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AN ITERATIVE METHOD FOR SYMMETRIC INDEFINITE LINEAR SYSTEMS

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ABSTRACT. For solving symmetric systems of linear equations, it is shown that a new Krylov subspace method can be obtained. The new approach is one of the projection methods, and we call it the projection method for convenience in this paper. The projection method maintains the residual vector like simpler GMRES, symmetric QMR, SYMMLQ, and MINRES. By studying the quasi-minimal residual method, we show that an extended projection method and the scaled symmetric QMR method are equivalent.

1. Introduction

The GMRES method [7] is a Krylov subspace method for solving a linear system

$$Ax = b, \quad \text{where} \quad A \in R^{n\times n} \quad \text{is nonsingular.} \tag{1}$$

The GMRES method characterizes the kth iterate as $x_k = x_0 + z_k$ for a given initial guess $x_0 \in R^n$, with the correction z_k chosen to minimize the norm of the residual $r(z) = b - A(x_0 + z) = r_0 - Az$, where $r_0 = b - Ax_0$, over the kth Krylov subspace $K_k(r_0, A) \equiv \text{span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$, i.e.,

$$\|r_0 - Az_k\|_2 = \min_{z\in K_k(r_0,A)} \|r_0 - Az\|_2. \tag{2}$$

Most implementations of GMRES rely on the Arnoldi process, given in Gram–Schmidt form as follows:

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ALGORITHM 1.1. Arnoldi Process
Initialize: Choose an initial vector v_1 with $\|v_1\|_2 = 1$.
Iterate: For $k = 1, 2, \ldots$, do:
Set $h_{i,k} = v_i^T A v_k, i = 1, 2, \ldots, k$.
Set $\tilde{v}_{k+1} = Av_k - \sum_{i=1}^{k} h_{i,k} v_i$.
Set $h_{k+1,k} = \|\tilde{v}_{k+1}\|_2$.
If $h_{k+1,k} = 0$, stop; otherwise, $v_{k+1} = \tilde{v}_{k+1}/h_{k+1,k}$.
The standard implementation (cf. [7]) is obtained with $v_1 = r_0/\|r_0\|_2$.

Simpler GMRES implementations of Walker and Zhou [8] are obtained if the Arnoldi process is applied with $v_1 = Ar_0/\|Ar_0\|_2$. Suppose $r_0 \neq 0$. Then $v_1 = Ar_0/\|Ar_0\|_2$ is well-defined, since A is nonsingular. Setting $\rho_{1,1} = \|Ar_0\|_2$ gives the equation

$$Ar_0 = \rho_{1,1} v_1, \tag{3}$$

and the following equation is satisfied by the Arnoldi process:

$$Av_{k-1} = \sum_{i=1}^{k} \rho_{i,k} v_i \text{ for unique } \rho_{i,k}'\text{s with } \rho_{k,k} > 0 \text{ for } k > 1. \tag{4}$$

Equations (3) and (4) give the relation

$$AU_k = V_k R_k, \tag{5}$$

where $U_k = (r_0, v_1, \ldots, v_{k-1})$, $V_k = (v_1, \ldots, v_k)$, and

$$R_k = \begin{pmatrix} \rho_{1,1} & \cdots & \rho_{1,k} \\ & \ddots & \vdots \\ & & \rho_{k,k} \end{pmatrix}.$$

The relation (5) reduces the least-squares problem (2) directly to an upper triangular least-squares problem by setting $r_0 = r_k + V_k V_k^T r_0$, where $r_k = \Pi_k^{\perp} r_0$ and $\Pi_k^{\perp}$ is the orthogonal projection onto the orthogonal complement of the space $K_k(v_1, A)$, because $\|r_0 - Az_k\|_2^2 = \|r_0 - AU_k y_k\|_2^2 = \|r_k + V_k(V_k^T r_0 - R_k y_k)\|_2^2 = \|r_k\|_2^2 + \|V_k^T r_0 - R_k y_k\|_2^2$. Then the kth iterate x_k of simpler GMRES is defined by $x_k = x_0 + U_k y_k$, where $y_k = R_k^{-1} w_k$ and $w_k = V_k^T r_0$. Therefore, simpler GMRES implementations given by Walker and Zhou are obtained in that the least-squares problem (2) can be solved without maintaining a factorization of an upper-Hessenberg matrix as in the standard GMRES implementation.

