.2,\ \ldots,\ 0.9$ and presented in Table 2. Also, we compute the values of SREF $\rho_{A,B}^D(\alpha)$ at the same level α for different possible choices of A and B and given in Tables 2-7. The values of MREF $\xi_{A,B}^D$ are calculated for different possible choices of A and B. Table 8 shows such values. Furthermore, we plot the reliability functions of the original system and the improved designs obtained by improving set B components according to redundancy methods. Figures 5, 6 show these functions when $|B| = 1$ and $|B| = 2$, respectively. From now and henceforth we use A_i and B_i instead of $A, B \in \mathcal{S}_i$, respectively, $i = 1, \ldots, 5$.

According to the the results shown in Table 2, one can find out that follows:

1. $L(\alpha) < L_{B_1}^H(\alpha) < L_{B_1}^C(\alpha) < L_{B_2}^H(\alpha) < L_{B_3}^H(\alpha) < L_{B_2}^C(\alpha)$ for all α, *2.* $L^C_{B_2}(\alpha)\left(L^H_{B_4}(\alpha)\right) < L^H_{B_5}(\alpha)\left(L^C_{B_3}(\alpha)\right) < L^C_{B_4}(\alpha) < L^C_{B_5}(\alpha)$ for all α,

3. $L^C_{B_2}(\alpha) < (>) L^H_{B_4}(\alpha)$ for $\alpha > (\leq) 0.2$,

3. $L^C_{B_3}(\alpha) < (<) L^H_{B_5}(\alpha)$ for $\alpha > (\leq) 0.2$.

Table 2. The α–fractiles $L(\alpha)$ and $L^D_B(\alpha)$.

α	$L(\alpha)$	$L^H_{B_1}$	$L^C_{B_1}$	$L^H_{B_2}$	$L^C_{B_2}$	$L^H_{B_3}$	$L^C_{B_3}$	$L^H_{B_4}$	$L^C_{B_4}$	$L^H_{B_5}$	$L^C_{B_5}$
0.1	1.549	1.593	1.634	1.713	1.867	1.754	1.957	1.836	2.081	1.925	2.438
0.2	1.197	1.244	1.278	1.350	1.464	1.395	1.556	1.463	1.652	1.549	1.956
0.3	0.985	1.031	1.058	1.124	1.217	1.171	1.303	1.232	1.383	1.313	1.652
0.4	0.825	0.868	0.890	0.955	1.029	1.000	1.110	1.053	1.177	1.134	1.424
0.5	0.693	0.732	0.748	0.813	0.873	0.855	0.946	0.903	1.002	0.984	1.233
0.6	0.576	0.611	0.624	0.685	0.733	0.725	0.798	0.767	0.846	0.849	1.061
0.7	0.467	0.497	0.505	0.565	0.602	0.599	0.655	0.636	0.693	0.719	0.897
0.8	0.360	0.381	0.386	0.442	0.468	0.471	0.510	0.501	0.539	0.584	0.729
0.9	0.239	0.253	0.255	0.304	0.318	0.323	0.344	0.343	0.361	0.429	0.535

Table 3. The SREF $\rho^D_{A,B}(\alpha)$ for different B when $A \in \mathcal{S}_1$.

α	ρ^H_{A,B_1}	ρ^C_{A,B_1}	ρ^H_{A,B_2}	ρ^C_{A,B_2}	ρ^H_{A,B_3}	ρ^C_{A,B_3}	ρ^H_{A,B_4}	ρ^C_{A,B_4}	ρ^H_{A,B_5}	ρ^C_{A,B_5}
0.1	0.632	0.407	0.110	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.2	0.568	0.356	0.037	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.3	0.518	0.318	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.4	0.473	0.285	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.5	0.429	0.254	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.6	0.384	0.223	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.7	0.335	0.190	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.8	0.278	0.154	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
0.9	0.202	0.109	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve

Table 4. The SREF $\rho^D_{A,B}(\alpha)$ for different B when $A \in \mathcal{S}_2$.

