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PRIMAL-DUAL 내부점법에 관한 연구

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A Study on Primal-Dual Interior-Point Method

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Abstract : The Primal-Dual Interior-Point (PDIP) method is currently one of the fastest emerging topics in optimization. This method has become an effective solution algorithm for large scale nonlinear optimization problems, such as the electric Optimal Power Flow (OPF) and natural gas and electricity OPF. This study describes major theoretical developments of the PDIP method as well as practical issues related to implementation of the method. A simple quadratic problem with linear equality and inequality constraints

Key words : Newton's method (뉴우톤 방법), primal-dual interior-point (PDIP) method (프라이멀-듀얼 내부점법), central path (중앙경로), barrier function (장벽함수), barrier parameter (장벽 파라미터), interior-point method (내부점법), KKT conditions (KKT 조건)

1. INTORDUCTION

Optimization models try to represent, in mathematical form, the objective of solving a problem in the best way[1]. That can be defined as the process of running a business to maximize profit, minimize loss, maximize efficiency, or minimize risk. It can be the process of providing the required electric power and line losses by allocating generation among a set of on-line generating units such that total generation cost is minimized[2]. It might mean designing a ship to minimize weight or maximize cargo space.

Optimization models arise in almost every area of application since the desire to solve a problem in an optimal way is so common. In recent years, they have become more essential as businesses become larger and more complicated, and as engineering designs become more challenging.

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Remarkable advances in computer hardware and software have been achieved for the last few decades, and these improvements have made optimiza -tion models a practical tool in various applications.

Linear programming has dominated paradigm in optimization since its formulation in the 1930s and 1940s and the development of the simplex algorithm by Dantzig in the mid 1940s[3]. The simplex method is the most widely used method for linear programming and one of the most widely used of all numerical algorithms. It is only in recent years with the development of interior-point methods that the simplex method has had a serious challenge for dominance in linear programming. A general feature common to the interior-point methods is that each iteration is strictly feasible. Thus, in contrast to the simplex algorithm where the movement is along the boundary of the feasible region, the points generated by these new approaches lie in the interior of the feasible region[1]. The interior-point methods are much faster than the simplex method on large linear programs. A thorough theoretical understanding of these methods has emerged for the last two decades, and the theory has been extended to other nonlinear optimization problems.

This tutorial introduces the Primal-Dual Interior-Point (PDIP)method, which has been recognized as the best available algorithm for non-linear optimization programs. It describes major theoretical developments of the PDIP method based on the logarithmic barrier function method, which mainly deals with inequality constraints. Updating the barrier parameter associated with the central path is also discussed for practical issues related to implementation of the PDIP method.

2. NEWTON'S METHOD

Newton's method has been the standard solution algorithm to solve for optimiza -tion problems with various equality and inequality constraints for several decades [4], [5]. Newton's method is a very powerful algorithm because of its rapid convergence near the solution. This property is especially beneficial for power system applications because an initial guess close to the solution is easily obtained[6]. For example, voltage magni -tude at each bus is presumably near the rated system value, generator outputs can be estimated from historical data, and transformer tap ratios are near 1.0 p.u. during steady-state operation.

The solution of the constrained optimization problem requires the mathema -tical formation of the Lagrangian by

$$\mathcal{L}(Y,\lambda,\mu)=C(Y)+\sum_{i=1}^{n}\lambda_i h_i(Y)+\sum_{i\in\mathcal{A}}\mu_i g_i(Y) \quad (1)$$

where λ_i is the Lagrange multiplier for the i^{th} equality constraint. Assuming that we know which inequality constraints are binding, and have put them in the set $\mathcal{A}$, then the inequality constraints can now be enforced as equality constraints. Thus the $\mu_i's$ in (1) have the same property as $\lambda_i's$

and they are the Lagrange multipliers for binding inequality constraints. However, we need $\mu_i \geq 0$ for every i[2]. We can ignore the inequality constraints that are not binding since their $\mu's$ are known to be zero by complementary slackness condition[2]. That is

$$g_i(Y) \leq 0 \Rightarrow \mu_i = 0,$$
$$g_i(Y) = 0 \Rightarrow \mu_i \geq 0.$$

Therefore, only binding inequality constraints are included in the Lagrangian function (1) with corresponding nonzero $\mu's$.

