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EXTENDED DIRECTED TRIPLE SYSTEMS WITH A GIVEN AUTOMORPHISM

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ABSTRACT. An *extended directed triple system of order* v, denoted by $EDTS(v)$, is a pair $(V, \mathfrak{B})$ where V is a v-set and $\mathfrak{B}$ is a set of transitive triples of elements of V such that every ordered pair of elements of V is contained in exactly one member of $\mathfrak{B}$. We obtain a necessary and sufficient condition for the existence of cyclic $EDTS(v)$s, and when $k = 1$ or 2, we also obtain a necessary and sufficient condition for the existence of k-rotational $EDTS(v)$s.

1. Introduction

A collection of (not necessarily distinct) three objects a, b, c, $\{a, b, c\}$ in order, is called a *transitive triple* (*triple* or *cyclic triple*) if it is a collection of three ordered pairs $(a, b), (b, c), (a, c)$ (three unordered pairs $\{a, b\}, \{a, c\}, \{b, c\}$ or three ordered pairs $(a, b), (b, c), (c, a)$, respectively). A *directed* (*Steiner* or *Mendelsohn*) *triple system of order* v is a pair $(V, \mathfrak{B})$ where V is a v-set of elements and $\mathfrak{B}$ is a set of transitive triples (triples or cyclic triples, respectively) of distinct elements of V, called *blocks*, such that every ordered pair (unordered pair or ordered pair, respectively) of distinct elements of V is contained in exactly one block of $\mathfrak{B}$. A system is said to be *extended* if both the blocks and the pairs are allowed repeated elements.

In an extended system, each block is one of three types: it consists of (i) three same elements, (ii) two same elements and one different element, or (iii) three distinct elements. An element a is called an *idempotent* if there is a block consisting of three a's; and a *nonidempotent* if there is a block consisting of two a's and one another element. We denote by $ESTS(v, \rho)$ (or $EMTS(v, \rho)$) an extended Steiner (Mendelsohn) triple system of order v with ρ idempotents.

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THEOREM 1.1 [5]. *There exists an $ESTS(v,\rho)$ if and only if*
(i) $v \equiv 0$ *(mod 3) and* $\rho \equiv 0$ *(mod 3) or*
(ii) $v \equiv 1, 2$ *(mod 3) and* $\rho \equiv 1$ *(mod 3), but*
(iii) *when v is even,* $\rho \leq \frac{v}{2}$ *and*
(iv) *when* $\rho = v - 1$, $v = 2$.

THEOREM 1.2 [1]. *There exists an $EMTS(v,\rho)$ if and only if*
(i) $v \equiv 0$ *(mod 3) and* $\rho \equiv 0$ *(mod 3),* $(v,\rho) \neq (6,6)$, *or*
(ii) $v \equiv 1, 2$ *(mod 3) and* $\rho \equiv 1$ *(mod 3).*

If a collection of three objects a, b, c, $\{a, b, c\}$ in order, is a transitive triple, we denote it by $[a, b, c]$ which consists of three ordered pairs $(a, b), (b, c)$ and (a, c). In an extended directed triple system, there are five types of blocks:

$$[a,a,a],\ [a,a,b],\ [a,b,a],\ [b,a,a],\ [a,b,c]$$

where a, b, c are distinct elements. We always deem that each block of the form $[a, a, a]$ contains only one ordered pair (a, a), and each block of the form $[a, a, b]$ (or $[b, a, a]$) with $a \neq b$ contains just two ordered pairs $(a, a), (a, b)$ $((a, a), (b, a))$. If $[a, a, b]$ ($[a, b, a]$ or $[b, a, a]$) is a block, then we say that a is a nonidempotent of *type 1* (*type 2* or *type 3*, respectively). We denote by $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ an extended directed triple system of order v with ρ idempotents, η_1 nonidempotents of type 1, η_2 nonidempotents of type 2, and η_3 nonidempotents of type 3. Obviously, we have

$$0 \leq \rho, \eta_1, \eta_2, \eta_3 \leq v \quad \text{and} \quad \rho + \eta_1 + \eta_2 + \eta_3 = v.$$

In general, the existence of $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$s is in doubt. In this paper, we deal with the existence of special classes of $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$s, so-called, one is *cyclic* and the other is *rotational* systems. We obtain a necessary and sufficient condition for the existence of cyclic extended directed triple systems, and when $k = 1$ or 2, we also obtain a necessary and sufficient condition for the existence of k-rotational extended directed triple systems.

2. Cyclic extended directed triple systems

An *automorphism* of an $EDTS(v)$, $(V, \mathfrak{B})$, is a permutation α of V, which maps the block-set $\mathfrak{B}$ onto itself, and α is said to be *cyclic* if it consists of a single cycle of length v. A *cyclic* $EDTS(v)$ is one which admits a cyclic automorphism.

Suppose that $(V, \mathfrak{B})$ is a cyclic $EDTS(v)$ with α as a cyclic automorphism. Then the block-set $\mathfrak{B}$ is partitioned into disjoint orbits under the group $< \alpha >$ which is generated by α. We say that a set of blocks which are taken exactly one, called a *starter block*, from each of the orbits is called a *set of starter blocks* for the cyclic $EDTS(v)$. The *length* of a starter block is the number of blocks of the orbit containing the starter block. It is easy to see that the length of each starter block of a cyclic $EDTS(v)$ is equal to v. Thus, in a cyclic $EDTS(v)$, if there is an idempotent then each element must be an idempotent, or if there is a nonidempotent then each element should be a nonidempotent. Therefore, if there exists a cyclic $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$, it is one of the systems

$$EDTS(v, v, 0, 0, 0), EDTS(v, 0, v, 0, 0),$$
$$EDTS(v, 0, 0, v, 0), EDTS(v, 0, 0, 0, v).$$

REMARK 2.1. We see that $\mathfrak{B}$ is a set of blocks for an $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ if and only if $\{[c, b, a] | [a, b, c] \in \mathfrak{B}\}$ is a set of blocks for an $EDTS(v, \rho, \eta_3, \eta_2, \eta_1)$. Therefore, there exists an $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ if and only if there exists an $EDTS(v, \rho, \eta_3, \eta_2, \eta_1)$.