α	ρ^H_{A,B_1}	ρ^C_{A,B_1}	ρ^H_{A,B_2}	ρ^C_{A,B_2}	ρ^H_{A,B_3}	ρ^C_{A,B_3}	ρ^H_{A,B_4}	ρ^C_{A,B_4}	ρ^H_{A,B_5}	ρ^C_{A,B_5}
0.1	0.888	0.798	0.651	0.436	0.585	0.338	0.472	0.228	0.372	0.003
0.2	0.849	0.756	0.589	0.384	0.500	0.259	0.385	0.153	0.270	-ve
0.3	0.821	0.730	0.541	0.346	0.436	0.204	0.319	0.100	0.191	-ve
0.4	0.800	0.714	0.498	0.312	0.380	0.158	0.261	0.056	0.120	-ve
0.5	0.782	0.703	0.456	0.281	0.327	0.118	0.208	0.019	0.050	-ve
0.6	0.769	0.698	0.412	0.250	0.276	0.081	0.155	-ve	-ve	-ve
0.7	0.760	0.700	0.365	0.217	0.223	0.047	0.102	-ve	-ve	-ve
0.8	0.760	0.713	0.309	0.180	0.167	0.013	0.450	-ve	-ve	-ve
0.9	0.779	0.750	0.234	0.132	0.100	-ve	-ve	-ve	-ve	-ve

Table 5. The SREF $\rho^D_{A,B}(\alpha)$ for different B when $A \in \mathcal{S}_3$.

α	ρ^H_{A,B_1}	ρ^C_{A,B_1}	ρ^H_{A,B_2}	ρ^C_{A,B_2}	ρ^H_{A,B_3}	ρ^C_{A,B_3}	ρ^H_{A,B_4}	ρ^C_{A,B_4}	ρ^H_{A,B_5}	ρ^C_{A,B_5}
0.1	0.910	0.836	0.712	0.529	0.657	0.446	0.561	0.352	0.475	0.164
0.2	0.882	0.807	0.672	0.501	0.598	0.397	0.502	0.308	0.406	0.115
0.3	0.862	0.791	0.638	0.478	0.552	0.360	0.455	0.274	0.349	0.076
0.4	0.847	0.779	0.607	0.455	0.510	0.327	0.413	0.243	0.296	0.037
0.5	0.834	0.772	0.574	0.431	0.469	0.269	0.370	0.214	0.240	-ve
0.6	0.823	0.768	0.538	0.405	0.426	0.265	0.326	0.184	0.180	-ve
0.7	0.815	0.768	0.497	0.374	0.379	0.231	0.277	0.152	0.108	-ve
0.8	0.812	0.775	0.443	0.334	0.323	0.192	0.220	0.116	0.016	-ve
0.9	0.821	0.797	0.362	0.273	0.245	0.140	0.143	0.071	-ve	-ve

Table 6. The SREF $\rho^D_{A,B}(\alpha)$ for different B when $A \in \mathcal{S}_4$.

α	ρ^H_{A,B_1}	ρ^C_{A,B_1}	ρ^H_{A,B_2}	ρ^C_{A,B_2}	ρ^H_{A,B_3}	ρ^C_{A,B_3}	ρ^H_{A,B_4}	ρ^C_{A,B_4}	ρ^H_{A,B_5}	ρ^C_{A,B_5}
0.1	0.941	0.890	0.797	0.639	0.751	0.558	0.668	0.459	0.587	0.233
0.2	0.920	0.866	0.760	0.610	0.698	0.506	0.611	0.410	0.515	0.169
0.3	0.905	0.852	0.731	0.587	0.656	0.468	0.565	0.372	0.456	0.115
0.4	0.893	0.843	0.704	0.565	0.618	0.434	0.524	0.339	0.399	0.059
0.5	0.884	0.837	0.676	0.544	0.581	0.403	0.483	0.307	0.339	-ve
0.6	0.877	0.835	0.647	0.522	0.543	0.372	0.441	0.275	0.346	-ve
0.7	0.872	0.837	0.614	0.497	0.502	0.340	0.394	0.240	0.179	-ve
0.8	0.721	0.845	0.573	0.465	0.453	0.302	0.337	0.200	0.032	-ve
0.9	0.883	0.866	0.510	0.417	0.386	0.254	0.257	0.146	-ve	-ve

Table 7. The SREF $\rho^D_{A,B}(\alpha)$ for different B when $A \in \mathcal{S}_5$.