Solution of a constrained optimization problem can be solved by adjusting control and state variables, and Lagrange multip -liers to satisfy the following first-order necessary optimality conditions:

$$\begin{aligned} &1) \quad \frac{\partial \mathcal{L}}{\partial Y_i} = 0, \\ &2) \quad \frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i = 0, \\ &3) \quad \frac{\partial \mathcal{L}}{\partial \mu_i} = g_i \leq 0, \\ &4) \quad \mu_i \geq 0 \text{ and } \mu_i g_i = 0. \end{aligned} \tag{2}$$

The above equations are also called the Karush-Kuhn-Tucker (KKT) conditions.

Let us define

$$\omega(z) = \nabla_z \mathcal{L}(z) = \left[\frac{\partial \mathcal{L}}{\partial Y} \quad \frac{\partial \mathcal{L}}{\partial \lambda} \quad \frac{\partial \mathcal{L}}{\partial \mu_{\mathcal{A}}} \right]^T = 0 \tag{3}$$

where z is a vector of $[Y^T \quad \lambda^T \quad \mu_{\mathcal{A}}^T]^T$, and $\mathcal{A}$ represents the binding inequality constraints.

To solve the KKT conditions, Newton's method is applied by using the Taylor's series expansion around a current point z^p as:

$$\omega(z) = \omega(z^p) + \left.\frac{\partial \omega(z)}{\partial z}\right|_{z=z^p} \cdot (z - z^p) + \frac{1}{2}(z - z^p)^T \cdot \left.\frac{\partial^2 \omega(z)}{\partial z^2}\right|_{z=z^p} \cdot (z - z^p) + \cdots = 0.$$

H.O.T

The current point z^p can either be an initial guess in the first iteration of the computation, or the estimate solution from the prior iteration. Recall that we want $\omega(z) = 0$. By ignoring the high order terms (H.O.T) and defining $\Delta z = z - z^p$, the above equation can be rewritten as:

$$\left.\frac{\partial \omega(z)}{\partial z}\right|_{z=z^p} \cdot \Delta z = -\omega(z^p) \tag{4}$$

The quantity Δz is the update vector, or the Newton step, and it tells how far and in which direction the variables and multipliers should move from this current point z^p to get closer to the solution. Since $\omega(z)$ is the gradient of the Lagrangian function $\mathcal{L}(z)$, equation (4) can be written in terms of the Lagrangian function $\mathcal{L}(z)$ as:

$$\nabla_z \nabla_z^T \mathcal{L}(z) \cdot \Delta z = -\nabla_z \mathcal{L}(z) \tag{5}$$

Or, simply

$$W \cdot \Delta z = -\omega(z) \tag{6}$$

where W denotes the second order derivatives (or the Hessian matrix) and ω is the gradient, both of the Lagrangian function with respect to z evaluated at

the current point. Equation (6) can be written in matrix form as

$$\begin{bmatrix} H_Y & J^T & A^T \\ J & 0 & 0 \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta Y \\ \Delta\lambda \\ \Delta\mu_A \end{bmatrix} = -\begin{bmatrix} \nabla_Y \mathcal{L} \\ \nabla_\lambda \mathcal{L} \\ \nabla_{\mu_A} \mathcal{L} \end{bmatrix} \tag{7}$$

where the Hessian matrix H_y and Jacobian matrices J and A are given as follows:

$$H_Y = \frac{\partial^2 \mathcal{L}(z)}{\partial Y^2},$$
$$J = \frac{\partial^2 \mathcal{L}(z)}{\partial \lambda \partial Y} = \frac{\partial h(Y)}{\partial Y},$$
$$A = \frac{\partial^2 \mathcal{L}(z)}{\partial \mu_A \partial Y} = \frac{\partial g_A(Y)}{\partial Y},$$

where g_A is a set of binding inequality constraints. The Newton step can be obtained by solving (6). Then a vector of estimated solution for the next iteration is updated as:

$$z^{p+1} = z^p + \alpha \Delta z \tag{8}$$

where α is usually 1, but can be adjusted to values above or below 1 to speed up convergence or cause convergence in a divergent case. It is important that special attention be paid to the inequality constraints. Equation (1) only includes binding inequality constraints being enforced as equality constraints. Thus, after obtaining an updated set of variables and multipliers, a new set of binding inequality constraints (or what we think is the active set) should be determined as follows [4]:

- If the updated μ's of the constraint functions in the current active set are zero or have become negative, then the corresponding constraints must be released from the current active set because $\mu_i < 0$ implies that $g_i = 0$ keeps the trial solution at the edge of the feasible region instead of allowing the trial solution to move into the interior of the feasible region.
- If other constraint functions evaluated at the updated variables violate their limits, then those constraints must be included in the new active set. The variable α may be chosen to prevent constraint violations, but $a < 1$ to avoid infeasibility implies that the constraint would otherwise be violated.

