Boundary Control of an Axially Moving Belt System in a Thin-Metal Production Line

Keum-Shik Hong, Chang-Won Kim, and Kyung-Tae Hong

Abstract: In this paper, an active vibration control of a translating steel strip in a zinc galvanizing line is investigated. The control objectives in the galvanizing line are to improve the uniformity of the zinc deposit on the strip surfaces and to reduce the zinc consumption. The translating steel strip is modeled as a moving belt equation by using Hamilton's principle for systems with moving mass. The total mechanical energy of the strip is considered to be a Lyapunov function candidate. A nonlinear boundary control law that assures the exponential stability of the closed loop system is derived. The existence of a closed-loop solution is shown by proving that the closed-loop dynamics is dissipative. Simulation results are provided.

Keywords: Asymptotic stability, axially moving system, boundary control, hyperbolic partial differential equation, Lyapunov method, nonlinear vibrations, zinc galvanizing line.

1. INTRODUCTION

The examples of axially moving systems are found in various engineering areas: axially moving steel strips in the thin-metal production line, power transmission belts, textile fibers, aerial cable thread lines, magnetic tapes, band saw blades, paper sheets during processing, etc. In such systems, undesirable vibrations caused by moving objects occur during the process due to circumstances such as the eccentricity of a pulley, and/or non-stationary speed of a driving motor, and/or aerodynamic excitation from the surrounding environment, and/or non-uniformity of the material. Such vibrations cause costly defects on the final products. Fig. 1 is a picture taken from Kwang Yang Work, Korea, which shows a steel strip moving upward in a zinc galvanizing line. The rectangular box in the middle is an air cooler. Fig. 2 is a schematic of the entire line and depicts the control strategy of using a hydraulic touch-roll actuator. The preheated steel strip is passed through a hot zinc tank for zinc coating and is then pulled up vertically. The control objectives in the galvanizing line are to improve the uniformity of the zinc deposit on the strip surfaces and to reduce the zinc consumption. Therefore, an active control of the vibrations, with a minimal use of actuators and sensors, is currently the main research focus in the area of axially moving systems.

During the past several decades, axially moving systems in the form of hyperbolic partial differential equations have been extensively studied by many researchers including, and in chronological order, Carrier [3], Mote [17], Bapat and Srinivasan [2], Wickert and Mote [29, 30], Morgul [16], Wickert [28], Rebarber and Townley [23], Laousy *et al.* [11], Lee and Mote [12], Moon and Wickert [15], Oshima *et al.* [19], Pellicano *et al.* [21, 22], Shahruz [24, 25], Ahmed *et al.* [1], Fung *et al.* [8, 9], Oostveen and Curtain [18], Winkin *et al.* [31], Li *et al.* [13], Ryu and Park [33], Lee *et al.* [34], and Yang *et al.* [35]. Recently, Wickert [28] analyzed free nonlinear vibrations of an axially moving elastic tensioned beam over the sub- and super-critical transport speed ranges. Rebarber and Townley [23] analyzed the spectrum of a class of abstract partial differential equations with boundary feedback control. Laousy *et al.* [11] proposed a stabilizing boundary feedback control law for a rotating body-beam system and demonstrated that the beam vibrations are forced to decay exponentially to zero. Lee and Mote [12] investigated a boundary control technique to control the transversal vibration of an axially moving string and proved the exponential stability of the boundary-controlled string. Moon and Wickert [15] analyzed nonlinear vibrations

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of a prototypical power transmission belt system, which is excited by pulleys having a slight eccentricity, through analytical and experimental methods. Oshima *et al.* [19] showed that a self-sensing actuator driver exhibits excellent performance in suppressing the vibration of a cantilever. Pellicano and Zirilli [22] analyzed the non-linear oscillations of a one-dimensional axially moving beam with vanishing flexural stiffness and weak non-linearities. Ahmed *et al.* [1] presented a boundary layer control of a moving belt system for studies on wing-in-ground-effect. Fung *et al.* [8, 9] controlled the vibrations of an axially moving string system using a boundary controller derived by including the actuator dynamics in the plant model. Oostveen and Curtain [18] studied the problem of robust stabilization with respect to dissipative systems with collocated sensors and actuators. Shahruz [25] also showed that a non-linear string could be stabilized by a linear boundary control. Winkin *et al.* [31] dedicated their studies to the dynamical analysis of distributed parameter tubular reactors. Li *et al.* [13] developed a novel vibration control system that uses control inputs to regulate the displacement of a distributed axial moving string model

The axially moving systems can be modeled in three different ways: a string equation [8, 9, 12, 13, 25], a beam equation [22, 28], and a belt equation [15, 21] depending on the flexibility of the system and the objectives of control. In this paper, a belt equation is used because the length of the steel strip between two supporting points (see Fig. 2) is 17.5 m long and therefore it allows some longitudinal deformation as well as transversal displacement. Hamilton's principle [14] is used to derive the motion equations of the moving strip for systems of changing mass. A nonlinear right-boundary control law to suppress the vibrations of the moving strip is determined so that the total vibration energy dissipates at the right boundary. The asymptotic stability of the closed loop system is guaranteed by Lyapunov second method.

The contributions of this paper are the first time analysis of the zinc galvanizing line, and the derivation of a control-oriented model for the translating steel strip. Considering the flexibility of the strip, the model is obtained in the form of coupled hyperbolic partial differential equations that describe the transversal and longitudinal motions, which are then simplified by decoupling the two-time scale behavior of the dual motions.

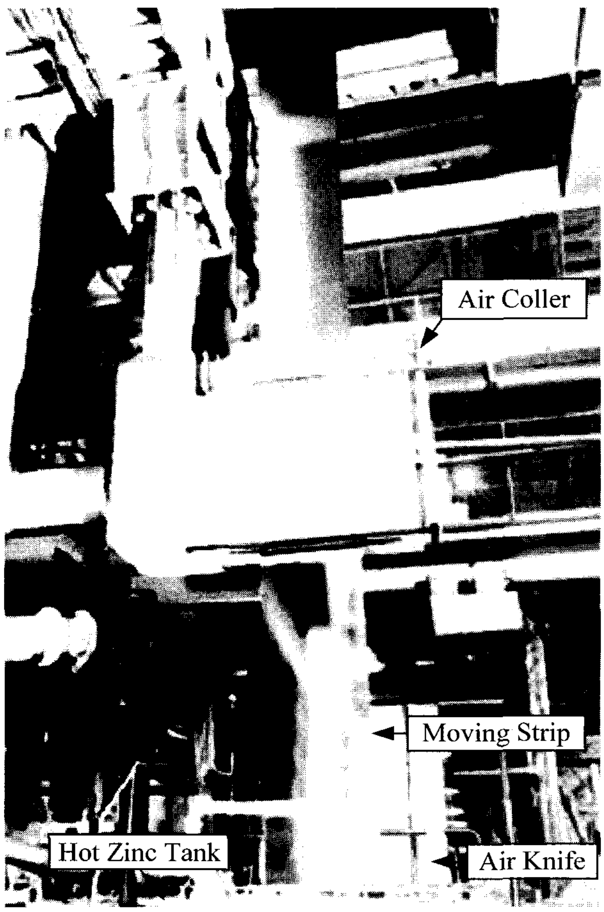


Fig. 1. A vertically moving steel strip in the zinc galvanizing line.