By Remark 2.1 and the above observation, it is enough to consider the existence of cyclic $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ for the three cases (i) $\rho = v$, (ii) $\eta_1 = v$ (or $\eta_3 = v$), and (iii) $\eta_2 = v$ with others zero in each case.

REMARK 2.2. Suppose there exists a cyclic $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ and let n be the number of blocks consisting of three distinct elements. By counting the number of ordered pairs which occur in the system, we have the following relations:
(i) if $\rho = v$, then $v + 3n = v^2$; so $v \equiv 0$ or $1 \pmod 3$,
(ii) if $\eta_1 = v$, then $2v + 3n = v^2$; so $v \equiv 0$ or $2 \pmod 3$,
(iii) if $\eta_2 = v$, then $3v + 3n = v^2$; so $v \equiv 0 \pmod 3$.

It is easy to see that there exists a cyclic $EDTS(v, v, 0, 0, 0)$ if and only if there exists a cyclic directed triple system of order v, which is equivalent to $v \equiv 1, 4$ or $7 \pmod{12}$ for the existence [4]. Thus we have the following theorem.

THEOREM 2.3. *There exists a cyclic $EDTS(v, v, 0, 0, 0)$ if and only if $v \equiv 1$, 4 or 7 (mod 12).*

LEMMA 2.4. *If there exists a cyclic $EDTS(v, 0, v, 0, 0)$, then $v \equiv 2$ (mod 3).*

PROOF. Since each starter block must have length v, the total number of blocks is divisible by v, but there are $\frac{v^2+v}{3}$ blocks and this should be divided by v; so $v \equiv 2 \pmod 3$. □

Hereafter, we assume that our cyclic $EDTS(v)$ has the element-set $V = Z_v$, the additive abelian group of residue classes, $0, 1, \dots, v-1$, of integers modulo v and the permutation $\alpha = (0, 1, \dots, v-1)$ as a cyclic automorphism, unless other stated.

REMARK 2.5. Let we have a cyclic $EDTS(v, 0, v, 0, 0)$ or a cyclic $EDTS(v, 0, 0, v, 0)$. With each orbit which contains a block $[a, b, c]$, we associate a unique *difference triple* (x, y, z) defined by

$$x \equiv b - a, y \equiv c - b, z \equiv c - a \pmod v$$

which satisfy the equation $x+y \equiv z \pmod v$. With each difference triple (x, y, z) where all x, y, z are nonzero, we associate the orbit containing the block $[0, x, x+y]$. If $(x, -x, 0)$ is a difference triple, we correspond the orbit containing the block $[0, x, 0]$, and if $(0, x, x)$ is a difference triple, we correspond the orbit containing the block $[0, 0, x]$.

We see that the existence of a cyclic $EDTS(v, 0, 0, v, 0)$ is equivalent to the existence of a set of difference triples (x, y, z) with $x + y \equiv z \pmod v$, which is a partition of Z_v, and the existence of a cyclic $EDTS(v, 0, v, 0, 0)$ is equivalent to the existence of a set of difference triples (x, y, z) with $x+y \equiv z \pmod v$, which is a partition of $Z_v \setminus \{0, x\}$ for some x. For the differences $0, x$, we correspond the orbit containing the block $[0, 0, x]$.

LEMMA 2.6. *If $v \equiv 2$ (mod 3), then there exists a cyclic*

$$EDTS(v, 0, v, 0, 0).$$

PROOF. If $v = 6t - 1$ and $t \geq 1$, then the following ordered triples

$$\begin{array}{ll} (2r, 3t-1-r, 3t-1+r), & r = 1, 2, \dots, t-1, \\ (2r-1, 5t-1-r, 5t-2+r), & r = 1, 2, \dots, t \end{array}$$

form a partition of the set $Z_{6t-1} \setminus \{0, 3t-1\}$ with $2t-1$ difference triples (x, y, z) so that $x + y \equiv z \pmod{6t-1}$.

Let $v = 6t + 2$ and $t \geq 0$. If $t = 0$, $[0, 0, 1], [1, 1, 0]$ form a cyclic $EDTS(2, 0, 2, 0, 0)$. If $t \geq 1$, then the following ordered triples

$$\begin{aligned}&(2r, 3t + 1 - r, 3t + 1 + r), \qquad r = 1, 2, \ldots, t,\\ &(2r - 1, 5t + 2 - r, 5t + 1 + r), \quad r = 1, 2, \ldots, t\end{aligned}$$

form a partition of the set $Z_{6t+2} \setminus \{0, 3t + 1\}$ with $2t$ difference triples (x, y, z) so that $x + y \equiv z \pmod{6t + 2}$. □

Now, Lemmas 2.4 and 2.6 together yield the following theorem.

THEOREM 2.7. *There exists a cyclic $EDTS(v, 0, v, 0, 0)$ if and only if $v \equiv 2 \pmod 3$.*

REMARK 2.8. For $v \equiv 0 \pmod 3$, we see that there exists a partition of the set Z_v into difference triples (x, y, z) with $x + y \equiv z \pmod v$ if and only if there exists a partition of the set $Z_v \setminus \{a, b, 0\}$ into difference triples (x, y, z) with $x + y \equiv z \pmod v$ and $a + b = v$ for some a, b. If $v = 6t$, then the later is equivalent to exist a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \ldots, 2t - 1\}$ with the property that

$$\{a_r, b_r | r = 1, 2, \ldots, 2t - 1\} = Z_{6t} \setminus \{0, x_1, x_2, \ldots, x_{2t-1}, x, y\}$$

where $0, x_1, x_2, \ldots, x_{2t-1}, x, y$ are distinct such that

$$b_r - a_r = x_r \text{ for } r = 1, 2, \ldots, 2t - 1 \text{ and } x + y = 6t.$$

If such a set of ordered pairs exists, then we have

$$\begin{aligned}\sum_{r=1}^{2t-1} (a_r + b_r) =& \frac{6t(6t - 1)}{2} - (x_1 + x_2 + \cdots + x_{2t-1} + x + y),\\ \sum_{r=1}^{2t-1} (b_r - a_r) =& x_1 + x_2 + \cdots + x_{2t-1}.\end{aligned}$$

Adding both sides, respectively, we have

$$2 \sum_{r=1}^{2t-1} b_r = 3t(6t - 1) - 6t$$

since $x + y = 6t$. Thus $3t(6t - 1) - 6t \equiv 0 \pmod 2$; so t must be even.