We introduce another approach to Krylov subspace methods for solving symmetric indefinite linear systems, which is called the projection method in this paper. The projection method is closely related to the

simpler GMRES method in that the projection and simpler GMRES methods use the same initial basis vector $v_1 = Ar_0/\|Ar_0\|_2$ in applying the symmetric Lanczos and Arnoldi processes, respectively, and, in the symmetric case, the projection method can be derived from the simpler GMRES method by finding a search direction p_k such that $Ap_k = v_k$ for each k. Both simpler GMRES and the projection method maintain orthonormal bases of the space $AK_k(r_0, A)$, which permit residual minimization through projection of the residual onto $[AK_k(r_0, A)]^{\perp}$. With simpler GMRES, the kth approximate solution is obtained by solving a $k \times k$ upper triangular system. This is also done with the projection method, but only implicitly. Because the projection method is based on the short recurrence symmetric Lanczos process, the triangular system is tridiagonal and, therefore, one can update the approximate solution using a three-term short recurrence formula. In contrast to simpler GMRES, the usual GMRES implementation maintains an orthonormal basis of $K_k(r_0, A)$ through the Arnoldi process, and, consequently, achieves residual minimization through the solution of an upper Hessenberg least-squares problem. MINRES [6] can be viewed as a specialization of the usual GMRES approach to the symmetric case, in which the short recurrence symmetric Lanczos process is used to generate an orthonormal basis of $K_k(r_0, A)$. The upper Hessenberg system is tridiagonal, and so solution of the upper Hessenberg least-squares problem is done implicitly in MINRES by implementing a three-term short recurrence formula for updating the approximate solution. In the symmetric indefinite case without preconditioning, symmetric QMR [2] is obtained using the same approach as MINRES. However, in solving the systems of the preconditioned system

$$(6)\quad A'x' = b', \text{ where } A' = M_1^{-1}AM_2^{-1},\ x' = M_2x, \quad \text{and} \quad b' = M_1^{-1}b,$$

symmetric QMR is implemented by solving a quasi-minimization problem. Thus the approach of the projection method is similar to that of simpler GMRES, while standard GMRES, MINRES, and symmetric QMR follow an alternative approach.

The projection method can be extended to solve preconditioned systems of the form (6) with a suitable inner product defined by using a symmetric positive definite preconditioner. The MINRES and symmetric QMR methods can also be applied to symmetric problems with symmetric positive definite preconditioning. It will be shown that, with a symmetric positive definite preconditioner, the projection method is equivalent to the scaled symmetric QMR method with suitable scaling factors. In section 2, we give a derivation of the projection method. In

section 3, we establish a theoretical result showing the equivalence of the extended projection method and the scaled symmetric QMR method, and we discuss breakdowns of the projection and extended projection methods. Finally, we present the results of numerical experiments in section 4.

2. Derivation of the projection method

The kth residual vector $r(z_k) = r_0 - Az_k$ in simpler GMRES is the same as $r_k = \Pi_k^{\perp} r_0 = (I - V_k V_k^T) r_0 = r_{k-1} - (v_k^T r_0) v_k = r_{k-1} - [v_k^T (r_{k-1} + V_{k-1} V_{k-1}^T r_0)] v_k = r_{k-1} - (r_{k-1}^T v_k) v_k$, i.e., the kth residual vector r_k can be obtained by orthogonalizing r_{k-1} against v_k. Suppose we have a set $\{v_1, \ldots, v_k\}$ of orthonormal basis vectors of the space $K_k(v_1, A)$ that are generated by Arnoldi's method starting with $v_1 = Ar_0/\|Ar_0\|_2$ and have a vector p_k such that $Ap_k = v_k$ for each k. Then the kth residual vector r_k in the simpler GMRES method is

$$
\begin{aligned}
r_k &= r_{k-1} - (r_{k-1}^T v_k) v_k \\
&= r_0 - Az_{k-1} - (r_{k-1}^T v_k) Ap_k \\
&= r_0 - A[z_{k-1} + (r_{k-1}^T v_k) p_k].
\end{aligned} \tag{7}
$$

By the last expression in equation (7) it is natural to define the kth iterate x_k of the projection method as $x_k = x_{k-1} + (r_{k-1}^T v_k) p_k$. Setting $P_k = (p_1, \ldots, p_k)$ and $V_k = (v_1, \ldots, v_k)$ we want $AP_k = V_k$ by the requirement of $Ap_i = v_i$ for each i. By the relation $AU_k = V_k R_k$ in (5), the equation $AP_k = V_k$ is equivalent to

$$
U_k = P_k R_k. \tag{8}
$$

The search direction p_k is then defined as

$$
p_k = \begin{cases} r_0/\rho_{1,1} & \text{if } k = 1 \\ \frac{1}{\rho_{k,k}} (v_{k-1} - \rho_{1,k} p_1 - \cdots - \rho_{k-1,k} p_{k-1}) & \text{if } k > 1. \end{cases}
$$