α	ρ^H_{A,B_1}	ρ^C_{A,B_1}	ρ^H_{A,B_2}	ρ^C_{A,B_2}	ρ^H_{A,B_3}	ρ^C_{A,B_3}	ρ^H_{A,B_4}	ρ^C_{A,B_4}	ρ^H_{A,B_5}	ρ^C_{A,B_5}
0.1	0.944	0.898	0.823	0.712	0.789	0.662	0.731	0.606	0.679	0.493
0.2	0.924	0.876	0.790	0.684	0.744	0.619	0.685	0.564	0.625	0.446
0.3	0910	0.863	0.765	0.663	0.710	0.589	0.649	0.535	0.582	0.410
0.4	0.899	0.855	0.742	0.645	0.680	0.564	0.618	0.510	0.543	0.378
0.5	0.890	0.849	0.720	0.628	0.653	0.541	0.589	0.489	0.506	0.348
0.6	0.883	0.847	0.797	0.611	0.625	0.521	0.561	0.470	0.467	0.318
0.7	0.879	0.848	0.673	0.594	0.597	0.503	0.532	0.452	0.424	0.286
0.8	0.879	0.855	0.644	0.575	0.568	0.485	0.503	0.437	0.373	0.249
0.9	0.889	0.874	0.606	0.551	0.534	0.470	0.472	0.427	0.302	0.199

Table 8. The MREF $\xi^D_{A,B}$ for different A and B.

A	B									
	B_1		B_2		B_3		B_4		B_5	
	H	C	H	C	H	C	H	C	H	C
A_1	0.516	0.325	-ve	-ve	-ve	-ve	-ve	-ve	-ve	-ve
A_2	0.839	0.764	0.551	0.365	0.453	0.241	0.342	0.140	0.192	-ve
A_3	0.874	0.809	0.632	0.475	0.552	0.371	0.456	0.284	0.328	0.065
A_4	0.916	0.871	0.731	0.589	0.661	0.484	0.571	0.388	0.438	0.100
A_5	0.920	0.880	0.771	0.675	0.721	0.612	0.663	0.559	0.586	0.424

Note that a negative value of $\rho^D_{A_1,B}(\alpha)$ $\left(\text{or } \xi^D_{A,B}\right)$ means that there is no possibility to get an equivalence between two different designs of the system that may be obtained by using reduction and duplication methods.

Based on the results given in Tables 2-8, one can conclude that:

1. $\rho^H_{A,B}(\alpha) > \rho^C_{A,B}(\alpha)$ for all possible values of α and subsets A and B (as it was expected).

2. Sometimes, the equivalence between improved designs obtained by following the reduction method and duplication methods is not possible for all subsets of the system components. For example, the negative sign of $\rho^H_{A_1,B_2}(\alpha)$ $\left(\text{or } \xi^H_{A_1,B_2}\right)$ means that the improved design of the system that obtained by improving the most structural importance component according to hot duplication method can not be equivalent, in the sense of having the same reliability function (or the same MTTF), with that improved system which can be derived by improving a subset of the system components via the reduction method.

3. The 0.1−fractile of the original system increases from $L(0.1) = 1.549$ to $L^H_{B_1}(0.1) = 1.593$ $\left(L^C_{B_1} = 1.634\right)$ by improving the component with smallest structural importance according to hot (cold) duplication method, see Table 2. The same result can be done by doing one of the following:

3.1 reducing the failure rate of the same component by the SREF $\rho^H_{A_1,B_1}(0.1) = 0.632$ $\left(\rho^C_{A_1,B_1}(0.1) = 0.407\right)$, see Table 3.

3.2 reducing the failure rate of the component which has the most structural importance by the SREF $\rho^H_{A_2,B_1}(0.1) = 0.888$ $\left(\rho^C_{A_2,B_1}(0.1) = 0.798\right)$, see Table 4.

3.3 reducing the failure rate of two components which satisfy that one of them becomes less important when the other is functioning by the SREF $\rho^H_{A_3,B_1}(0.1) = 0.910$ $\left(\rho^C_{A_3,B_1}(0.1) = 0.836\right)$, see Table 5.