From Remarks 2.2 and 2.8, we have the following lemma.

Lemma 2.9. *If there exists a cyclic $EDTS(v,0,0,v,0)$, then $v \equiv 0$, 3 or 9 (mod 12).*

Lemma 2.10. *If $v \equiv 0,3,9$ (mod 12), then there exists a cyclic $EDTS(v,0,0,v,0)$.*

Proof. Let $v = 6t+3$. Then a set of starter blocks for a cyclic $EDTS(v,0,0,v,0)$:

$v = 3$: $[0,1,0]$.

$v = 9$: $[0,1,3],[3,1,0],[0,4,0]$.

For $t \geq 2$, there exists a cyclic $STS(6t+3)$ based on Z_{6t+3} [6] and if $\mathfrak{B}$ is a set of its starter blocks, then it must contain the starter block $\{0,2t+1,4t+2\}$. For each $\{a,b,c\} \in \mathfrak{B} \setminus \{\{0,2t+1,4t+2\}\}$, we define

$$[a,b,c],[c,b,a]$$

which form a set of starter blocks for a cyclic $EDTS(6t+3,0,0,6t+3,0)$ together with $[0,2t+1,0]$.

If $v = 12t$, then the following ordered triples

$$\begin{array}{ll}
(3t,6t,9t) & (2t,10t,0),\\
(2r,3t-r,3t+r), & r=1,2,\ldots,t-1,\\
(2r-1,5t-r,5t-1+r), & r=1,2,\ldots,t,\\
(12t-2r,9t+r,9t-r), & r=1,2,\ldots,t-1,\\
(12t+1-2r,7t+r,7t+1-r), & r=1,2,\ldots,t
\end{array}$$

form a partition of the set Z_{12t} with $4t$ difference triples (x,y,z) so that $x+y \equiv z$ (mod $12t$). □

Lemmas 2.9 and 2.10 together yield the following theorem.

Theorem 2.11. *There exists a cyclic $EDTS(v,0,0,v,0)$ if and only if $v \equiv 0$, 3 or 9 (mod 12).*

Now, we can conclude the following theorem.

Theorem 2.12. *There exists a cyclic $EDTS(v,\rho,\eta_1,\eta_2,\eta_3)$ if and only if*

(i) $\rho = v \equiv 1$, 4, or 7 (mod 12) and $\eta_1=\eta_2=\eta_3=0$, or

(ii) $\eta_1 = v \equiv 2$ (mod 3) and $\rho=\eta_2=\eta_3=0$, or

(iii) $\eta_2 = v \equiv 0$, 3 or 9 (mod 12) and $\rho=\eta_1=\eta_3=0$, or

(iv) $\eta_3 = v \equiv 2$ (mod 3) and $\rho=\eta_1=\eta_2=0$.

3. 1-rotational extended directed triple systems

An $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ is said to be *k-rotational* if it admits an automorphism α consisting of a single fixed element and k disjoint cycles of length $\frac{v-1}{k}$. In a 1-rotational $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$, each orbit of a block has length either 1 or $v-1$, and hence ρ must be either 1 or v. Thus if there exists a 1-rotational $EDTS(v, 1, \eta_1, \eta_2, \eta_3)$, then we have (i) $\eta_1 = v-1$, (ii) $\eta_2 = v-1$, or (iii) $\eta_3 = v-1$, with others zero. By Remark 2.1, it is enough to consider the existence of 1-rotational $EDTS(v, 1, \eta_1, \eta_2, \eta_3)$ for the two cases (i) and (ii).

LEMMA 3.1. (i) *If there exists a 1-rotational $EDTS(v, 1, v-1, 0, 0)$, then $v \equiv 1$ (mod 3).*

(ii) *If there exists a 1-rotational $EDTS(v, 1, 0, v-1, 0)$, then $v \equiv 2$ (mod 3).*

PROOF. Let n be the number of blocks consisting of three distinct elements. By counting the number of ordered pairs which appear in the system, if it is (i), we have

$$1 + 2(v-1) + 3n = v^2;$$

so $3n = (v-1)^2$ and hence $v \equiv 1 \pmod 3$; if it is (ii), we have

$$1 + 3(v-1) + 3n = v^2;$$

so $3n = (v-2)(v-1)$ and hence $v \equiv 2 \pmod 3$ since $(v-2)(v-1)$ must be divisible by both 3 and $v-1$. □

LEMMA 3.2. *If $v \equiv 2$, 5 or 8 (mod 12), then there exists a 1-rotational*

$$EDTS(v, 1, 0, v-1, 0).$$

PROOF. It comes from the existence of cyclic $DTS(v-1)$ [3]. Let $(V, \mathfrak{B})$ be a cyclic $DTS(v-1)$ and let ∞ be a new element. Then $(V \cup \{\infty\}, \{[\infty, \infty, \infty], [x, \infty, x] | x \in V\} \cup \mathfrak{B})$ is a 1-rotational $EDTS(v, 1, 0, v-1, 0)$ which fixes ∞. □

A $(S_1, 4t-2)$-*system* is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \ldots, 4t-2\}$ such that $\{a_r, b_r | r = 1, 2, \ldots, 4t-2\} = \{4t, 4t+1, \ldots, 6t-2, 6t, \ldots, 12t-4\}$ and $b_r - a_r = r + 1$ for $r = 1, 2, \ldots .4t-2$.