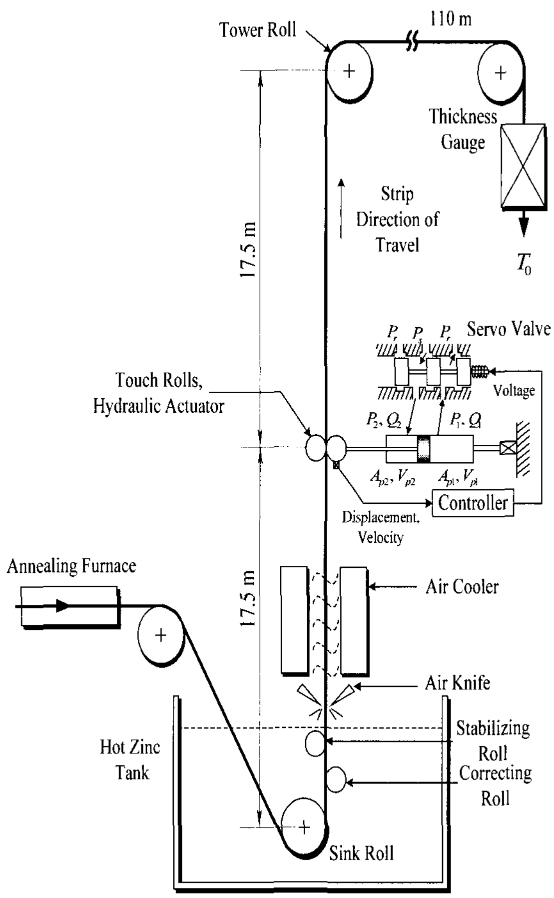


Fig. 2. A translating steel strip in the zinc galvanizing line: control strategy.

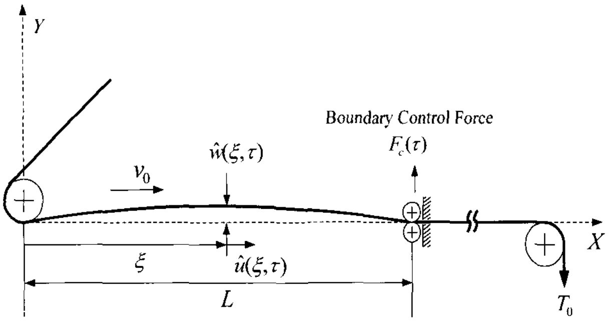


Fig. 3. A schematic of the axially moving steel strip with two intermediate touch rolls.

To the author's best knowledge, the paper is the first attempt for boundary control of a belt equation, while many researches have previously focused on investigating the analytic solutions of belt equations. The boundary control law derived is implementable and assures the exponential stability of the closed loop system.

The structure of this paper is as follows. In Section 2, the translating steel strip in the zinc galvanizing line is mathematically modeled. A normalized equation for the transversal motion is derived. In Section 3, a nonlinear boundary control law is derived. In Section 4, the existence and exponential convergence of the closed-loop solution are shown. In Sections 5 and 6, numerical simulations and conclusions are given.

2. MODELING OF THE TRANSLATING STEEL STRIP

Fig. 2 depicts a continuous hot-dip zinc galvanizing process with an active vibration control. The steel strip, which may vary in width from 800 mm to 1,400 mm and in thickness from 1.2 mm to 4.5 mm, is preheated in a continuous annealing furnace and then introduced at the speed of about 1 m/sec into a pot of molten zinc at about 450°C. The steel strip passes under a sink roll and through a pair of stabilizing and correcting rolls to emerge vertically from the pot coated with a layer of zinc. Because there is an increasing demand for greater consistency in the thickness of zinc film, a pair of air knives located approximately 0.5 m above the surface of the zinc tank, which direct a long thin wedge-shaped jet of high-velocity air toward the strip, are used to control the deposited mass by removing excess zinc and dumping it back into the pot. The strip then travels 35m in a vertical direction, while the solidification of the deposited film is enhanced with the aid of an air cooler, and 110m in a horizontal direction, cooling as it goes, to a gauge that measures the zinc mass deposited on the strip surfaces.

In order to achieve uniformity of the zinc deposit on the strip surfaces and to reduce zinc consumption, the strip must pass at equidistance from each of the air knives. However, due to the shifting and vibration of the strip, a discrepancy between the averaged deposited masses on the left and right strip surfaces and a non-uniformity of the deposited mass across the strip occur. These variations in deposited mass will degrade the quality of the product. Also, the actual measurement of the deposited mass is being performed at a downstream gauge and therefore there is a large measurement time-delay ranging from 1.8 min to 5 min depending on the translating speed of the strip. Hence, by the time a defect is detected, a lengthy strip has already been processed.

It is known that the vibration of the strip is due to the eccentricity of the sink roll. This eccentricity is again due to the wear of the bushing in the sink roll. Therefore, by means of an active control, if the transversal (lateral) vibration of the strip (at the air knives location) can be reduced, not only the uniformity of the deposited mass is achieved but also the maintenance interval of the sink roll can be extended, which means that the frequent halt of the entire production line can be prevented.

Fig. 3 shows a schematic of the axially moving steel strip for control system design purposes. Depending on the thickness of the strip and the distance between two support points, the strip can be modeled as one out of three models: a moving beam, a moving string, and a moving belt. In the zinc galvanizing line, the distance between two-support points is quite large compared to the strip thickness. Therefore, the modeling as a beam can be excluded. Also, if only the transversal displacement of the strip is concerned, a string model would suffice. In this paper, however, both the transversal and longitudinal displacements of the strip are considered, i.e., the steel strip is modeled as a moving belt. But, in the second stage, assuming that the longitudinal wave speed is far faster than the transversal speed, a decoupling of the transversal motion from the longitudinal speed is pursued, i.e., only the static relationship between the two motions is incorporated in the final "control-oriented" model, which would allow the use of a single actuator.

Now, the left boundary at the stabilizing roll in Fig. 3 is assumed to be fixed in the sense that a transverse motion is not allowed but an axial movement of the strip itself is allowed. The two touch rolls located in the middle section of the strip will play the right boundary, where the control input force is applied. Let τ be the time, ξ be the spatial coordinate along the longitude of motion, v_0 be the axial speed of the strip, $\hat{u}(\xi,\tau)$ and $\hat{w}(\xi,\tau)$ be the longitudinal and transversal displacements of the strip, respectively, and L be the length of the strip. Also, let ρ represent the mass per unit area of the strip, A be the cross section area, E be the elastic modulus, and T_0 be the tension applied to the strip. Then, with a control force $F_c(\tau)$ at the right boundary, the following equations of motion are derived: See Appendix A.

$$\rho\left(v_0^2\hat{u}_{\xi\xi}+2v_0\hat{u}_{\xi\tau}+\hat{u}_{\tau\tau}\right)-E\left(\hat{u}_\xi+\frac{1}{2}\hat{w}_\xi^2\right)_\xi=0,$$
$$\hat{u}(\xi,0)=\hat{u}_0(\xi),\ \hat{u}_\tau(\xi,0)=\hat{u}_{\tau 0}(\xi),\qquad (1)$$
$$\hat{u}(0,\tau)=\hat{u}(L,\tau)=0,$$

$$\rho A\left(v_0^2\hat{w}_{\xi\xi}+2v_0\hat{w}_{\xi\tau}+\hat{w}_{\tau\tau}\right)-T_0\hat{w}_{\xi\xi}$$
$$-AE\left\{\hat{w}_\xi\left(\hat{u}_\xi+\frac{1}{2}\hat{w}_\xi^2\right)\right\}_\xi=0,\qquad (2)$$
$$\hat{w}(\xi,0)=\hat{w}_0(\xi),\ \hat{w}_\tau(\xi,0)=\hat{w}_{\tau 0}(\xi),\ \hat{w}(0,\tau)=0,$$

and

$$F_c(\tau)-T_0\hat{w}_\xi(L,\tau)$$
$$-AE\hat{w}_\xi(L,\tau)\left\{\hat{u}_\xi(L,\tau)+\frac{1}{2}\hat{w}_\xi^2(L,\tau)\right\}=0,\qquad (3)$$

where (1)-(3) represent the equations of the longitudinal motion, the transversal motion, and the right transversal boundary condition, respectively. $(\cdot)_\xi$ and $(\cdot)_\tau$ denote $\partial(\cdot)/\partial\xi$ and $\partial(\cdot)/\partial\tau$, respectively.