As a result, if μ_i is positive, continued enforcement will result in an improvement of the objective function, and enforcement is maintained. If μ_i is negative, then enforcement will result in an decrease of the objective function, and enforcement is stopped.

Once the active set has been updated, $\omega(z^{p+1})$ is checked for convergence. There are several criteria for checking convergence of Newton's method. The convergent tolerance may be set on the maximum absolute value of elements in $\omega(z)$, or on its norm. If the updated z^{p+1} does not satisfy the desired convergence criterion, the Newton step calculation is repeated.

3. PRIMAL-DUAL INTERIOR POINT METHOD

One of disadvantages of Newton's method is to identify a set of binding

inequality constraints, or active constraints. Among several methods to avoid the difficulty associated with guessing the correct active set, the PDIP method has been acknowledged as one of the most successful [4], [7], [8]. Interior point methods for optimization have been widely known since the publication of Karmarkar's seminal paper in 1984 [9]. Barrier function methods were proposed much earlier in Russia but little attention was paid because the algorithm was so slowin implementation. Later, this method was shown to be equivalent to the interior point methods. Karmarkar's method results in numerical ill-conditioning although this problem is not so bad with the PDIP method.

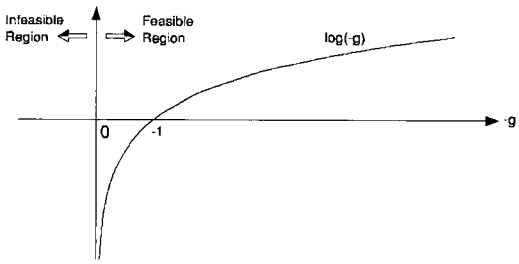

Fig. 1 Effect of barrier term

The method uses a barrier function that is continuous in the interior of the feasible set, and becomes unbounded as the boundary of the set is approached from its interior. Two examples of such a function [1], [3] are the logarithmic function, as shown in Fig. 1

$$\phi(Y)=-\sum_{i=1}^{m}\ln(-g_i(Y)) \tag{9}$$

and the inverse of the inequality function

$$\phi(Y)=\sum_{i=1}^{m}\frac{1}{-g_i(Y)} \tag{10}$$

where

$$g_i(Y)\le 0 .$$

The barrier method generates a sequence of strictly feasible iterates that converge to a solution of the problem from the interior of the feasible region [1], [3].

To apply the primal-dual interior point algorithm to the OPF problem that has equality and inequality constraints, we construct the nonlinear equality and inequality-constrained optimization problem as

$$\min_{Y} f(Y) \tag{11}$$

subject to

$$h_i(Y)=0, \qquad i=1,\cdots,n,$$
$$g_i(Y)\le 0, \qquad i=1,\cdots,m.$$

To solve the problem, we first form the *logarithmic barrier function* as

$$B=f(Y)-\nu\sum_{i=1}^{m}\ln(-g_i(Y)) \tag{12}$$

where the parameter ν is referred to as the barrier parameter, a positive number that is reduced to approach to zero as the algorithm converges to the optimum. Then we solve a sequence of constrained minimization problems of the form

$$\min_{(Y,\lambda,\nu)} B(Y,\lambda,\nu) \tag{13}$$

subject to

$$h_i(Y)=0, \qquad i=1,\cdots,n,$$

for a sequence $\{\nu_k\}$ of positive barrier parameters that decrease monotonically to zero. The solution of this problem by Newton's method requires the formulation of the Lagrangian function

$$\mathcal{L}=f(Y)+\sum_{i=1}^{n}\lambda_i h_i(Y)-\nu\sum_{i=1}^{m}\ln(-g_i(Y)) \quad (14)$$

Because the barrier term is infinite on the boundary of the feasible region, as shown in Fig. 1, it acts as a repelling force that drives the current trial solution away from the boundary into the interior of the feasible region. As the barrier parameter ν is decreased, the effect of the barrier term is diminished, so that the iterates can gradually approach the constraint boundaries of the feasible region for those constraints which eventually turn out to be binding.