LEMMA 3.3. *For each positive integer t, there exists a $(S_1, 4t-2)$-system.*

PROOF. The following ordered pairs form a $(S_1, 4t-2)$-*system*:

$$\begin{aligned}&(6t, 10t-2),\\&(4t-1+r, 8t-r), && r = 1, 2, \ldots, 2t-1,\\&(8t-1+r, 12t-3-r), && r = 1, 2, \ldots, 2t-2.\end{aligned}$$

□

Throughout, we assume that our 1-rotational $EDTS(v)$ has the element-set $V = Z_{v-1} \cup \{\infty\}$ and the permutation $\alpha = (\infty)(0, 1, \ldots, v-1)$ as a 1-rotational automorphism.

LEMMA 3.4. *If $v \equiv 11 \pmod{12}$, then there exists a 1-rotational $EDTS(v, 1, 0, v-1, 0)$.*

PROOF. Let $v = 12t - 1$ and let $\{(a_r, b_r) | r = 1, 2, \ldots, 4t-2\}$ be a $(S_1, 4t-2)$-system. Then the following transitive triples

$$\begin{aligned}&[\infty, \infty, \infty], [0, \infty, 6t-1], [0, 1, 0],\\&[0, r+1, b_r], \quad r = 1, 2, \ldots, 4t-2\end{aligned}$$

form a set of starter blocks for a 1-rotational $EDTS(12t-1, 1, 0, 12t-2, 0)$. □

Lemmas 3,1, 3.2 and 3.4 together yield the following theorem.

THEOREM 3.5. *There exists a 1-rotational $EDTS(v, 1, 0, v-1, 0)$ if and only if $v \equiv 2 \pmod{3}$.*

A $(S_2, 2t-1)$-*system* is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \ldots, 2t-1\}$ such that $\{a_r, b_r | r = 1, 2, \ldots, 2t-1\} = \{2t+1, 2t+2, \ldots, 3t-1, 3t+1, \ldots, 6t-1\}$ and $b_r - a_r = r$ for $r = 1, 2, \ldots .2t-1$.

LEMMA 3.6. *For each positive integer t, there exists a $(S_2, 2t-1)$-system.*

PROOF. The following ordered pairs form a $(S_2, 2t-1)$-*system*:

$$\begin{aligned}&(3t-r, 3t+r), && r = 1, 2, \ldots, t-1,\\&(4t-1+r, 6t-r), && r = 1, 2, \ldots, t.\end{aligned}$$

□

LEMMA 3.7. *If $v \equiv 1$ (mod 6), then there exists a 1-rotational $EDTS(v, 1, v-1, 0, 0)$.*

PROOF. Let $v = 6t + 1$ and let $\{(a_r, b_r) | r = 1, 2, \ldots, 2t-1\}$ be a $(S_2, 2t-1)$-system. Then the following transitive triples

$$[\infty, \infty, \infty], [0, \infty, 3t], [0, 0, 2t],$$
$$[0, r, b_r], \quad r = 1, 2, \ldots, 2t-1$$

form a set of starter blocks for a 1-rotational $EDTS(6t+1, 1, 6t, 0, 0)$.□

LEMMA 3.8. *If $v \equiv 4$ (mod 6), then there exists a cyclic $EDTS(v, 1, v-1, 0, 0)$.*

PROOF. Let $v = 6t+4$. Then a set of starter blocks for a 1-rotational $EDTS(v, 1, v-1, 0, 0)$:

$v = 4$: $[\infty, \infty, \infty], [0, \infty, 1], [0, 0, 2]$.

$v = 9$: $[\infty, \infty, \infty], [0, \infty, 4], [0, 1, 3], [3, 1, 0], [0, 0, 5]$.

For $t \geq 2$, there exists a cyclic $STS(6t+3)$ based on Z_{6t+3} [4] and if $\mathfrak{B}$ is a set of its starter blocks, then we may say that it contains the starter block $\{0, 2t+1, 4t+2\}$. For each $\{a, b, c\} \in \mathfrak{B} \setminus \{\{0, 2t+1, 4t+2\}\}$, we define

$$[a, b, c], [c, b, a]$$

which form a set of starter blocks for a cyclic $EDTS(6t+3, 0, 0, 6t+3, 0)$ together with $[\infty, \infty, \infty], [0, \infty, 4t+2], [0, 0, 2t+1]$. □

Lemmas 3.1,3.7 and 3.8 together yield the following theorem.

THEOREM 3.9. *There exists a 1-rotational $EDTS(v, 1, v-1, 0, 0)$ if and only if $v \equiv 1$ (mod 3).*

The following theorem is a consequence of the existence of a 1-rotational $DTS(v)$ [3].

THEOREM 3.10. *There exists a 1-rotational $EDTS(v, v, 0, 0, 0)$ if and only if $v \equiv 0$ (mod 3).*

Now, we can conclude the following theorem.

THEOREM 3.11. *There exists a 1-rotational $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ if and only if*

(i) $v \equiv 0$ *(mod 3)*, $\rho = v$ *and* $\eta_1 = \eta_2 = \eta_3 = 0$, *or*

(ii) $v \equiv 1$ *(mod 3)*, $\rho = 1$, $\eta_1 = v - 1$ *and* $\eta_2 = \eta_3 = 0$, *or*

(iii) $v \equiv 2$ *(mod 3)*, $\rho = 1$, $\eta_2 = v - 1$ *and* $\eta_1 = \eta_3 = 0$, *or*

(iv) $v \equiv 1$ *(mod 3)*, $\rho = 1$, $\eta_3 = v - 1$ *and* $\eta_1 = \eta_2 = 0$.

4. 2-rotational extended directed triple systems

In a 2-rotational $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$, ρ must be 1, $\frac{v+1}{2}$ or v. The following theorem is a consequence of the existence of a 2-rotational $DTS(v)$ [3].