Remark 1: If only the linear terms in (1)-(2) are retained, the equations represent the dynamics of a traveling rod and a tensioned string, respectively. Over a technically useful range of parameter values, the speed of the longitudinal waves is significantly faster than the transversal ones [17]. On the time scales of lower transversal modes, the tension variations propagate almost instantaneously as the influence of longitudinal inertial is small [28].

Now, the control problem of transverse vibrations is formulated: First, note that the dynamics of the two variables $\hat{u}$ and $\hat{w}$ evolve in two different time scales. From a physical point of view, the speed of the longitudinal waves is significantly faster than the speed of the transversal ones. Also, considering the practicality of implementation, i.e., the use of one actuator, a decoupling of (1) and (2) is pursued. Hence, following the work of [22], the time derivatives of $\hat{u}$ are neglected, i.e., $\hat{u}_\tau=\hat{u}_{\tau\tau}=\hat{u}_{\xi\tau}\cong 0$. Also, since $E/\rho >> v_0^2$ (for example, $E/\rho=2.5\times 10^5$ and $v_0^2=2.79$ for a steel strip), (1) can be approximated as

$$0\cong\left(\hat{u}_\xi+\frac{1}{2}\hat{w}_\xi^2\right)_\xi=\left(\psi(\tau)\right)_\xi,$$

where

$$\psi(\tau):=\hat{u}_\xi+\frac{1}{2}\hat{w}_\xi^2,\qquad (4)$$

which implies that the strain equation is not a function of ξ. By integrating both sides of (4) with respect to the spatial coordinate from 0 to ξ and using $\hat{u}(0,\tau)=0$,

$$\hat{u}(\xi,\tau)=\psi(\tau)\xi-\frac{1}{2}\int_0^\xi\hat{w}_\theta^2(\theta,\tau)d\theta,\qquad (5)$$

is derived. Now, observing that the touch rolls at $\xi=L$ do not permit any longitudinal displacement, i.e., $\hat{u}(L,\tau)=0$,

$$\psi(\tau)=\frac{1}{2L}\int_0^L\hat{w}_\theta^2(\theta,\tau)d\theta\ \geq\ 0,\quad \text{for all }\tau\geq 0\qquad (6)$$

is derived. Therefore, the $\hat{u}$-term in (2) and (3) can be approximated in terms of the transversal one as follows:

$$\hat{u}(\xi,\tau)=\frac{\xi}{2L}\int_0^L\hat{w}_\theta^2(\theta,\tau)d\theta-\frac{1}{2}\int_0^\xi\hat{w}_\theta^2(\theta,\tau)d\theta.\qquad (7)$$

Finally, the substitution of (7) into (2)–(3) yields the following "control-oriented" model for the belt equation (1)-(3):

$$\rho A\left(\hat{w}_{\tau\tau}+2v_0\hat{w}_{\xi\tau}+v_0^2\hat{w}_{\xi\xi}\right)-T_0\hat{w}_{\xi\xi}$$
$$-\frac{AE}{2}\int_0^L\hat{w}_\theta^2(\theta,\tau)d\theta\cdot\hat{w}_{\xi\xi}=0,\qquad (8)$$
$$\hat{w}(\xi,0)=\hat{w}_0(\xi),\ \hat{w}_\tau(\xi,0)=\hat{w}_{\tau 0}(\xi),\ \hat{w}(0,\tau)=0,$$

and

$$F_c(\tau)=\left(T_0+\frac{AE}{2}\int_0^L\hat{w}_\theta^2(\theta,\tau)d\theta\right)\hat{w}_\xi(L,\tau).\qquad (9)$$

In (8), only the "weak" nonlinearity from $\hat{u}$ dynamics has been incorporated. Note that (8) is a non-linear integro-differential equation.

The following new dimensionless variables are introduced.

$$x=\frac{\xi}{L},\quad t=\tau\sqrt{\frac{T_0}{\rho AL^2}},\quad w=\frac{\hat{w}}{L},\quad v=\frac{1}{\sqrt{T_0/\rho A}}v_0,$$
$$v_T=\sqrt{\frac{EA}{T_0}},\ \text{and}\ f_c(t)=\frac{F_c(t)}{T_0}.\qquad (10)$$

The substitution of (10) into (8)–(9) yields the normalized equations of the transversal motion and the right boundary condition, respectively, as follows:

$$w_{tt}+2vw_{xt}+\left(v^2-1\right)w_{xx}=\frac{1}{2}v_T^2\int_0^1 w_x^2dx\cdot w_{xx},$$
$$w(x,0)=w_0(x),\ w_t(x,0)=w_{t0}(x),\ w(0,t)=0,\qquad (11)$$

and

$$f_c(t) = \left\{1 + \frac{1}{2} v_T^2 \int_0^1 w_x^2 dx\right\} w_x(1,t), \tag{12}$$

where the terms w_{tt}, $2vw_{xt}$, and w_{xx} represent the local, the Coriolis, and the centrifugal acceleration components, respectively, and $f_c(t)$ is the control input force to be designed. Note that (12) involves an integral term, which may not be easily implementable. However, through the boundary control law design in the upcoming Section 3, the control force in (12), $f_c(t)$, will be re-formulated in an implementable form.

Remark 2: The equilibrium solutions $w(x)$ of (11) satisfy:

$$\left\{\left(v^2 - 1\right) + \frac{1}{2} v_T^2 \int_0^1 w_x^2 dx\right\} w_{xx} = 0,$$

$w(0) = 0$, and $w_x(1) = 0$.

Note that the trivial solution $w(x) = 0$ is always a solution.

Remark 3: In (11), a critical speed exists at $v = 1$. That is, the fundamental natural frequency vanishes and divergence instability occurs [17]. Thus, in this paper it is assumed that v belongs to a sub-critical speed range, i.e., $|v| < 1$. In simulations in Section V, $v = 0.0119$ will be used.

Remark 4: The normalized form of (4)-(6) using (10) becomes:

$$u_x(x,t) + \frac{1}{2} w_x^2(x,t) = \frac{1}{2} \int_0^1 w_x^2(x,t)\, dx. \tag{13}$$

3. BOUNDARY CONTROL LAW

The objective is to design a right-boundary control law that guarantees the asymptotic stabilization of the axially moving strip. It is assumed that the dynamics of the strip are well represented by the non-linear integro-differential equation (11). For proving the stability of the closed-loop system, the Lyapunov second method is pursued.

As a positive definite function, the total mechanical energy $V(t)$ of the strip is considered as follows:

$$V(t) = \frac{1}{2} \int_0^1 \left\{(v + vu_x + u_t)^2 + (w_t + vw_x)^2\right\} dx + \frac{v_T^2}{2} \int_0^1 \left(u_x + \frac{1}{2} w_x^2\right)^2 dx + \int_0^1 \left(u_x + \frac{1}{2} w_x^2\right) dx. \tag{14}$$

Note that the kinetic and potential energies of both w-dynamics and u-dynamics have been incorporated in (14).