Then we have a long recursion formula to generate p_k in general. If A is symmetric, then an orthonormal basis $\{v_1, \ldots, v_k\}$ of the space $K_k(v_1, A)$ can be generated by the symmetric Lanczos process. Then

the upper-triangular matrix R_k in (5) reduces to the form

$$R_k = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} & 0 & \cdots & 0 \\ 0 & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} & \ddots & \vdots \\ \vdots & \ddots & \rho_{3,3} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \rho_{k-2,k} \\ \vdots & & & \ddots & \rho_{k-1,k-1} & \rho_{k-1,k} \\ 0 & \cdots & \cdots & \cdots & 0 & \rho_{k,k} \end{pmatrix}$$

Therefore, we have a short recursion formula for p_k by (8), i.e.,

$$p_k = \frac{1}{\rho_{k,k}}(v_{k-1} - \rho_{k-1,k}p_{k-1} - \rho_{k-2,k}p_{k-2}) \quad \text{for} \quad k > 1, \tag{9}$$

where $\rho_{k-2,k} = v_{k-2}^T A v_{k-1}$, $\rho_{k-1,k} = v_{k-1}^T A v_{k-1}$, $\rho_{k,k} = \|\tilde{v}_k\|_2$, and $\tilde{v}_k = Av_{k-1} - \rho_{k-1,k}v_{k-1} - \rho_{k-2,k}v_{k-2}$.

For a symmetric matrix A, the projection method is as follows:

ALGORITHM 2.1. Projection method (symmetric A)

Initialize: Choose x_0 and set $r_0 = b - Ax_0$.

Set $z = Ar_0, \alpha_1 = \|z\|_2$, and $v_1 = z/\alpha_1$.

Compute $\beta_1 = r_0^T v_1$.

Set $r_1 = r_0 - \beta_1 v_1$, $p_1 = r_0/\alpha_1$, and set

$x_1 = x_0 + \beta_1 p_1$.

Iterate: For $k = 2, 3, \ldots,$ do:

Set $v_k = Av_{k-1}$.

For $i = \max\{k-2, 1\}, \ldots, k-1$, do:

Set $\bar{\alpha}_i = v_k^T v_i$.

Update $v_k \leftarrow v_k - \bar{\alpha}_i v_i$.

Set $\alpha_k = \|v_k\|_2$.

Update $v_k \leftarrow v_k/\alpha_k$.

Compute $\beta_k = r_{k-1}^T v_k$.

Set $r_k = r_{k-1} - \beta_k v_k$.

Set $p_k = \frac{1}{\alpha_k}\left(v_{k-1} - \sum_{i=\max\{k-2,1\}}^{k-1} \bar{\alpha}_i p_i\right)$ and set

$x_k = x_{k-1} + \beta_k p_k$.

REMARK. One might consider extending the projection method to solve nonsymmetric linear systems by similarly using the nonsymmetric Lanczos process to get a short recursion formula for the search directions. However, we found in experiments that the projection method with the

nonsymmetric Lanczos process is very unstable for solving nonsymmetric linear systems. Therefore, we consider only symmetric indefinite systems in this paper.

In the projection method, we use the Euclidean inner product and norm, i.e., $< u, v >_2 = u^T v$ and $\|v\|_2 = (v^T v)^{1/2}$ to determine orthogonal projections. However, Algorithm 2.1 still works[1] for any inner product and norm as long as A is symmetric with respect to that inner product, i.e., $< Au, v >=< u, Av >$ for all u and v. From the above observation we may extend the projection method to apply to preconditioned systems of the form (6) under the assumption that $M = M_1 M_2$ is symmetric positive definite. If $M = M_1 M_2$ is symmetric positive definite, then we have

$$(10)\quad \begin{aligned}(M_2^{-T} M_1)^T &= M_1^T M_2^{-1} = M_2^{-T} M_2^T M_1^T M_2^{-1} \\ &= M_2^{-T} (M_1 M_2)^T M_2^{-1} = M_2^{-T} (M_1 M_2) M_2^{-1} \\ &= M_2^{-T} M_1\end{aligned}$$

and

$$(11)\quad \begin{aligned}x^T M_2^{-T} M_1 x &= (M_2^{-1} x)^T (M_1 x) = (M_1^{-T} M_2^{-T} M_1 x)^T (M_1 x) \\ &= (M_1 x)^T M^{-1} (M_1 x) > 0 \quad \text{for all nonzero } x.\end{aligned}$$