THEOREM 4.1. *There exists a 2-rotational* $EDTS(v, v, 0, 0, 0)$ *if and only if* $v \equiv 1$ *(mod 6).*

In a 2-rotational $EDTS\left(v, \frac{v+1}{2}, \eta_1, \eta_2, \eta_3\right)$, we have (i) $\eta_1 = \frac{v-1}{2}$, (ii) $\eta_2 = \frac{v-1}{2}$ or (iii) $\eta_3 = \frac{v-1}{2}$, with others zero. By Remark 2.1, it is enough to consider the existence of 2-rotational $EDTS\left(v, \frac{v+1}{2}, \eta_1, \eta_2, \eta_3\right)$ for the two cases (i) and (ii).

LEMMA 4.2. (i) *If there exists a 2-rotational* $EDTS(v, \frac{v+1}{2}, \frac{v-1}{2}, 0, 0)$, *then* $v \equiv 5$ *(mod 6).*

(ii) *If there exists a 2-rotational* $EDTS\left(v, \frac{v+1}{2}, 0, \frac{v-1}{2}, 0\right)$, *then* $v \equiv 1$ *(mod 6).*

PROOF. First of all, v must be odd. Let n be the number of blocks consisting of three distinct elements. By counting the number of ordered pairs which appear in the system, if it is (i), we have

$$\frac{v+1}{2} + (v-1) + 3n = v^2;$$

so $3n = \frac{(2v-1)(v-1)}{2}$ and hence $v \equiv 5$ (mod 6) since v is odd; and if it is (ii), we have

$$\frac{v+1}{2} + \frac{3(v-1)}{2} + 3n = v^2;$$

so $3n = (v-1)^2$ and hence $v \equiv 1$ (mod 6) since v is odd. □

LEMMA 4.3. *There exists a 2-rotational* $EDTS\left(v, \frac{v+1}{2}, 0, \frac{v-1}{2}, 0\right)$ *for* $v \equiv 1$ *(mod 6).*

PROOF. If $v \equiv 1$ (mod 6), there exists a 2-rotational $ESTS\left(v, \frac{v+1}{2}\right)$ [2]. If we replace each block of a 2-rotational $ESTS\left(v, \frac{v+1}{2}\right)$ as follows:

$$\begin{aligned} &\{a,a,a\} \text{ by } [a,a,a], \\ &\{a,b,c\} \text{ by } [a,b,c] \text{ and } [c,b,a], \\ &\{a,a,b\} \text{ by } [a,b,a] \end{aligned}$$

the resulting transitive triples form a 2-rotational $EDTS(v, \frac{v+1}{2}, 0, \frac{v-1}{2}, 0)$. □

From Lemmas 4.2 and 4.3, we have the following theorem.

THEOREM 4.4. *There exists a 2-rotational* $EDTS\left(v, \frac{v+1}{2}, 0, \frac{v-1}{2}, 0\right)$ *if and only if* $v \equiv 1 \pmod{6}$.

LEMMA 4.5. *There exist a 2-rotational* $EDTS(v, \frac{v+1}{2}, \frac{v-1}{2}, 0, 0)$ *for* $v \equiv 5 \pmod{6}$.

PROOF. Let $v = 6t+5$ and $t \geq 0$. Then the following transitive triples

$$\begin{array}{lll} [\infty,\infty,\infty], & [0,\infty,5t+1], & [1,\infty,5t+2], \\ [0,0,0], & [1,1,6t+3], & [0,6t+3,6t+2], \\ [0,2r-1,3t+r], & [0,2r,5t+1+r], & r=1,2,\ldots,t, \\ [1,2r,3t+1+r], & [1,2r+1,5t+2+r], & r=1,2,\ldots,t \end{array}$$

are a set of starter blocks for a 2-rotational $EDTS(v, \frac{v+1}{2}, \frac{v-1}{2}, 0, 0)$ based on $Z_{v-1} \cup \{\infty\}$ with

$$\alpha = \left(0, 2, \ldots, \frac{v}{2} - 2\right)\left(1, 3, \ldots, \frac{v}{2} - 1\right)$$

as a 2-rotational automorphism. □

From Lemmas 4.2 and 4.5, we have the following theorem.

THEOREM 4.6. *There exists a 2-rotational* $EDTS\left(v, \frac{v+1}{2}, \frac{v-1}{2}, 0, 0\right)$ *if and only if* $v \equiv 5 \pmod{6}$.

Now, In a 2-rotational $EDTS\,(v, 1, \eta_1, \eta_2, \eta_3)$, we have (i) $\eta_1 = v-1$, (ii) $\eta_2 = v-1$, (iii) $\eta_3 = v-1$, (iv) $\eta_1 = \eta_2 = \frac{v-1}{2}$, (v) $\eta_1 = \eta_3 = \frac{v-1}{2}$, or (vi) $\eta_2 = \eta_3 = \frac{v-1}{2}$, with others zero. Also, by Remark 2.1, it is enough to consider the existence of 2-rotational $EDTS\,(v, 1, \eta_1, \eta_2, \eta_3)$ for the two cases (i), (ii), (iv) and (v).

LEMMA 4.7. (i) *If there exists a 2-rotational* $EDTS(v, 1, v-1, 0, 0)$, *then* $v \equiv 1 \pmod{6}$.

(ii) *If there exists a 2-rotational* $EDTS(v, 1, 0, v-1, 0)$, *then* $v \equiv 5 \pmod{6}$.