3.4 reducing the failure rate of two components which satisfy that the importance of one of them is unchanged by functioning the other by the SREF $\rho^H_{A_4,B_1}(0.1) = 0.941$ $\left(\rho^C_{A_4,B_1}(0.1) = 0.890\right)$, see Table 6.

3.5 reducing the failure rate of two components which satisfy that one of them becomes more important when the other is functioning by the SREF $\rho^H_{A_5,B_1}(0.1) = 0.944$ $\left(\rho^C_{A_5,B_1}(0.1) = 0.898\right)$, see Table 7.

4. In the same manner one can explain the reset of the results presented in Tables 2-7.

According to the results presented in Table 8, one can conclude that follows:

1. $\xi^H_{A,B} > \xi^C_{A,B}$ for all subsets A and B (as for SREF).

2. Improving the smallest structural importance component according to hot (cold) duplication method increases the MTTF of the system form $\frac{0.817}{\lambda}$ to $\text{MTTF}^H_{B_1} = \frac{0.850}{\lambda}$ $\left(\text{MTTF}^C_{B_1} = \frac{0.869}{\lambda}\right)$, see Table 1. The same amount of increase in the MTTF can be derived by doing any one of the following:

1.1 reducing the failure rate of a similar component by MREF $\xi^H_{A_1,B_1} = 0.516$ $(\xi^C_{A_1,B_1} = 0.325)$, see Table 8.

1.2 reducing the failure rate of the most structural important component by MREF $\xi^H_{A_2,B_1} = 0.839$ $(\xi^C_{A_2,B_1} = 0.746)$, see Table 8.

1.3 reducing the failure rates of two components one of them becomes less important when the other is functioning by the same MREF $\xi^H_{A_3,B_1} = 0.874$ $(\xi^C_{A_3,B_1} = 0.809)$, see Table 8.

1.4 reducing the failure rates of two components such that the importance of any of them does not depend on the stat of the other by the same MREF $\xi^H_{A_4,B_1} = 0.916$ $(\xi^C_{A_4,B_1} = 0.871)$, see Table 8.

1.5 reducing the failure rates of two components one of them becomes more important when the other is functioning by the same MREF $\xi^H_{A_5,B_1} = 0.92$ $(\xi^C_{A_5,B_1} = 0.88)$, see Table 8.

2. Following the hot (cold) duplication method to improve the system via improving two of its components one of them becomes more importance if the other is functioning increases the system MTTF from $\frac{0.817}{\lambda}$ to $\text{MTTF}^H_{B_4} = \frac{1.102}{\lambda}$ $\left(\text{MTTF}^C_{B_4} = \frac{1.387}{\lambda}\right)$, see Table 1. Reduction method can not be used to get an improved design of the system with the same $\text{MTTF}^D_{B_4}$ by improving a component with lowest structural importance since $\xi^D_{A_1,B_5} < 0$, for $D = H, C$. It is not possible to get an improved design with MTTF equals $\text{MTTF}^C_{B_4}$ if we decide to use the reduction method to improve the most structural importance component since $\xi^C_{A_2,B_5} < 0$. On the other hand, an improved design that has the the same MTTF as $\text{MTTF}_{B_4}^H$ $\left(\text{MTTF}_{B_4}^C\right)$ can be reached by using the reduction method when:

2.1 the failure rates of two component one of them becomes less importance when the other is functioning are reduced by MREF $\xi_{A_3,B_5}^H = 0.328$ ($\xi_{A_3,B_5}^C = 0.065$), see Table 8.

2.2 the failure rates of two components such that the importance of any of them does not depend on whether the other is functioning are reduced by the same MREF $\xi_{A_4,B_5}^H = 0.438$ ($\xi_{A_4,B_5}^C = 0.1$), see Table 8.

2.3 the failure rates of same components or another similar components are reduced by the same MREF $\xi_{A_5,B_5}^H = 0.586$ ($\xi_{A_5,B_1}^C = 0.424$), see Table 8.

3. In the same manner on can analyze the rest of the results booked in Table 8.

Finally, the plots of the reliability function of the original and improved designs of the system shown in Figures 4,5 agree with the theoretical results stated in section 3 and with the numerical analysis previously concluded.