It follows from (10) and (11) that $M_2^{-T} M_1$ is also symmetric positive definite. Therefore, we can define an inner product by using the symmetric positive definite matrix $M_2^{-T} M_1$. Similarly, it can be easily shown that the converse of the above result also holds. We summarize this result in the following lemma:

LEMMA 2.1. *$M_2^{-T} M_1$ is symmetric positive definite if and only if $M = M_1 M_2$ is. Then we have an inner product defined by $< u, v >_* \equiv u^T M_2^{-T} M_1 v$ for all u and v.*

Furthermore, if A is symmetric, it can be shown that $A' \equiv M_1^{-1} A M_2^{-1}$ is symmetric with respect to the inner product $< \cdot, \cdot >_*$. Then the desired extension of the projection method is obtained by using this inner product in place of the Euclidean inner product in the projection method, and we give the algorithm below. We call the result the extended projection method. It can be summarized as follows:

ALGORITHM 2.2. Extended projection method (symmetric A)
Initialize: Choose x_0 and set $r_0 = b - Ax_0$,

[1]The v_j's are orthonormal with respect to $< \cdot, \cdot >$, the residuals are minimized in the norm $\| \cdot \|$ over the Krylov subspace, etc.

$z = M^{-1}r_0, u_1 = Az, w_1 = M^{-1}u_1$, and $\alpha_1 = \sqrt{u_1^T w_1}$.
Update $u_1 \leftarrow u_1/\alpha_1$ and $w_1 \leftarrow w_1/\alpha_1$.
Compute $\beta_1 = r_0^T w_1$.
Set $r_1 = r_0 - \beta_1 u_1, p_1 = z/\alpha_1$, and set $x_1 = x_0 + \beta_1 p_1$.
Iterate: For $k = 2, 3, \ldots$, do:
Set $u_k = Aw_{k-1}$.
For $i = \max\{k-2, 1\}, \ldots, k-1$, do:
Set $\bar{\alpha}_i = u_k^T w_i$.
Update $u_k \leftarrow u_k - \bar{\alpha}_i u_i$.
Set $w_k = M^{-1}u_k$ and $\alpha_k = \sqrt{u_k^T w_k}$.
Update $u_k \leftarrow u_k/\alpha_k$ and $w_k \leftarrow w_k/\alpha_k$.
Compute $\beta_k = r_{k-1}^T w_k$ and set $r_k = r_{k-1} - \beta_k u_k$.
Set $p_k = \frac{1}{\alpha_k}\left(w_{k-1} - \sum_{i=\max\{k-2,1\}}^{k-1} \bar{\alpha}_i p_i\right)$ and set
$x_k = x_{k-1} + \beta_k p_k$.

Note that in Algorithm 2.2 only M and not M_1 or M_2 appears explicitly. It follows in particular that the iterates x_k are independent of the decomposition $M = M_1 M_2$. In addition, note that the extended projection method allows use of only symmetric positive definite preconditioners.

3. On theoretical behavior and breakdowns

We first give a brief description of symmetric QMR, introduced by Freund and Nachtigal [2] for solving symmetric indefinite systems of linear equations. The nonsymmetric Lanczos process requires the multiplication of vectors with A^T as well as A. Freund and Zha [4] observed that the nonsymmetric Lanczos process can be simplified by finding a nonsingular matrix P such that $A^T P = PA$ and setting the initial left Lanczos vector $w_1 = Pv_1/\|Pv_1\|_2$. In fact, it has been shown that the left Lanczos vector w_k can be updated by using P and v_k only, i.e., $w_k = \frac{Pv_k}{\|Pv_k\|_2}$ for each k.