PROOF. First of all, since $\frac{v-1}{2}$ is an integer, v must be odd. Let n be the number of blocks consisting of three distinct elements. By counting the number of ordered pairs which appear in the system, if it is (i), we have

$$1 + 2(v-1) + 3n = v^2;$$

so $3n = (v-1)^2$ and hence $v \equiv 1 \pmod 3$; so $v \equiv 1 \pmod 6$ since v is odd; if it is (ii), we have

$$1 + 3(v-1) + 3n = v^2;$$

so $3n = (v-2)(v-1)$ and hence $v \equiv 5 \pmod 6$ since $(v-2)(v-1)$ must be divisible by both 3 and $\frac{v-1}{2}$, and v is odd. □

It is easy to see that if there exists a 1-rotational $EDTS(v, \rho, \eta_1, \eta_2, \eta_3)$ with α as a 1-rotational automorphism, then it is also a 2-rotational $EDTS\,(v, \rho, \eta_1, \eta_2, \eta_3)$ with α^2 as a 2-rotational automorphism, provided v is odd. Thus the following theorem follows from Theorems 3.5 and 3.9 together with Lemma 4.7.

THEOREM 4.8. (i) *There exists a 2-rotational* $EDTS(v, 1, v-1, 0, 0)$ *if and only if* $v \equiv 1$ *(mod 6)*.

(ii) *There exists a 2-rotational* $EDTS(v, 1, 0, v-1, 0)$ *if and only if* $v \equiv 5$ *(mod 6)*.

LEMMA 4.9. *If there exists a 2-rotational* $EDTS\left(v, 1, \frac{v-1}{2}, 0, \frac{v-1}{2}\right)$, *then* $v \equiv 1$ *(mod 6)*.

PROOF. First of all, v is odd. Let n be the number of blocks consisting of three distinct elements. Then we have

$$1 + 2(v-1) + 3n = v^2;$$

so $3n = (v-1)^2$ which is divisible by $\frac{v-1}{2}$ and hence $v \equiv 1 \pmod 3$. Since v is odd, $v \equiv 1 \pmod 6$. □

LEMMA 4.10. *There exists a 2-rotational* $EDTS\left(v, 1, \frac{v-1}{2}, 0, \frac{v-1}{2}\right)$ *for* $v \equiv 1$ *(mod 6)*.

PROOF. Let $v = 6t+1$ and let $\mathfrak{B}$ be the set of blocks for a 1-rotational $EDTS(v, 1, v-1, 0, 0)$ constructed in Lemma 3.7, with

$$\alpha = (\infty)(0, 1, \ldots, v-2)$$

as a 1-rotational automorphism. if we replace the blocks

$$[1, 1, 2t+1], [3, 3, 2t+3], \ldots, [6t-1, 6t-1, 2t-1]$$

in $\mathfrak{B}$ by

$$[1, 2t+1, 2t+1], [3, 2t+3, 2t+3], \ldots, [6t-1, 2t-1, 2t-1],$$

then the resulting blocks form a set of blocks for a 2-rotational $EDTS$ $\left(v, 1, \frac{v-1}{2}, 0, \frac{v-1}{2}\right)$ with

$$\alpha^2 = (\infty)(0, 2, \ldots, 6t-2)(1, 3, \ldots, 6t-1)$$

as a 2-rotational automorphism. □

From Lemmas 4.9 and 4.10, we have the following theorem.

THEOREM 4.11. *There exists a 2-rotational* $EDTS\left(v, 1, \frac{v-1}{2}, 0, \frac{v-1}{2}\right)$ *if and only if* $v \equiv 1 \pmod 6$.

LEMMA 4.12. *If there exists a 2-rotational* $EDTS\left(v, 1, \frac{v-1}{2}, \frac{v-1}{2}, 0\right)$, *then* $v \equiv 3 \pmod 6$.

PROOF. First of all, v is odd. Let n be the number of blocks consisting of three distinct elements. Then we have

$$1 + (v-1) + \frac{3(v-1)}{2} + 3n = v^2;$$

so $3n = \frac{(2v-3)(v-1)}{2}$ which is divisible by $\frac{v-1}{2}$ and hence $v \equiv 0 \pmod 3$. Since v is odd, $v \equiv 3 \pmod 6$. □

REMARK 4.13. It is easy to see that there is no 2-rotational $EDTS(3, 1, 1, 1, 0)$.

We assume that our 2-rotational $EDTS(v)$ has the element-set $V = Z_{\frac{v-1}{2}} \times Z_2 \cup \{\infty\}$ and the permutation $\alpha = (\infty)\left(0_0, 1_0, \ldots, \left(\frac{v-1}{2}-1\right)_0\right)$ $\left(0_1, 1_1, \ldots, \left(\frac{v-1}{2}-1\right)_1\right)$ as a 2-rotational automorphism. For brevity, we write x_i for the ordered pair $(x, i) \in Z_{\frac{v-1}{2}} \times Z_2$.

A $(S_3, 3t+1)$-*system* is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \ldots, 3t+1\}$ such that $\{a_r, b_r | r = 1, 2, \ldots, 3t+1\} = \{0, 1, \ldots, 6t+1\}$ and $b_r - a_r = r$ for $r = 1, 2, \ldots, 3t+1$.

LEMMA 4.14. *If* $t \equiv 0$ *or 1* $\pmod 4$, *then there exists a* $(S_3, 3t+1)$-*system*.

PROOF. Obviously, $\{(0,1)\}$ is a $(S_3,1)$-system. If $t \equiv 0 \pmod 4$ and $t \geq 4$, then the following ordered pairs form a $(S_3, 3t+1)$-system:

$$\begin{aligned}
&(3t+r, 6t+2-r), && r = 1,2,\ldots,\frac{3t}{2},\\
&(r-1, 3t-r), && r = 1,2,\ldots,\frac{3t}{4},\\
&\left(\frac{3t+4}{4}+r, \frac{9t}{4}-r\right), && r = 1,2,\ldots,\frac{3t-8}{4},\\
&\left(\frac{3t}{4},\frac{3t+4}{4}\right), \left(\frac{3t}{2},\frac{9t+2}{2}\right), \left(\frac{3t+2}{2}, 3t\right).
\end{aligned}$$