The time derivative of an energy function of a moving material, which involves a fixed control volume and a control surface through which mass flows in and out, can be computed by applying the Reynold's transport theorem for the systems with changing mass [7] as follows:

$$\begin{aligned}\frac{d}{dt} V(t) &= \frac{d}{dt} \int_0^1 \tilde{V}(x,t) dx = \int_0^1 \frac{d}{dt} \tilde{V}(x,t) dx \\ &= \int_0^1 \left[\frac{\partial}{\partial t} \tilde{V}(x,t) + v \frac{\partial}{\partial x} \tilde{V}(x,t)\right] dx \\ &= \frac{\partial}{\partial t} \int_0^1 \tilde{V}(x,t) dx + v\tilde{V}(x,t)\Big|_0^1 = V_t + vV_x,\end{aligned} \tag{15}$$

where

$$\tilde{V}(x,t) = \frac{1}{2}\left\{(v + vu_x + u_t)^2 + (w_t + vw_x)^2\right\} + \frac{v_T^2}{2}\left(u_x + \frac{1}{2} w_x^2\right)^2 + \left(u_x + \frac{1}{2} w_x^2\right),$$

$V_t = \frac{\partial}{\partial t} \int_0^1 \tilde{V}(x,t) dx$, and

$V_x = \int_0^1 \frac{\partial}{\partial x} \tilde{V}(x,t) dx$.

Note that V_t in (15) represents the local rate of change in the fixed control volume and vV_x denotes the resultant energy flux across the boundaries. In deriving (15), the material derivative $df(x,t)/dt = \partial f(x,t)/\partial t + v\, \partial f(x,t)/\partial x$ has been utilized. Now, the first term in (15) becomes

$$\begin{aligned}V_t = \int_0^1 \{&(v + vu_x + u_t)(vu_{xt} + u_{tt}) \\ &+ (w_t + vw_x)(w_{tt} + vw_{xt}) + v_T^2\left(u_x + \frac{1}{2} w_x^2\right) \\ &\times (u_{xt} + w_x w_{xt}) + (u_{xt} + w_x w_{xt})\}\, dx.\end{aligned} \tag{16}$$

The substitution of (11) into (16), with $\psi(t) = u_x + \frac{1}{2} w_x^2$ and $u_{xt} = u_{tt} \cong 0$, yields:

$$\begin{aligned}V_t\big|_{(11)} &= \int_0^1 \{(w_t + vw_x)(w_{tt} + vw_{xt}) \\ &\quad + \psi v_T^2 w_x w_{xt} + w_x w_{xt}\} dx \\ &= \int_0^1 \{(w_t + vw_x)(\psi v_T^2 w_{xx} + w_{xx} - v^2 w_{xx} - vw_{xt}) \\ &\quad + (\psi v_T^2 + 1) w_x w_{xt}\} dx.\end{aligned} \tag{17}$$

Also, V_x in (15) can be expressed by using (13) as follows:

$$\begin{aligned}V_x = \int_0^1 \{&(v + vu_x + u_t)(vu_{xx} + u_{xt}) \\ &+ (w_t + vw_x)(w_{xt} + vw_{xx})\} dx\end{aligned}$$

$$
\begin{aligned}
&+ v_T^2 \int_0^1 \Big\{ \left(u_x + \frac{1}{2} w_x^2 \right) \left(u_x + \frac{1}{2} w_x^2 \right)_x \\
&+ \left(u_x + \frac{1}{2} w_x^2 \right)_x \Big\} dx \\
&= \int_0^1 \left\{ v^2 (1+u_x) u_{xx} + (w_t + v w_x)(w_{xt} + v w_{xx}) \right\} dx,
\end{aligned} \tag{18}
$$

because ψ is independent of x. The substitution of (17) and (18) into (15) with $u_x = \psi(t) - w_x^2/2$ yields:

$$
\begin{aligned}
\dot{V}\Big|_{(11)} &= \int_0^1 \Big\{ (\psi v_T^2 + 1) w_t w_{xx} + (\psi v_T^2 + 1) v w_x w_{xx} \\
&+ (\psi v_T^2 + 1) w_x w_{xt} + v^3 (u_x u_{xx} + u_{xx}) \Big\} dx \\
&= (\psi v_T^2 + 1) w_t w_x \Big|_0^1 + \frac{(\psi v_T^2 + 1)}{2} v \cdot w_x^2 \Big|_0^1 \\
&+ \frac{v^3}{2} \cdot u_x^2 \Big|_0^1 + v^3 \cdot u_x \Big|_0^1 \\
&= (\psi v_T^2 + 1) w_t w_x \Big|_0^1 + \frac{(\psi v_T^2 + 1)}{2} v \cdot w_x^2 \Big|_0^1 \\
&- \frac{\psi v^3}{2} \cdot w_x^2 \Big|_0^1 + \frac{v^3}{8} \cdot w_x^4 \Big|_0^1 - \frac{v^3}{2} \cdot w_x^2 \Big|_0^1 \\
&= (\psi v_T^2 + 1) \{ w_t(1,t) w_x(1,t) - w_t(0,t) w_x(0,t) \} \\
&+ \frac{v}{2} \left\{ \psi (v_T^2 - v^2) + 1 - v^2 \right\} \cdot \{ w_x^2(1,t) \\
&- w_x^2(0,t) \} + \frac{v^3}{8} \{ w_x^4(1,t) - w_x^4(0,t) \}.
\end{aligned} \tag{19}
$$

Using $w_t(0,t) = 0$, (19) is further simplified.

$$
\begin{aligned}
\dot{V}\Big|_{(11)} &= -\Gamma w_x^2(0,t) - \frac{v^3}{8} \cdot w_x^4(0,t) + w_x(1,t) \\
&\times \left\{ (\psi v_T^2 + 1) w_t(1,t) + \Gamma w_x(1,t) + \frac{v^3}{8} \cdot w_x^3(1,t) \right\},
\end{aligned} \tag{20}
$$

where $\Gamma := \frac{v}{2} \left\{ \psi (v_T^2 - v^2) + (1 - v^2) \right\} > 0$, because $v_T^2 >> v^2$, $|v| < 1$, and $\psi \geq 0$.

The first two terms in (20) are always negative. Thus, the following feedback control law, which relates the two terms, $w_x(1,t)$ in (12) and $w_t(1,t)$, will make (20) negative semi-definite.

$$
\begin{aligned}
w_t(1,t) &= -K_1 (1 + \psi v_T^2) w_x(1,t) \\
&- K_2 (1 + \psi v_T^2)^3 w_x^3(1,t).
\end{aligned} \tag{21}
$$

To see this, (21) is substituted into (20). Then,

$$
\begin{aligned}
\dot{V}\Big|_{(11),(21)} &= -\Gamma w_x^2(0,t) - \frac{v^3}{8} \cdot w_x^4(0,t) \\
&- K_1 (\psi v_T^2 + 1)^2 w_x^2(1,t) - K_2 (\psi v_T^2 + 1)^4 w_x^4(1,t) \\
&+ \Gamma w_x^2(1,t) + \frac{v^3}{8} \cdot w_x^4(1,t) \\
&= -\Gamma w_x^2(0,t) - \frac{v^3}{8} \cdot w_x^4(0,t) \\
&- \left(K_1 (\psi v_T^2 + 1)^2 - \Gamma \right) w_x^2(1,t) \quad (18) \\
&- \left(K_2 (\psi v_T^2 + 1)^4 - \frac{v^3}{8} \right) w_x^4(1,t) \leq 0,
\end{aligned} \tag{22}
$$

where $K_1 > \Gamma / (\psi v_T^2 + 1)^2 > 0$, $K_2 > v^3 / 8(\psi v_T^2 + 1)^4 > 0$ are assumed. The existence of such K_1 and K_2 is apparent from the following equations:

$$
\begin{aligned}
\frac{\Gamma}{(\psi v_T^2 + 1)^2} &= \frac{v \left\{ \psi (v_T^2 - v^2) + (1 - v^2) \right\}}{2 (\psi v_T^2 + 1)^2} \\
&= \frac{v}{2} \left\{ \frac{1}{\psi v_T^2 + 1} - \frac{\psi v^2 + v^2}{(\psi v_T^2 + 1)^2} \right\} < \frac{v}{2} \leq K_1,
\end{aligned} \tag{23}
$$

$$
\frac{v^3}{8 (\psi v_T^2 + 1)^4} < \frac{1}{8} \leq K_2. \tag{24}
$$

Therefore, from (23) and (24), K_1 and K_2 should satisfy

$$
4 v K_2 \leq K_1 \text{ and } K_2 \geq 0.125. \tag{25}
$$

It is remarked that (25) is a sufficient condition for assuring that the time derivative of $V(t)$ along (11) and (21) is negative.