Let $M \in R^{n\times n}$ be any symmetric nonsingular matrix. Suppose M can be factored as $M = M_1 M_2$. Then $M = M_1 M_2 = M_2^T M_1^T = M^T$, where M_1 and M_2 need not be the transposes of each other. With $P = M_1^T M_2^{-1}$ and $A' = M_1^{-1} A M_2^{-1}$, it is easily shown that $(A')^T P = PA'$, since A and M are symmetric. With the above observations of Freund and Zha,

this implies that the QMR method applied to the preconditioned system (6) can be simplified. The resulting QMR method using a symmetric preconditioner M for symmetric indefinite systems is referred to as the symmetric QMR method. This can be summarized as follows:

ALGORITHM 3.1. Symmetric QMR
Initialize: Set $r_0 = b - Ax_0, t = M_1^{-1}r_0, \tau = \|t\|_2, q_0 = M_2^{-1}t$,
$\theta_0 = 0$, and set $\rho_0 = r_0^T q_0$.
Iterate: For $k = 1,2, \ldots,$ do:
Compute $t = Aq_{k-1}$ and $\sigma_{k-1} = q_{k-1}^T t$.
If $\sigma_{k-1} = 0$, then stop; otherwise, set
$\alpha_{k-1} = \frac{\rho_{k-1}}{\sigma_{k-1}}$ and $r_k = r_{k-1} - \alpha_{k-1}t$.
Set $t = M_1^{-1}r_k, \theta_k = \frac{\|t\|_2}{\tau_{k-1}}, c_k = \frac{1}{\sqrt{1+\theta_k^2}}, \tau_k = \tau_{k-1}\theta_k c_k$,
$d_k = c_k^2\theta_{k-1}^2 d_{k-1} + c_k^2\alpha_{k-1}q_{k-1}$, and set $x_k = x_{k-1} + d_k$.
If x_k has converged, then stop.
If $\rho_{k-1} = 0$, then stop; otherwise, set
$u_k = M_2^{-1}t, \rho_k = r_k^T u_k, \beta_k = \frac{\rho_k}{\rho_{k-1}}$, and $q_k = u_k + \beta_k q_{k-1}$.

If we apply the nonsymmetric Lanczos process to the preconditioned system (6) to generate a basis of the space $K_k(M_1^{-1}r_0, A')$ starting with the initial Lanczos vectors $\tilde{v}_1 = M_1^{-1}r_0/\mu$ and $\tilde{w}_1 = P\tilde{v}_1/\|P\tilde{v}_1\|_2$, where $\mu = \|M_1^{-1}r_0\|_2$ and $P = M_1^T M_2^{-1}$, one can readily see that the Lanczos basis vectors $\{\tilde{v}_1, \ldots, \tilde{v}_k\}$ for the space $K_k(M_1^{-1}r_0, A')$ generated by using the Euclidean inner product and norm are orthogonal with respect to the inner product $< \cdot, \cdot >_*$. This follows from the fact that the Lanczos vectors satisfy $\tilde{w}_j = P\tilde{v}_j/\|P\tilde{v}_j\|_2$ for each j and the biorthogonality of the vectors $\{\tilde{v}_1, \ldots, \tilde{v}_k\}$ and $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ generated by the nonsymmetric Lanczos process. Set $\Omega_k \equiv \mathrm{diag}(\bar{\omega}_1, \ldots, \bar{\omega}_{k+1})$, where $\bar{\omega}_i \equiv \tilde{v}_i^T M_2^{-T} M_1 \tilde{v}_i$ for each i. If we choose $\Omega_k^{1/2}$, which is obtained by taking the square root of $\bar{\omega}_i$ for each i, as a diagonal weight matrix in the QMR method [1], then symmetric QMR chooses $z_k' = \tilde{V}_k y_k$, where y_k solves the following minimization problem:

$$\min_{y \in R^k} \|\Omega_k^{1/2}(\mu e_1^{k+1} - \tilde{H}_k y)\|_2,$$

where $\tilde{V}_k = (\tilde{v}_1, \ldots, \tilde{v}_k)$, e_1^{k+1} is the first column of the identity matrix I_{k+1}, and $\tilde{H}_k$ is a $(k+1) \times k$ tridiagonal matrix that is obtained by applying the nonsymmetric Lanczos process to the system (6). Then

one can see that

$$\begin{aligned}\|\Omega_k^{1/2}(\mu e_1^{k+1} - \tilde{H}_k y)\|_2 &= \|\tilde{V}_{k+1}(\mu e_1^{k+1} - \tilde{H}_k y)\|_* \\ &= \|M_1^{-1} r_0 - A' z'\|_*.\end{aligned}$$