$\{(0,1),(4,6),(2,5),(3,7)\}$ is a $(S_3,4)$-system. If $t \equiv 1 \pmod 4$ and $t \geq 5$, then the following ordered pairs form a $(S_3, 3t+1)$-system:

$$\begin{aligned}
&(3t-1+r, 6t+2-r), && r = 1,2,\ldots,\frac{3t+1}{2},\\
&(r-1, 3t-1-r), && r = 1,2,\ldots,\frac{3t-3}{4},\\
&\left(\frac{3t+1}{4}+r, \frac{9t-1}{4}-r\right), && r = 1,2,\ldots,\frac{3t-7}{4},\\
&\left(\frac{3t-3}{4},\frac{3t+1}{4}\right), \left(\frac{3t-1}{2}, 3t-1\right), \left(\frac{3t+1}{2},\frac{9t+1}{2}\right).
\end{aligned}$$

□

LEMMA 4.15. *There exists a 2-rotational* $EDTS(v,1,\frac{v-1}{2},\frac{v-1}{2},0)$ *for* $v \equiv 9$ *or 21 (mod 48).*

PROOF. Let $v = 12t+9$, $t \equiv 0$ or $1 \pmod 4$, and let $\{(a_r,b_r) | r = 1,2,\ldots,3t+1\}$ be a $(S_3,3t+1)$-system. Then the following transitive triples

$$\begin{aligned}
&[\infty,\infty,\infty], [0_0, 0_0, (3t+2)_0], [\infty, (6t+2)_1, 0_0], [0_0, (6t+2)_1, \infty],\\
&[0_0, r_0, (b_r)_1], [(b_r)_1, r_0, 0_0] \;\; r = 1,2,\ldots,3t+1,\\
&[(6t+3)_1, 0_0, (6t+3)_1]
\end{aligned}$$

together with a set of starter blocks for a cyclic $DTS(6t+4)$ based on $Z_{6t+4} \times \{1\}$ form a set of starter blocks for a 2-rotational $EDTS(v,1,\frac{v-1}{2},\frac{v-1}{2},0,0)$. □

A $(S_4, 3t+1)$-*system* is a set of ordered pairs $\{(a_r,b_r) | r = 1,2,\ldots,3t+1\}$ such that $\{a_r, b_r | r = 1,2,\ldots,3t+1\} = \{0,1,\ldots,6t,6t+2\}$ and $b_r - a_r = r$ for $r = 1,2,\ldots,3t+1$.

LEMMA 4.16. *If $t \equiv 2$ or 3 (mod 4), then there exists a $(S_4, 3t+1)$-system.*

PROOF. If $t \equiv 2 \pmod 4$, then the following ordered pairs form a $(S_4, 3t+1)$-system:

$$\begin{aligned}
&(3t+1+r, 6t+1-r), && r = 1,2,\ldots,\frac{3t-2}{2},\\
&(r-1, 3t-r), && r = 1,2,\ldots,\frac{3t-2}{4},\\
&\left(\frac{3t+2}{4}+r, \frac{9t+2}{4}-r\right), && r = 1,2,\ldots,\frac{3t-6}{4},\ (t>2),\\
&\left(\frac{3t-2}{4}, \frac{3t+2}{4}\right), \left(\frac{3t}{2}, 3t\right), \left(\frac{3t+2}{2}, \frac{9t+2}{2}\right), (3t+1, 6t+2).
\end{aligned}$$

If $t \equiv 3 \pmod 4$, then the following ordered pairs form a $(S_4, 3t+1)$-system:

$$\begin{aligned}
&(3t+1+r, 6t+1-r), && r = 1,2,\ldots,\frac{3t-5}{4},\\
&(r-1, 3t-r), && r = 1,2,\ldots,\frac{3t-1}{2},\\
&\left(\frac{15t-1}{4}+r, \frac{21t+1}{4}-r\right), && r = 1,2,\ldots,\frac{3t-5}{4},\\
&\left(\frac{3t-1}{2}, \frac{9t-1}{2}\right), \left(3t, \frac{9t+1}{2}\right), \left(\frac{21t+1}{4}, \frac{21t+5}{4}\right), (3t+1, 6t+2).
\end{aligned}$$

□

LEMMA 4.17. *There exists a 2-rotational EDTS $\left(v, 1, \frac{v-1}{2}, \frac{v-1}{2}, 0\right)$ for $v \equiv 33$ or 45 (mod 48).*

PROOF. Let $v = 12t+9$, $t \equiv 2$ or $3 \pmod 4$, and let $\{(a_r, b_r) | r = 1, 2, \ldots, 3t+1\}$ be a $(S_4, 3t+1)$-system. Then the following transitive triples

$$\begin{aligned}
&[\infty,\infty,\infty], [0_0, 0_0, (3t+2)_0], [\infty, (6t+1)_1, 0_0], [0_0, (6t+1)_1, \infty],\\
&[0_0, r_0, (b_r)_1], [(b_r)_1, r_0, 0_0]\ \ r = 1,2,\ldots,3t+1,\\
&[(6t+3)_1, 0_0, (6t+3)_1]
\end{aligned}$$

together with a set of starter blocks for a cyclic $DTS(6t+4)$ based on $Z_{6t+4} \times \{1\}$ form a set of starter blocks for a 2-rotational $EDTS(v, 1, \frac{v-1}{2}, \frac{v-1}{2}, 0, 0)$. □

A $(S_5, 3t)$*-system* is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \ldots, 3t\}$ such that $\{a_r, b_r | r = 1, 2, \ldots, 3t+1\} = \{0, 1, \ldots, 6t-1\}$ and $b_r - a_r = r$ for $r = 1, 2, \ldots, 3t$.