To solve equation (13), we need to find the gradient of $\mathcal{L}$ with respect to Y:

$$\begin{aligned}\nabla_Y\mathcal{L}&=\nabla_Y f(Y)+\sum_{i=1}^{n}\lambda_i\nabla_Y h_i(Y)+\nu\sum_{i=1}^{m}\frac{1}{-g_i(Y)}\nabla_Y g_i(Y)\\&=\nabla_Y f(Y)+\sum_{i=1}^{n}\lambda_i\nabla_Y h_i(Y)+\nu\sum_{i=1}^{m}\frac{1}{s_i}\nabla_Y g_i(Y)\\&=\nabla_Y f(Y)+\sum_{i=1}^{n}\lambda_i\nabla_Y h_i(Y)+\sum_{i=1}^{m}\mu_i\nabla_Y g_i(Y)\\&=\nabla_Y f(Y)+h_Y^T\lambda+g_Y^T\mu,\end{aligned} \quad (15)$$

where

$$\begin{aligned}-g_i(Y)&=s_i, \qquad i=1,\cdots,m,\\ \mu_i s_i&=\nu, \qquad i=1,\cdots,m.\end{aligned}$$

h_Y^T is a matrix consisting of the gradient $\nabla_Y h_i(Y)$ as columns, and g_Y^T is a matrix consisting of the gradient $\nabla_Y g_i(Y)$ as columns, or

$$\begin{aligned}h_Y^T&=\left[\nabla_Y h_1(Y) \;\; \nabla_Y h_2(Y) \;\; \cdots \;\; \nabla_Y h_n(Y)\right],\\ g_Y^T&=\left[\nabla_Y g_1(Y) \;\; \nabla_Y g_2(Y) \;\; \cdots \;\; \nabla_Y g_m(Y)\right].\end{aligned}$$

Also, λ is a vector containing the elements λ_i, and μ is a vector containing the elements μ_i. The set of equations we must solve is

$$\begin{aligned}\nabla_Y f+\sum_{i=1}^{n}\lambda_i\nabla h_i+\sum_{i=1}^{m}\mu_i\nabla g_i&=0,\\ h_i(Y)&=0, \qquad i=1,\cdots,n\\ g_i(Y)+s_i&=0, \qquad i=1,\cdots,m\\ \mu_i s_i-\nu&=0, \qquad i=1,\cdots,m\\ s_i&>0, \qquad i=1,\cdots,m\\ \mu_i&>0, \qquad i=1,\cdots,m.\end{aligned} \quad (16)$$

These equations can be solved using the Newton iterative method. Our four sets of variables for which we must solve are Y, λ, μ, and s. The equation for a first order approximation of a Taylor series of a function, F, whose independent variables are Y, λ, μ and s is

$$\begin{aligned}F(Y,\lambda,\mu,s)\cong F(Y_o,\lambda_o,\mu_o,s_o)&+\frac{\partial F}{\partial Y}\Delta Y+\frac{\partial F}{\partial\lambda}\Delta\lambda\\&+\frac{\partial F}{\partial\mu}\Delta\mu+\frac{\partial F}{\partial s}\Delta s.\end{aligned} \quad (17)$$

We want to find the values of Y, λ, μ and s where the expressions on the left side of the equations we want to solve evaluate to zero. We use Newton-Raphson to do so.

Taking the first order approximation to the Taylor series for each of the four expressions given in (16) and setting them equal to zero (the desired value for each) give us

$$\begin{aligned}(\nabla_Y f + h_Y^T\lambda + g_Y^T\mu) + (\nabla_Y^2 f + \sum_{i=1}^{n}\lambda_i\nabla_Y^2 h_i \\ + \sum_{i=1}^{m}\mu_i\nabla_Y^2 g_i)\Delta Y + h_Y^T\Delta\lambda + g_Y^T\Delta\mu = 0, \\ h + h_Y\Delta Y = 0, \\ (g+s) + g_Y\Delta Y + I\Delta s = 0, \\ (MSe - \nu e) + S\Delta\mu + M\Delta s = 0,\end{aligned} \tag{18}$$

where

I : an identity matrix,

S : a diagonal matrix constructed from $(s_1, s_2, \cdots, s_m)$,

M : a diag. matrix constructed from $(\mu_1, \mu_2, \cdots, \mu_m)$,

e : a column vector with all elements 1.