Finally, $w_x(1,t)$ in (21) can be solved in terms of $w_t(1,t)$ as follows:

$$
\begin{aligned}
&(1 + \psi v_T^2) w_x(1,t) \\
&= \sqrt[3]{-\frac{w_t(1,t)}{2K_2} + \sqrt{\frac{K_1^3}{27 K_2^3} + \frac{w_t^2(1,t)}{4 K_2^2}}} \\
&+ \sqrt[3]{-\frac{w_t(1,t)}{2K_2} - \sqrt{\frac{K_1^3}{27 K_2^3} + \frac{w_t^2(1,t)}{4 K_2^2}}},
\end{aligned} \tag{26}
$$

where the other two complex conjugate roots are excluded. Therefore, by substituting (26) into (12), the final control input becomes

$$
\begin{aligned}
f_c(t) &= \sqrt[3]{-\frac{w_t(1,t)}{2K_2} + \sqrt{\frac{K_1^3}{27 K_2^3} + \frac{w_t^2(1,t)}{4 K_2^2}}} \\
&+ \sqrt[3]{-\frac{w_t(1,t)}{2K_2} - \sqrt{\frac{K_1^3}{27 K_2^3} + \frac{w_t^2(1,t)}{4 K_2^2}}},
\end{aligned} \tag{27}
$$

where the control input is a function of only $w_t(1,t)$ and gains K_1 and K_2.

All above developments are summarized in the following theorem.

Theorem 1: Consider the axially moving system (11)-(12). Then, the closed-loop system with the right boundary control law (21) is uniformly asymptotically stable.

Proof: The proof is apparent from the previous developments: (14) is a Lyapunov function, whose time-derivative is negative semi-definite. Since system (11) is autonomous and the largest invariant set contained in $\dot{V}=0$ is $w(x,t)=0$, uniform asymptotic stability is achieved [27]. □

Remark 5: The desired control force using a hydraulic touch-roll actuator in Fig. 2 can be implemented as follows. A practical control input to the hydraulic actuator is the voltage applied to the servo valve. Let $\varepsilon(t)$ be the control voltage directly proportional to the spool position. A non-linear force tracking control law for the hydraulic servo-system can be designed as follows:

$$\varepsilon(t)=\frac{1}{\zeta}\Big[\dot{f}_c(t)-k_p\{f(t)-f_c(t)\}$$
$$+w_t(1,t)\beta\left(\frac{A_{p2}^2}{V_{p2}}+\frac{A_{p1}^2}{V_{p1}}\right)\Big],$$

and

$$\zeta=\begin{cases}\beta\left(\dfrac{A_{p2}c_3}{V_{p2}}\sqrt{p_2-p_r}+\dfrac{A_{p1}c_1}{V_{p1}}\sqrt{p_s-p_1}\right), & \varepsilon\ge 0\\ \beta\left(\dfrac{A_{p2}c_4}{V_{p2}}\sqrt{p_s-p_2}+\dfrac{A_{p1}c_2}{V_{p1}}\sqrt{p_1-p_r}\right), & \varepsilon<0,\end{cases}$$

where $f_c(t)$ is the desired force in (27), f is the net force of the hydraulic fluid on the piston, k_p is a positive force error gain, β is the fluid bulk modulus, p_s is the supply pressure, p_r is the return pressure, $\{p_1, A_{p1}, V_{p1}\}$ and $\{p_2, A_{p2}, V_{p2}\}$ are the pressure, the cross-sectional area of the piston, and the fluid volume in chambers 1 and 2, respectively, while c_1, c_2, c_3 and c_4 are the valve orifice coefficients determined by the shape and size of the valve orifices. Those orifice coefficients are often provided by the manufacturer but a more practical way of determining these values is through off-line testing: see [26].

4. EXPONENTIAL STABILITY

In this section, the exponential stability of the transversal motion of the axially moving strip with the right boundary control law (21) and control gains (25) is further investigated. In order to analyze this, the state space Λ is introduced as follows:

$$\Lambda := \left\{(w,\dot{w})^T \,\middle|\, w\in H_{0,l}^1, \dot{w}\in L^2\right\}, \tag{28}$$

where L^2 and $H_{0,l}^k$ are defined as

$$L^2 := \left\{f:[0,1]\to R \,\middle|\, \int_0^1 f^2 dx<\infty\right\}, \tag{29}$$

$$H_{0,l}^k := \Big\{f\in L^2 \,\Big|\, f', f'', \cdots, f^{(k)}\in L^2,$$
$$\text{and } f(0)=0\Big\}. \tag{30}$$

The subscript l in H denotes that the function has a left support. Equation (11) can be written in the state space form as follows:

$$\dot{z}=Az,\quad z(0)\in\Lambda, \tag{31}$$

where $z=(w,\dot{w})^T\in\Lambda$, the operator $A:\Lambda\to\Lambda$ is a nonlinear operator defined as

$$Az=\begin{bmatrix} w_t+vw_x \\ w_{xx}+\dfrac{v_T^2}{2}\displaystyle\int_0^1 w_x^2dx\cdot w_{xx}\end{bmatrix}, \tag{32}$$

where $\dot{w}=w_t+vw_x$ and $\ddot{w}=w_{tt}+2vw_{xt}+v^2w_{xx}$ have been utilized. The domain $D(A)$ of the nonlinear operator A for the system with the right boundary control law is

$$D(A):=\Big\{(w,\dot{w})^T \,\Big|\, w\in H_{0,l}^2, \dot{w}\in H_{0,l}^1, K_2(1+\psi v_T^2)^3 w_x^3(1,t)$$
$$+K_1(1+\psi v_T^2)w_x(1,t)+w_t(1,t)=0\Big\}. \tag{33}$$

Lemma 1: The operator defined in (32) generates a C_0-semigroup of contraction. That is, the transverse dynamics of the axially moving system (11) with boundary control law (21) is dissipative.

Proof: The transversal energy of the strip is introduced as follows:

$$E(t)=\langle z,z\rangle_\Lambda=\|z(t)\|_\Lambda^2=\frac{1}{2}\int_0^1(w_t+vw_x)^2dx$$
$$+\frac{v_T^2}{8}\left(\int_0^1 w_x^2dx\right)+\frac{1}{2}\int_0^1 w_x^2dx. \tag{34}$$

The substitution of (13) into (34), also using the fact $\int_0^1 w_x^2\,dx=\int_0^1\left(\int_0^1 w_x^2\,dx\right)dx$, yields another expression as follows:

$$E(t)=\frac{1}{2}\int_0^1 (w_t+vw_x)^2dx+\frac{v_T^2}{2}\int_0^1\left(u_x+\frac{1}{2}w_x^2\right)^2dx + \int_0^1\left(u_x+\frac{1}{2}w_x^2\right)dx. \quad (35)$$