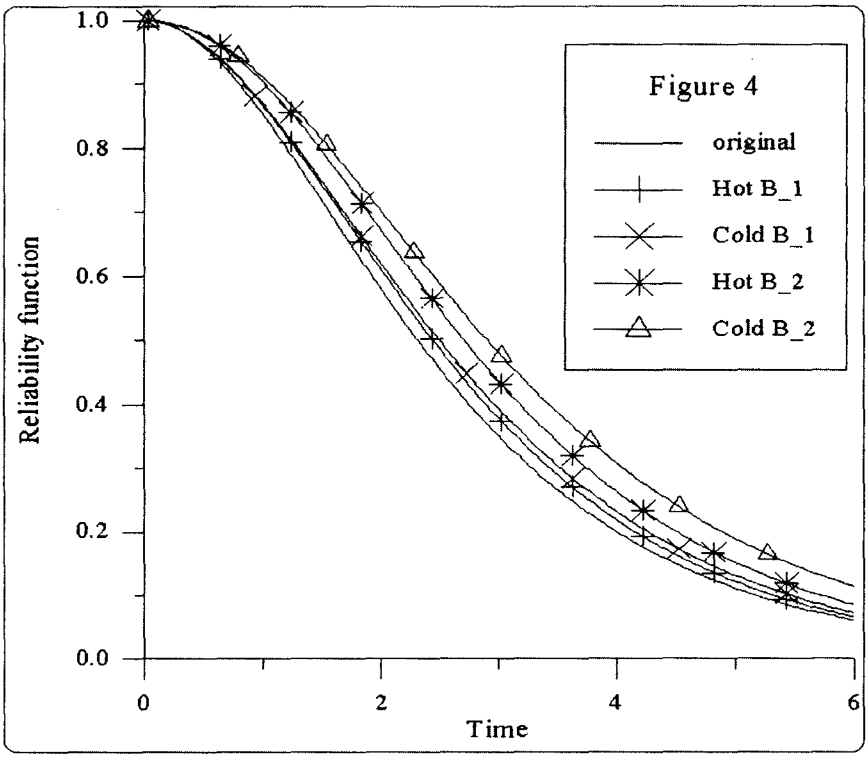

Figure 4. The reliability functions of the original and improved designs when $|B| = 1$.

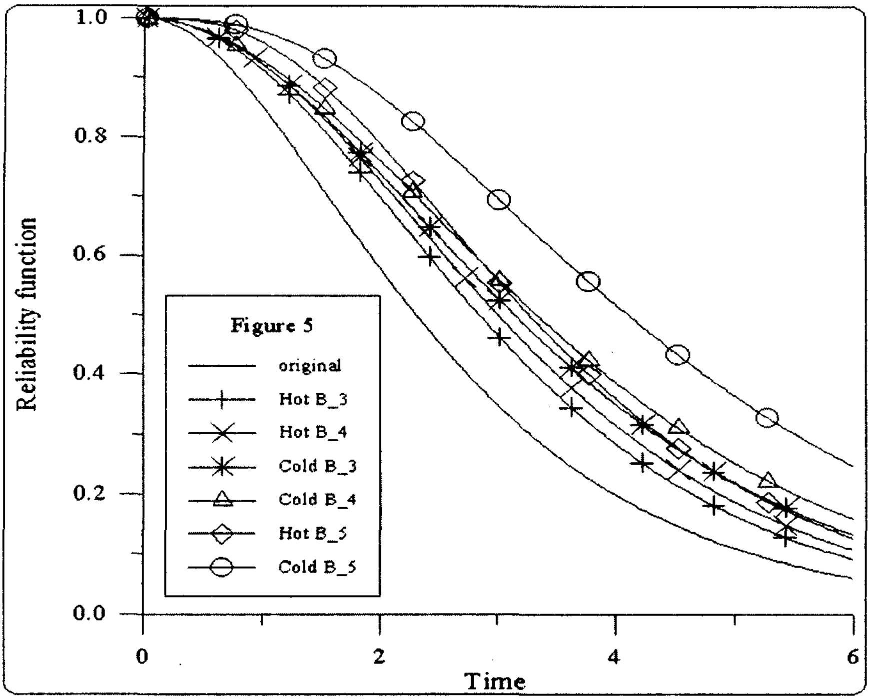

Figure 5. The reliability functions of the original and improved designs when $|B| = 2$.

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