Therefore, choosing y_k to minimize $\|\Omega_k^{1/2}(\mu e_1^{k+1} - \tilde{H}_k y)\|_2$ over R^k is equivalent to minimizing the residual norm $\|M_1^{-1} r_0 - A' z'\|_*$ over $K_k(M_1^{-1} r_0, A')$. Since the projection method applied to the linear system (6) using the inner product $< \cdot, \cdot >_*$ and norm $\| \cdot \|_*$ also minimizes the residual norm $\|M_1^{-1} r_0 - A' z'\|_*$ over $K_k(M_1^{-1} r_0, A')$, the residuals between the scaled symmetric QMR method with scaling determined by the $\bar{\omega}_i$'s and the extended projection method must be the same for each k. Then this proves the following theorem:

THEOREM 3.1. *Suppose the preconditioned system* (6) *is such that* $M \equiv M_1 M_2$ *is symmetric positive definite. Then the extended projection method applied to this system is equivalent to the symmetric QMR method with scaling factors* $\bar{\omega}_k$ *defined as above.*

In Algorithm 2.1, breakdown occurs when $\alpha_k = 0$. If $\alpha_k = 0$, then Av_{k-1} belongs to the space spanned by the vectors $v_1, \ldots, v_{k-1}$. This gives $K_k(v_1, A) = K_{k-1}(v_1, A)$, where $v_1 = Ar_0/\|Ar_0\|_2$, since $K_k(v_1, A) = \text{span}\{v_1, \ldots, v_{k-1}, Av_{k-1}\}$. Then spaces $K_k(r_0, A)$ and $K_{k-1}(r_0, A)$ have the same dimension. Therefore, we have $A^{-1}b - x_0 \in K_{k-1}(r_0, A)$, and it follows that $r_{k-1} = 0$.

If we apply the projection method to systems of the form

$$AM^{-1}y = b \quad \text{and} \quad x = M^{-1}y, \tag{12}$$

with M symmetric positive definite, then Algorithm 2.2 produces an orthonormal basis $\{u_1, \ldots, u_k\}$ of the space $K_k(u_1, \bar{A})$ with respect to the inner product $< \cdot, \cdot >_*$, where $u_1 = \bar{A}r_0/\|\bar{A}r_0\|_*$ and $\bar{A} = AM^{-1}$. In addition, it is clear that Algorithm 2.2 breaks down when $\alpha_k = \sqrt{u_k^T M^{-1} u_k} = 0$, i.e.,

$$0 = u_k = Aw_{k-1} - \sum_{i=\max\{k-2,1\}}^{k-1} \bar{\alpha}_i u_i = AM^{-1}u_{k-1} - \sum_{i=\max\{k-2,1\}}^{k-1} \bar{\alpha}_i u_i.$$

Then $\bar{A}u_{k-1}$ belongs to the space $\text{span}\{u_1, \ldots, u_{k-1}\}$, which implies that both spaces $K_k(u_1, \bar{A})$ and $K_{k-1}(u_1, \bar{A})$ are the same. The dimension of the space $K_k(r_0, \bar{A})$ is then the same as that of the space $K_{k-1}(r_0, \bar{A})$. Therefore, we have $\bar{A}^{-1}b - \bar{x}_0 \in K_{k-1}(r_0, \bar{A})$, where $\bar{x}_0 = Mx_0$, and it also follows that $r_{k-1} = 0$. Consequently, we do not need to worry

about breakdown of either Algorithm 2.1 or Algorithm 2.2 before the solution is reached, since the extended projection method applied to general systems of the form (6) is equivalent to the extended projection method applied to systems of the form (12).

4. Numerical experiments

We present numerical experiments that show the performance of the Krylov subspace methods discussed in the previous sections for symmetric indefinite systems. In our experiments, we also include the SYMMLQ method [6] for solving symmetric indefinite linear systems. Basically, the kth iterate of SYMMLQ can be obtained by orthogonalizing the residual vector $r(z) = r_0 - Az$ against $K_k(r_0, A)$, whereas that of MINRES is obtained by minimizing the residual vector over the space $K_k(r_0, A)$ for each k. For a symmetric positive definite preconditioner M, it can be shown that algorithms for the SYMMLQ, MINRES, symmetric QMR, and extended projection methods can be implemented with only one matrix-vector multiplication with A and one preconditioner-vector solve with M at each iteration if $M_1^T = M_2$ in implementing symmetric QMR. However, in implementing a preconditioner-vector solve of the form $Mw = r$, factoring the preconditioner M first, i.e., $M = M_1 M_2$, we may save floating-point operations by solving two preconditioning solves of the form $M_1 u = r$ and $M_2 w = u$ instead of performing a preconditioner-vector solve with M. At each iteration, in addition to one matrix-vector multiplication with A and two M_1 and M_2 preconditioner solves or one preconditioner solve with M, the algorithms for the symmetric QMR, MINRES, SYMMLQ, and extended projection methods use approximately $7n$, $10n$, $11n$, and $12n$ multiplications and divisions, respectively.