Note that (35) is the same form as (14) except the lack of the first term in (14). The time derivative of (35) becomes

$$\begin{aligned}\dot{E} &= E_t+vE_x\\ &= \int_0^1\Big\{(w_t+vw_x)(w_{tt}+vw_{xt})+v_T^2\left(u_x+\frac{1}{2}w_x^2\right)\\ &\quad\times(u_{xt}+w_xw_{xt})+(u_{xt}+w_xw_{xt})\Big\}dx \quad (36)\\ &\quad+v\int_0^1\Big\{(w_t+vw_x)(w_{xt}+vw_{xx})+v_T^2\left(u_x+\frac{1}{2}w_x^2\right)\\ &\quad\times\left(u_x+\frac{1}{2}w_x^2\right)_x+\left(u_x+\frac{1}{2}w_x^2\right)_x\Big\}dx.\end{aligned}$$

The substitution of (11) into (36), with $\psi(t)=u_x+\frac{1}{2}w_x^2$ and $u_{xt}=u_{tt}\cong 0$, yields:

$$\begin{aligned}\dot{E}\Big|_{(11)} &= \int_0^1\Big\{(w_t+vw_x)(w_{tt}+vw_{xt})+\psi v_T^2 w_xw_{xt}+w_xw_{xt}\Big\}dx\\ &\quad+v\int_0^1\{(w_t+vw_x)(w_{xt}+vw_{xx})\}dx\\ &= \int_0^1\Big\{(w_t+vw_x)\big(\psi v_T^2 w_{xx}+w_{xx}-v^2w_{xx}-vw_{xt}\big)\\ &\quad+\left(\psi v_T^2+1\right)w_xw_{xt}\Big\}dx\\ &\quad+v\int_0^1\{(w_t+vw_x)(w_{xt}+vw_{xx})\}dx \quad (37)\\ &= \int_0^1\Big\{\left(\psi v_T^2+1\right)w_tw_{xx}+\left(\psi v_T^2+1\right)vw_xw_{xx}\\ &\quad+\left(\psi v_T^2+1\right)w_xw_{xt}\Big\}dx\\ &= \left(\psi v_T^2+1\right)w_tw_x\Big|_0^1+\frac{\left(\psi v_T^2+1\right)}{2}v\cdot w_x^2\Big|_0^1\\ &= \left(\psi v_T^2+1\right)\{w_t(1,t)w_x(1,t)-w_t(0,t)w_x(0,t)\}\\ &\quad+\frac{v}{2}\left(\psi v_T^2+1\right)\{w_x^2(1,t)-w_x^2(0,t)\}.\end{aligned}$$

Again, the substitution of (21) into (37) yields:

$$\begin{aligned}\dot{E}\Big|_{(11),(21)} &= -K_1(1+\psi v_T^2)^2w_x^2(1,t)\\ &\quad-K_2(1+\psi v_T^2)^4w_x^4(1,t)+\frac{v}{2}\left(\psi v_T^2+1\right)w_x^2(1,t)\\ &\quad-\frac{v}{2}\left(\psi v_T^2+1\right)w_x^2(0,t).\\ &\le -\frac{v}{2}(1+\psi v_T^2)^2w_x^2(1,t)-\frac{1}{8}(1+\psi v_T^2)^4w_x^4(1,t)\\ &\quad+\frac{v}{2}\left(\psi v_T^2+1\right)w_x^2(1,t)-\frac{v}{2}\left(\psi v_T^2+1\right)w_x^2(0,t)\\ &= -\frac{v}{2}\psi v_T^2(1+\psi v_T^2)w_x^2(1,t)-\frac{1}{8}(1+\psi v_T^2)^4\\ &\quad\times w_x^4(1,t)-\frac{v}{2}\left(\psi v_T^2+1\right)w_x^2(0,t)\le 0. \quad (38)\end{aligned}$$

Now,

$$\begin{aligned}\frac{d}{dt}<z,z>_\Lambda\Big|_{(11),(21)} &= 2<z,Az>_\Lambda\Big|_{(11),(21)}\\ &= \dot{E}\Big|_{(11),(21)}\le 0. \quad (39)\end{aligned}$$

Hence, the closed-loop operator with (21) is dissipative on Λ. Therefore, a C_0 semigroup $S(t)$ of contraction on Λ is generated [5, 10, 20], where $S(t)$ is a bounded operator on Λ for $t\ge 0$. □

Theorem 2: The axially moving system (11) under boundary control law (21) with the control gains in (25) is exponentially stable. That is, there exist constants $\mu>0$ and $M>0$ such that

$$E(t)\le Me^{-\mu t},\ t\ge 0. \quad (40)$$

Proof: To prove that the system decays exponentially to zero, the following positive definite function, by following the approach in [12], is introduced.

$$\eta(t)=tE(t)+\int_0^1 x\{2w_x(w_t+vw_x)\}dx,\ t\ge 0, \quad (41)$$

where $E(t)$ is defined in (34). The last term of (41) satisfies the following inequalities:

$$\begin{aligned}&2\int_0^1 xw_x(w_t+vw_x)dx\\ &\le \int_0^1 (xw_x)^2dx+\int_0^1(w_t+vw_x)^2dx \quad (42)\\ &\le \int_0^1 w_x^2dx+\int_0^1(w_t+vw_x)^2dx\le 2E(t)\le CE(t),\end{aligned}$$

where $C>2$ is a constant. Hence, the following holds:

$$0\le(t-C)E(t)\le\eta(t)\le(t+C)E(t), \quad (43)$$

for $t>C$ sufficiently large. With the use of (11) and (13), the differentiation of (41) with respect to time yields:

$$\begin{aligned}\dot{\eta}(t)\Big|_{(11)} &= E(t)+t\dot{E}(t)\\ &\quad+2\int_0^1(xw_{xt}w_t+xw_xw_{tt}+2vxw_xw_{xt})dx\\ &= E(t)+t\dot{E}(t)+2\int_0^1\Big[xw_{xt}w_t+xw_x\{-2vw_{xt}\\ &\quad+\left(1-v^2+\psi v_T^2\right)w_{xx}\}+2vxw_xw_{xt}\Big]dx\end{aligned}$$

$$= E(t) + t\dot{E}(t) + 2\int_0^1 \left[x w_{xt} w_t + x\left(1 - v^2 + \psi v_T^2\right) w_x w_{xx} \right] dx$$
$$= E(t) + t\dot{E}(t) + 2\int_0^1 x w_{xt} w_t dx$$
$$+ 2\left(1 - v^2 + \psi v_T^2\right)\int_0^1 x w_x w_{xx} dx \quad (44)$$

Also, the last two terms of (44) satisfy the following equalities.

$$2\int_0^1 x w_{xt} w_t dx = \int_0^1 \frac{d}{dx}\left(x w_t^2\right) dx - \int_0^1 w_t^2 dx$$
$$= x w_t^2 \Big|_0^1 - \int_0^1 w_t^2 dx = w_t^2(1,t) - \int_0^1 w_t^2 dx, \quad (45)$$

$$2\int_0^1 x w_x w_{xx} dx = \int_0^1 \frac{d}{dx}\left(x w_x^2\right) dx - \int_0^1 w_x^2 dx$$
$$= x w_x^2 \Big|_0^1 - \int_0^1 w_x^2 dx = w_x^2(1,t) - \int_0^1 w_x^2 dx. \quad (46)$$

Finally, (44) is expressed as follows:

$$\dot{\eta}(t) = E(t) + t\dot{E}(t) + w_t^2(1,t) - \int_0^1 w_t^2 dx + (1 - v^2$$
$$+ \psi v_T^2) w_x^2(1,t) - (1 - v^2 + \psi v_T^2)\int_0^1 w_x^2 dx. \quad (47)$$