The above equations can be organized in matrix form as

$$\begin{bmatrix} H_Y & h_Y^T & g_Y^T & 0 \\ h_Y & 0 & 0 & 0 \\ g_Y & 0 & 0 & I \\ 0 & 0 & S & M \end{bmatrix}\begin{bmatrix}\Delta Y \\ \Delta\lambda \\ \Delta\mu \\ \Delta s\end{bmatrix} = \begin{bmatrix} -\nabla_Y f - h_Y^T\lambda - g_Y^T\mu \\ -h \\ -g - s \\ \nu e - MSe\end{bmatrix},$$

or

$$W\Delta z = \Delta F.$$

where

$$H_Y = \nabla_Y^2 f + \sum_{i=1}^{n}\lambda_i\nabla_Y^2 h_i + \sum_{i=1}^{m}\mu_i\nabla_Y^2 g_i.$$

We initially set ν to some relatively large number, such as 10. Starting with an initial guess

$$z_o = \begin{bmatrix} Y_o \\ \lambda_o \\ \mu_o \\ s_o \end{bmatrix},$$

we calculate the W matrix and the ΔF vector. If all the elements of ΔF are sufficiently close to zero, we have found the solution z_o. Otherwise we must solve for Δz by

$$\Delta z = W^{-1}\Delta F. \tag{19}$$

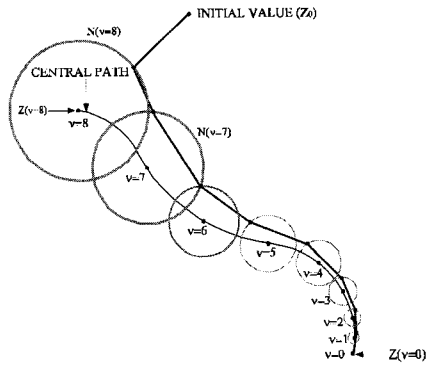

Fig. 2 Graphical representation of central path

Then the original z is updated by

$$\Delta z = W^{-1}\Delta F. \tag{20}$$

However, following conditions need to be satisfied when updating z :

$$\mu_i > 0,$$
$$s_i > 0.$$

Therefore, when calculating the updated μ and S, we must make sure that each μ_i and each S_i is still strictly greater than zero when $\Delta\mu_i$ and Δs_i are added to it, respectively. If adding $\Delta\mu_i$ or Δs_i violates this condition, all $\Delta\mu_i$ and all Δs_i must be scaled by some factor α less than one before adding them as in (20).

Now that we have a new guess for z, the W matrix and ΔF vector are calculated again. If the elements of the ΔF vector are sufficiently close to zero, we have found

the z which solves our problem. Otherwise we solve for Δz and update z again. This process is repeated until we find the z which makes ΔF very close to zero. After we solve for z, we reduce the barrier parameter ν by some factor κ. For example, if $\kappa = 0.4$, the barrier parameter ν is updated by letting the new ν be 0.4 times the old ν. Using the value of z obtained using the old ν as an initial guess z_o, we use the iterative procedure described above to solve for the value of z which makes ΔF approximately zero for the new barrier parameter. The process of solving for z, decreasing ν, and then solving for z again is repeated until ν becomes a very small number, such as 10-10. This process is illustrated in Fig. 2. When ν gets this small, we have found the z that solves our problem. When solving for z given a particular ν, it is not necessary to force ΔF to be zero. What we really want to know is how close we are to the central path. The central path is defined by a sequence of $\{y(\nu), \lambda(\nu), \mu(\nu), s(\nu)\}$ which make ΔF evaluate to zero for every possible value of ν. One way of measuring how close we are to the central path is by checking to see how close each product $\mu_i s_i$ is to the barrier parameter ν. One proposed way of doing this is by first calculating the average value of $\mu_i s_i$, also known as the duality factor [3], by