In our experiments, we used a discretization of

$$\begin{aligned} \Delta u + cu &= f \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D, \end{aligned}$$

for a test problem involving a symmetric linear system, where $D = [0,1] \times [0,1]$, and c is a constant. The usual centered difference approximations were used in the discretization. We set $f \equiv x(1-x)+y(1-y)$ and used $m = 64$, where m is the number of equally spaced interior points on each side of D, so that the resulting system has dimension 4096. For a symmetric positive definite preconditioner we used $\tilde{M} = -L + I$, where L is the discrete Laplacian matrix. Also, we used the vector

$(1, 1, \ldots, 1)^T \in R^n$ for the initial guess and used double precision on Sun Microsystems workstations in all experiments.

We look at the issue of whether the projection method given in Section 2 is numerically as sound as the SYMMLQ, MINRES, and symmetric QMR methods. We also address numerical aspects of the MINRES, SYMMLQ, symmetric QMR, and projection methods. In all experiments, the true residual norms $\|b - Ax_k\|_2$ are monitored in assessing the comparative performance. It is known that there exists a symmetric positive definite matrix Z such that $M = Z^2$ for a symmetric positive definite matrix M. Therefore, we may apply the MINRES and SYMMLQ methods to the following system:

$$(13) \quad \tilde{A}\tilde{x} = \tilde{b}, \quad \text{where} \quad \tilde{A} = Z^{-1}AZ^{-1}, \quad \tilde{x} = Zx, \quad \text{and} \quad \tilde{b} = Z^{-1}b.$$

Since the implementation of each method requires only solutions of systems involving M, without regard to any particular decomposition of M, it can be shown that MINRES applied to the system (13) is equivalent to the extended projection method and the scaled symmetric QMR method with scaling factors $\bar{\omega}_k$ defined in Section 3 when the latter two methods are applied to the system (6) with any decomposition $M = M_1M_2$, as long as M is symmetric positive definite. In all experiments of the SYMMLQ, MINRES, scaled symmetric QMR, and extended projection methods, we used Cholesky decomposition of the preconditioner $\tilde{M}$. Note that in this case $\bar{\omega}_k = 1$ for each k, i.e., scaled symmetric QMR is the same as symmetric QMR without scaling.

In our experiments, we used solid, dashed, dashdot, and dotted curves to distinguish the true (directly evaluated) residual norm curves generated by the MINRES, SYMMLQ, symmetric QMR, and projection methods, respectively. In the following Figures 1 and 2, the true residual norm curves generated by these methods are monitored using different values of c: 100 and 50.

In Figures 1 and 2, one sees that there are differences in the limits of reduction of the true residual norms. We regard these differences as insignificant, since they are small relative to levels of residual reduction that are typically satisfactory in practice. Overall, the figures suggest that the projection method is as numerically sound as MINRES, SYMMLQ, and symmetric QMR in these experiments.

With the value of $c = 100$ we also considered the cpu-times that symmetric QMR, MINRES, SYMMLQ, and Algorithm 2.2 take to reach 10^{-9} level of residual reduction. Symmetric QMR, MINRES, SYMMLQ,