The substitution of (21) into (47) yields:

$$\dot{\eta}(t)\big|_{(11),(21)} = E(t) + K_1^2(1 + \psi v_T^2)^2 w_x^2(1,t)$$
$$+ 2K_1K_2(1 + \psi v_T^2)^4 w_x^4(1,t) + K_2^2(1 + \psi v_T^2)^6$$
$$\times w_x^6(1,t) + (1 - v^2 + \psi v_T^2) w_x^2(1,t) + t\dot{E}(t) \quad (48)$$
$$- \int_0^1 w_t^2 dx - (1 - v^2 + \psi v_T^2)\int_0^1 w_x^2 dx,$$

where $\dot{E}(t) \le 0$. Thus, noting that $E(t)$ and $w_x(1,t)$ are bounded, (48) is negative for a sufficiently large time Ω. That is, when $t > \Omega$, (48) satisfies the following inequality.

$$\dot{\eta}(t) \le 0. \quad (49)$$

From (43) and (49), the following holds:

$$E(t) \le \frac{\eta(t)}{t - C}, \quad t > \Omega. \quad (50)$$

Thus, from (34), (50) and the semi-group property of the solution, the following inequality is obtained.

$$\int_0^\infty E^2(t)dt = \int_0^\Omega E^2(t)dt + \int_\Omega^\infty E^2(t)dt$$
$$\le \int_0^\Omega \left\|S(t)z(0)\right\|_\Lambda^4 dt + \int_\Omega^\infty \frac{\eta(t)^2}{(t - C)^2} dt < \infty, \quad (51)$$

where $z(0) \in D(A)$. (51) implies that

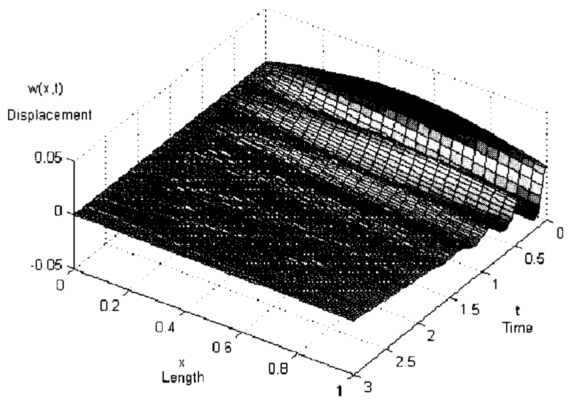


Fig. 4. 3D plot of the controlled response with control gains of $K_1 = 0.0714$, $K_2 = 10$ (Dimensionless).

$$\int_0^\infty \left\|S(t)z(0)\right\|_\Lambda^4 dt < \infty. \quad (52)$$

Then, by the semigroup theorem [20], there exist constants $\mu' > 0$ and $M' > 0$ such that $\left\|S(t)\right\|_\Lambda \le M' e^{-\mu' t}$. That is,

$$\left\|z(t)\right\|_\Lambda \le M' \left\|z(0)\right\|_\Lambda e^{-\mu' t}. \quad (53)$$

From (34),

$$E(t) = \left\|z(t)\right\|_\Lambda^2 \le (M')^2 e^{-2\mu' t}\left\|z(0)\right\|_\Lambda^2 \le M e^{-\mu t}, \quad (54)$$

where $M = \left\|z(0)\right\|_\Lambda^2 (M')^2$ and $\mu = 2\mu'$. Therefore, the theorem is proved. □

5. NUMERICAL SIMULATIONS

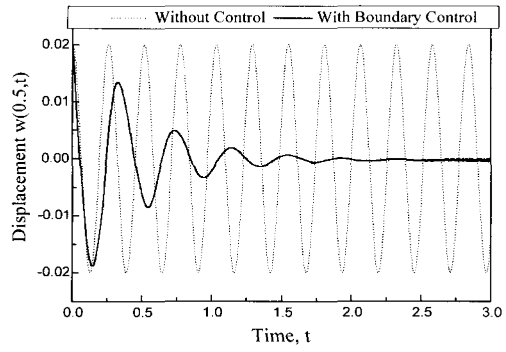
To illustrate the control performance of the proposed boundary control law, numerical simulations by using a finite difference scheme are performed. A mesh of $N = 100$ nodes along the length of the strip has been used. Typical parameter values of the steel strip are:

$$E = 2\times 10^{11} \text{ N/m}^2,\ A = 1.4\times 0.0045 \text{ m}^2,$$
$$T_0 = 9{,}800 \text{ kN},\ \rho = 7{,}850 \text{ kg/m}^2, \quad (55)$$
$$v_0 = 1.67 \text{ m/s, and } L = 17.5 \text{ m}.$$

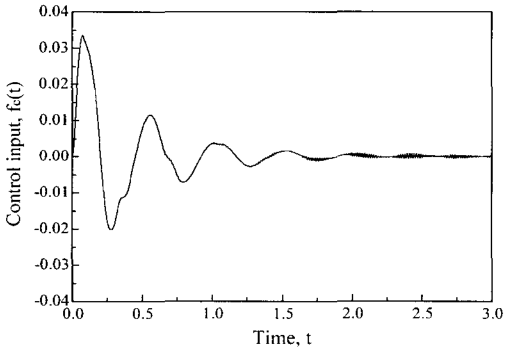
Therefore, the parameters in normalized equation (11) are

$v = 0.0119$, $v_T = 35.857$, and $x \in [0,1]$.

Fig. 4 shows the 3D plot of the controlled response of system (11), which demonstrates the exponentially decaying behavior of the system. The gains used are $K_1 =$ 0.0714 and $K_2 =$10, which satisfy condition



(a) Transversal displacement at $x = 0.5$, $w(0.5,t)$.



(b) Control input.

Fig. 5. Simulation results of the axially moving strip with control gains of $K_1 = 0.0714$, $K_2 = 10$ (Dimensionless).

(25). The initial condition used is $w(x,0) = 0.02 \sin \pi x$.

In Fig. 5, the transversal displacement at $x = 0.5$ and the control input used are shown. As compared in Fig. 5, the response of the uncontrolled system continuously oscillates with almost identical magnitude as the initial condition. This uncontrolled vibration will damp out eventually, but because the damping is so small it will last for a while. Conversely, with the boundary control, an acceptable level of vibration suppression was achieved within 1.5 sec.

6. CONCLUSIONS

In this paper, a boundary control law for the axially moving steel strip in a zinc galvanizing line was investigated. The boundary control law derived was in the form of a negative feedback control of the transversal velocity at the right end. The control law was derived in such a way that the total energy of the strip dissipates exponentially. The simulation results also reveal that the transversal vibrations die out exponentially. As future research issues related to the controls in the zinc galvanizing line, the consideration of the tension variation due to the eccentricity of the sink roll and the disturbance rejection from the air knives and air cooler needs to be investigated.

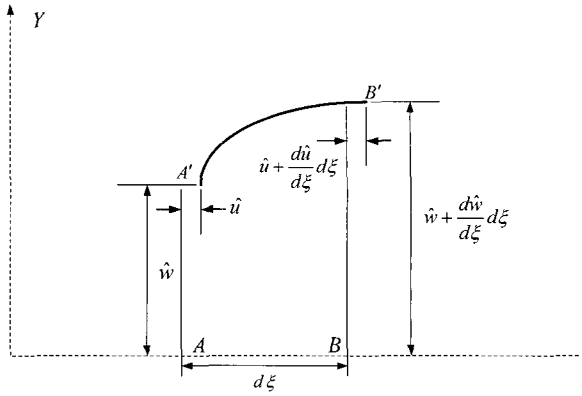


Fig. 6. The nonlinear strain relationship for a flexible material.