$$\hat{\nu} = \sum_{i=1}^{m} \frac{\mu_i s_i}{m},$$

which evaluates to zero when each product $\mu_i s_i$ is equal to ν. Instead of checking to see if ΔF is sufficiently close to zero, we check to see when the following is true:

$$\left\| \begin{matrix} \mu_1 s_1 - \hat{\nu} \\ \mu_2 s_2 - \hat{\nu} \\ \vdots \\ \mu_m s_m - \hat{\nu} \end{matrix} \right\|_2 \leq \tau\nu$$

where

$$0 < \tau < 1.$$

When this logical statement becomes true, we are sufficiently close to the central path, and we can reduce ν.

4. IMPLEMENTATION

Consider a quadratic objective function with linear equality and inequality constraints :

$$\min_{x,y} f(x,y) = y^2 - 4xy - x^2 + 8y + 14x$$

subject to:

$$h = -\frac{x}{5} - y + 3 = 0,$$
$$g_1 = 5x - y - 12 \leq 0,$$
$$g_2 = x - y \leq 0.$$

Note that in solving for x, y, λ, and μ, it is necessary to know (or assume) which constraints are active. Of course, this is usually not the case. But we can proceed by trial and error to find a candidate solution where the KKT conditions are all satisfied. Since we have two inequality constraints, there are $2 = 4$ possible solutions. Suppose that inequality const -raint g_1 is only active, and the associated

multiplier is defined by μ_1. We ignore inequality constraint g_2 since we assume that it is not binding. The Lagrangian is then

$$\mathcal{L}(x,y,\lambda,\mu_1)=\left(y^2-4xy-x^2+8y+14x\right)+\lambda\left(-\frac{x}{5}-y+3\right)+\mu_1(5x-y-12).$$

The first-order necessary optimality conditions become

$$0=\nabla_x\mathcal{L}=(-2x-4y+14)-0.2\lambda+5\mu_1,$$
$$0=\nabla_y\mathcal{L}=(-4x+2y+8)-\lambda-\mu_1,$$
$$0=\nabla_\lambda\mathcal{L}=h=-\frac{x}{5}-y+3,$$
$$0=\nabla_{\mu_1}\mathcal{L}=g_1=5x-y-12.$$

Collecting these terms into matrix-vector form, we obtain

$$\begin{bmatrix} -2 & -4 & \vdots & -0.2 & \vdots & 5 \\ -4 & 2 & \vdots & -1 & \vdots & -1 \\ \cdots & \cdots & & \cdots & & \cdots \\ -0.2 & -1 & \vdots & 0 & \vdots & 0 \\ \cdots & \cdots & & \cdots & & \cdots \\ 5 & -1 & \vdots & 0 & \vdots & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ \cdots \\ \lambda \\ \cdots \\ \mu_1 \end{bmatrix}=\begin{bmatrix} -14 \\ -8 \\ \cdots \\ -3 \\ \cdots \\ 12 \end{bmatrix},$$

which gives

$$\begin{bmatrix} x \\ y \\ \cdots \\ \lambda \\ \cdots \\ \mu_1 \end{bmatrix}=\begin{bmatrix} 2.8846 \\ 2.4231 \\ \cdots \\ 0.9763 \\ \cdots \\ 0.3314 \end{bmatrix}.$$

Since the above solution does not satisfy inequality constraint g_2, it is not a feasible solution. In a similar way, we can verify other cases, which are summarized in the following table.

g_1 binding?	g_2 binding?	x^*	y^*	f^*	KKT
No	No	-2.5	3.5	34	Yes
Yes	No	2.89	2.42		No
No	Yes	2.5	2.5	30	Yes
Yes	Yes	0	0		No

So two of the possible solutions satisfy the KKT conditions. The optimal solution is obtained with $x=2.5$ and $y=2.5$ when g_2 is only binding. This problem can be also solved by using the PDIP method. A Matlab code implementing the PDIP method for this problem can be provided upon a reader's request.

5. CONCLUSIONS

Despite several attractive features of the PDIP method, it has an inherent disadvantage that the size of problem is much bigger than the Newton active set method since the PDIP method includes all inequality constraints. This problem becomes even more involved when we consider large scale problems, such as the electric optimal power flow (OPF) and natural gas electricity OPF [4], [10], [11].

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