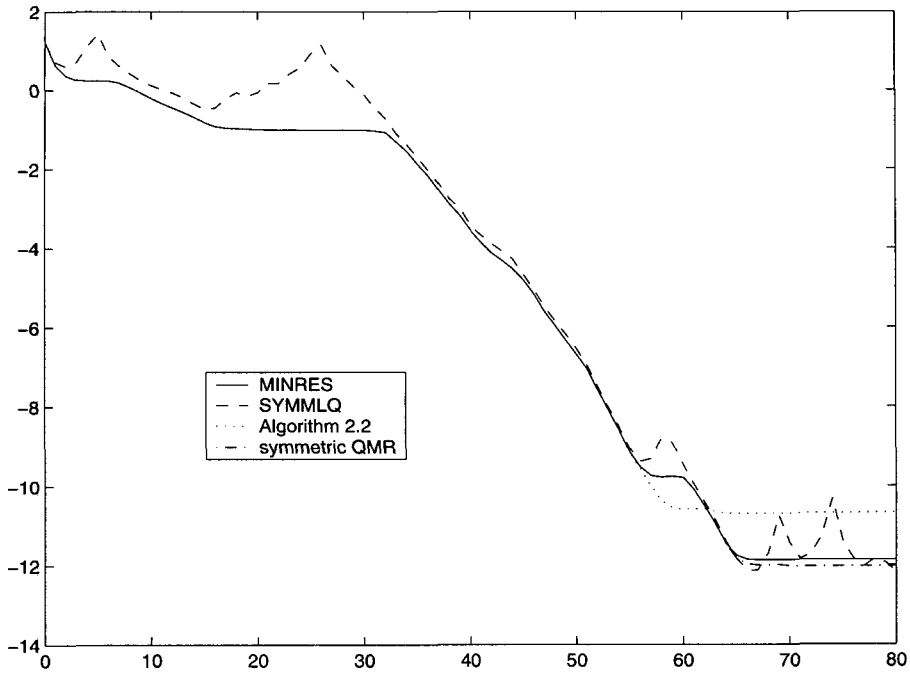

FIGURE 1. Log_{10} of the true residual norms vs. the number of iterations; $c = 100$ and $m = 64$.

and Algorithm 2.2 took approximately 167 ± 1 seconds, although this data is machine dependent.

5. Conclusion

In this paper, we have considered Krylov subspace methods for solving large symmetric indefinite linear systems and have introduced a new approach for solving them, which is called the projection method in this paper. Also, we showed the equivalence between the extended projection method, which allows the use of a symmetric positive definite preconditioner, and the scaled symmetric QMR method with scaling factors $\bar{\omega}_k$ defined in section 3. Our numerical experiments show that the projection method is as numerically sound as the MINRES, SYMMLQ, and symmetric QMR methods. Furthermore, on our test problems, these methods require roughly similar effort to achieve comparable residual norm reduction. However, only the symmetric QMR method allows use

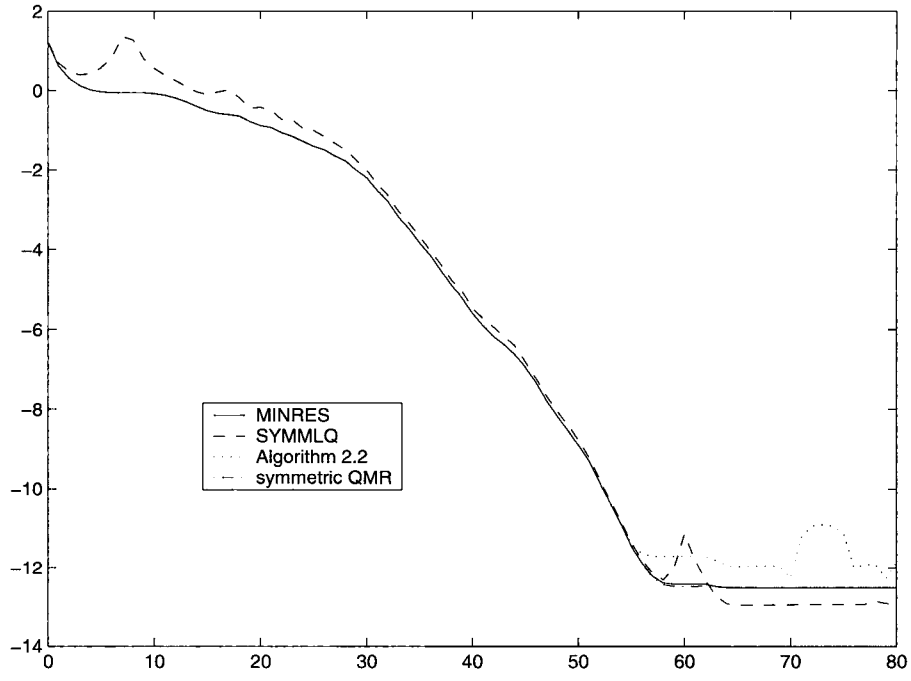

FIGURE 2. Log_{10} of the true residual norms vs. the number of iterations; $c = 50$ and $m = 64$.

of arbitrary nonsingular symmetric indefinite preconditioners, which is an advantage of this method over the other methods. In our experiments, there are instances of some disagreement of the true residual norms of the MINRES, extended projection, and scaled symmetric QMR methods near the limits of residual reduction; however, differences of this magnitude seem unlikely to be significant in practice.

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