APPENDIX A. EQUATIONS OF MOTION

In Fig. 3, the velocity of a point at ξ is given by

$$\begin{aligned} \bar{v}(\xi,\tau) &= \left\{ v_0 + \frac{d\hat{u}(\xi,\tau)}{d\tau} \right\} i + \left\{ \frac{d\hat{w}(\xi,\tau)}{d\tau} \right\} j \\ &= \left\{ v_0 + v_0 \cdot \hat{u}_\xi + \hat{u}_\tau \right\} i + \left\{ v_0 \cdot \hat{w}_\xi + \hat{w}_\tau \right\} j, \end{aligned} \tag{A.1}$$

where i and j denote the unit vectors in X and Y directions, respectively, and $(\cdot)_\xi$ and $(\cdot)_\tau$ represent $\partial(\cdot)/\partial\xi$ and $\partial(\cdot)/\partial\tau$, respectively. Mass enters at $\xi = 0$ and exits at $\xi = L$. Thus, the governing equation and boundary motion equation can be derived by applying Hamilton's principle for systems of changing mass [14] such that

$$\int_{\tau_1}^{\tau_2} (\delta T - \delta U + \delta W_c - \delta W_b) d\tau = 0, \tag{A.2}$$

where T is the kinetic energy, U is the strain energy, W_c is the non-conservative work, and W_b is the virtual momentum transport at both boundaries. The kinetic and strain energies are:

$$\begin{aligned} T &= \frac{1}{2} \rho A \int_0^L \bar{v} \cdot \bar{v} d\xi \quad \text{and} \\ U &= \frac{1}{2} A \int_0^L \sigma_\xi \cdot \varepsilon_\xi d\xi + T_0 \int_0^L \varepsilon_\xi d\xi, \end{aligned} \tag{A.3}$$

where ρ is the mass per unit area of the strip which is assumed to be constant, A is the cross section area, and T_0 is the tension applied to the strip.

For a highly flexible material, the strain ε_ξ in (A.3) has a nonlinear relationship with respect to deflections rather than having a linear relationship, which is normally accepted for less flexible materials.

Fig. 6 depicts the schematic of the infinitesimal deformation of a highly flexible strip. The non-linear strain equation is derived as follows:

$$\varepsilon_\xi = \frac{A'B' - AB}{AB}$$
$$= \frac{\left[\left(d\xi + \frac{d\hat{u}}{d\xi}d\xi\right)^2 + \left(\frac{d\hat{w}}{d\xi}d\xi\right)^2\right]^{1/2} - d\xi}{d\xi}$$
$$\cong \left[1 + 2\frac{d\hat{u}}{d\xi} + \left(\frac{d\hat{u}}{d\xi}\right)^2 + \left(\frac{d\hat{w}}{d\xi}\right)^2 + \frac{d\hat{u}}{d\xi}\left(\frac{d\hat{w}}{d\xi}\right)^2 + \frac{1}{4}\left(\frac{d\hat{w}}{d\xi}\right)^4\right]^{1/2} - 1 \tag{A.4}$$
$$= \left[\left\{1 + \frac{d\hat{u}}{d\xi} + \frac{1}{2}\left(\frac{d\hat{w}}{d\xi}\right)^2\right\}^2\right]^{1/2} - 1$$
$$= \hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2 .$$

Using (A.1), (A.4), and the stress-strain relationship, (A.3) is rewritten as follows:

$$T = \frac{1}{2}\rho A \int_0^L \left\{(v_0 + v_0\hat{u}_\xi + \hat{u}_\tau)^2 + (v_0\hat{w}_\xi + \hat{w}_\tau)^2\right\} d\xi , \tag{A.5}$$

$$U = \frac{1}{2}AE\int_0^L \left\{\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right\}^2 d\xi + T_0\int_0^L \left\{\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right\} d\xi , \tag{A.6}$$

where E is the elastic modulus. The variational forms of (A.5) and (A.6) are given as follows:

$$\begin{aligned}\delta T &= \frac{1}{2}\rho A \int_0^L \{2(v_0 + v_0\hat{u}_\xi + \hat{u}_\tau)(v_0\delta\hat{u}_\xi + \delta\hat{u}_\tau) + 2(v_0\hat{w}_\xi + \hat{w}_\tau)(v_0\delta\hat{w}_\xi + \delta\hat{w}_\tau)\} d\xi \\ &= \rho A \int_0^L \{(v_0{}^2 + v_0{}^2\hat{u}_\xi + v_0\hat{u}_\tau)\delta\hat{u}_\xi + (v_0 + v_0\hat{u}_\xi + \hat{u}_\tau)\delta\hat{u}_\tau \\ &\quad + (v_0^2\hat{w}_\xi + v_0\hat{w}_\tau)\delta\hat{w}_\xi + (v_0\hat{w}_\xi + \hat{w}_\tau)\delta\hat{w}_\tau\} d\xi ,\end{aligned} \tag{A.7}$$

$$\begin{aligned}\delta U &= \frac{1}{2}AE\int_0^L \left\{2\left(\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right)(\delta\hat{u}_\xi + \hat{w}_\xi\delta\hat{w}_\xi)\right\} d\xi + T_0\int_0^L (\delta\hat{u}_\xi + \hat{w}_\xi\delta\hat{w}_\xi) d\xi \\ &= AE\int_0^L \left\{\left(\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right)\delta u_x + \hat{w}_\xi\left(\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right)\delta\hat{w}_\xi\right\} d\xi \\ &\quad + T_0\int_0^L (\delta\hat{u}_\xi + \hat{w}_\xi\delta\hat{w}_\xi) d\xi .\end{aligned} \tag{A.8}$$

Furthermore, the variational forms of the non-conservative work and the virtual momentum transport in (A.2) are provided as follows:

$$\begin{aligned}\delta W_c &= F_c(\tau)\cdot\delta\hat{w}(L,\tau) , \\ \delta W_b &= \rho A v_0\{\hat{w}_\tau(L,\tau) + v_0\hat{w}_x(L,\tau)\}\cdot\delta\hat{w}(L,\tau) .\end{aligned} \tag{A.9}$$

The substitution of (A.7)-(A.9) into (A.2) yields:

$$\begin{aligned}&\int_{t_1}^{t_2}(\delta T - \delta U + \delta W_c - \delta W_b)d\tau \\ &= \int_{t_1}^{t_2}\int_0^L \Big\langle \Big[\rho A(-v_0^2\hat{u}_{\xi\xi} - 2v_0\hat{u}_{\xi\tau} - \hat{u}_{\tau\tau}) + AE\left(\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right)_\xi\Big]\delta\hat{u} \\ &\quad + \Big[\rho A(-v_0^2\hat{w}_{\xi\xi} - 2v_0\hat{w}_{\xi\tau} - \hat{w}_{\tau\tau}) + AE\left\{\hat{w}_\xi\left(\hat{u}_\xi + \frac{1}{2}\hat{w}_\xi^2\right)\right\}_\xi + T_0\hat{w}_{\xi\xi}\Big]\delta\hat{w}\Big\rangle d\xi d\tau \\ &\quad + \int_{t_1}^{t_2}\Big[F_c(\tau) - T_0\hat{w}_\xi(L,\tau) - AE\hat{w}_\xi(L,\tau) \times\left\{\hat{u}_\xi(L,\tau) + \frac{1}{2}\hat{w}_x^2(L,\tau)\right\}\Big]\delta\hat{w}(L,\tau)d\tau = 0 .\end{aligned} \tag{A.10}$$

Since (A.10) must be satisfied for all variation variables $\delta(\cdot)$, the term in front of $\delta(\cdot)$ must be zero. Therefore, (1)-(3) are derived.

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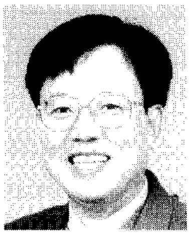

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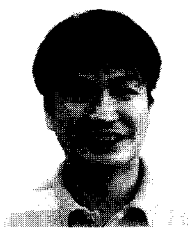

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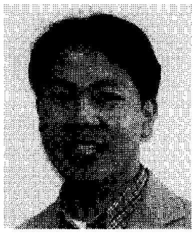

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