LEMMA 4.18. *If $t \equiv 0$ or 3 (mod 4), then there exists a $(S_5, 3t)$-system.*

PROOF. If $t \equiv 0 \pmod 4$, then the following ordered pairs form a $(S_5, 3t)$-system:

$$
\begin{aligned}
&(3t-2+r, 6t-r), && r = 1, 2, \ldots, \frac{3t}{2}, \\
&(r-1, 3t-2-r), && r = 1, 2, \ldots, \frac{3t-4}{4}, \\
&\left(\frac{3t}{4}+r, \frac{9t-4}{4}-r\right), && r = 1, 2, \ldots, \frac{3t-8}{4}, \\
&\left(\frac{3t-4}{4}, \frac{3t}{4}\right), \left(\frac{3t-2}{2}, 3t-2\right), \left(\frac{3t}{2}, \frac{9t-2}{2}\right).
\end{aligned}
$$

If $t \equiv 3 \pmod 4$, then the following ordered pairs form a $(S_5, 3t)$-system:

$$
\begin{aligned}
&(3t-1+r, 6t-r), && r = 1, 2, \ldots, \frac{3t-1}{2}, \\
&(r-1, 3t-1-r), && r = 1, 2, \ldots, \frac{3t-1}{4}, \\
&\left(\frac{3t+3}{4}+r, \frac{9t-3}{4}-r\right), && r = 1, 2, \ldots, \frac{3t-9}{4}, \ (t>3), \\
&\left(\frac{3t-1}{4}, \frac{3t+3}{4}\right), \left(\frac{3t-1}{2}, \frac{9t-1}{2}\right), \left(\frac{3t+1}{2}, 3t-1\right).
\end{aligned}
$$

□

LEMMA 4.19. *If $v \equiv 3$ or 39 (mod 48) and $v \neq 3$, then there exists a 2-rotational $EDTS\left(v, 1, \frac{v-1}{2}, \frac{v-1}{2}, 0\right)$.*

PROOF. Let $v = 12t + 3$, $t \equiv 0$ or $3 \pmod 4$, $t > 0$, and let $\{(a_r, b_r) | r = 1, 2, \ldots, 3t\}$ be a $(S_5, 3t)$-system. Then the following transitive triples

$$[\infty, \infty, \infty], [0_0, 0_0, (3t)_0], [\infty, (b_{3t})_1, 0_0], [(3t)_1, 0_0, \infty],$$
$$[0_0, r_0, (b_r)_1], [(b_r)_1, r_0, 0_0] \;\; r = 1, 2, \ldots, 3t - 1,$$
$$[(3t)_0, 0_0, (b_{3t})_1], [(6t)_1, 0_0, (6t)_1]$$

together with a set of starter blocks for a cyclic $DTS(6t + 4)$ based on $Z_{6t+1} \times \{1\}$ form a set of starter blocks for a 2-rotational $EDTS(v, 1, \frac{v-1}{2}, \frac{v-1}{2}, 0, 0)$. □

A $(S_6, 3t)$-*system* is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \ldots, 3t\}$ such that $\{a_r, b_r | r = 1, 2, \ldots, 3t+1\} = \{0, 1, \ldots, 6t-2, 6t\}$ and $b_r - a_r = r$ for $r = 1, 2, \ldots, 3t$

LEMMA 4.20. *If $t \equiv 1$ or $2 \pmod 4$, then there exists a $(S_6, 3t + 1)$-system.*

PROOF. $\{(0, 1), (2, 4), (3, 6)\}$ is a $(S_6, 3t)$-system. If $t \equiv 1 \pmod 4$ and $t > 1$, then the following ordered pairs form a $(S_6, 3t)$-system:

$$(3t + r, 6t - 1 - r), \qquad r = 1, 2, \ldots, \frac{3t-3}{2},$$
$$(r - 1, 3t - 1 - r), \qquad r = 1, 2, \ldots, \frac{3t-3}{4},$$
$$\left(\frac{3t+1}{4} + r, \frac{9t-1}{4} - r\right), \qquad r = 1, 2, \ldots, \frac{3t-7}{4},$$
$$\left(\frac{3t+1}{4}, \frac{3t+5}{4}\right), \left(\frac{3t-1}{2}, 3t - 1\right), \left(\frac{3t+1}{2}, \frac{9t-1}{2}\right), (3t, 6t).$$

If $t \equiv 2 \pmod 4$ and $t > 1$, then the following ordered pairs form a $(S_6, 3t)$-system:

$$(3t + r, 6t - 1 - r), \qquad r = 1, 2, \ldots, \frac{3t-6}{4}, \; (t > 2),$$
$$(r - 1, 3t - 1 - r), \qquad r = 1, 2, \ldots, \frac{3t-2}{2},$$
$$\left(\frac{15t-6}{4} + r, \frac{21t-6}{4} - r\right), \qquad r = 1, 2, \ldots, \frac{3t-6}{4}, \; (t > 2),$$
$$\left(\frac{3t-2}{2}, \frac{9t-4}{2}\right), \left(3t - 1, \frac{9t-2}{2}\right), \left(\frac{21t-6}{4}, \frac{21t-2}{4}\right), (3t, 6t).$$

□

LEMMA 4.21. *There exists a 2-rotational EDTS* $\left(v,1,\frac{v-1}{2},\frac{v-1}{2},0\right)$ *for* $v \equiv 15$ *or 27 (mod 48).*

PROOF. Let $v = 12t+3$, $t \equiv 1$ or 2 (mod 4), and let $\{(a_r,b_r)|r = 1,2,\ldots,3t\}$ be a $(S_6,3t)$-system. Then the following transitive triples

$$\begin{aligned}&[\infty,\infty,\infty],[0_0,0_0,(3t)_0],[\infty,(b_{3t})_1,0_0],[(3t)_1,0_0,\infty],\\&[0_0,r_0,(b_r)_1],[(b_r)_1,r_0,0_0]\ \ r=1,2,\ldots,3t-1,\\&[(3t)_0,0_0,(b_{3t})_1],[(6t-1)_1,0_0,(6t-1)_1]\end{aligned}$$

together with a set of starter blocks for a cyclic $DTS(6t+4)$ based on $Z_{6t+1}\times\{1\}$ form a set of starter blocks for a 2-rotational $EDTS(v,1,\frac{v-1}{2},\frac{v-1}{2},0,0)$. □

From Lemmas 4.12, 4.15, 4.19 and 4.21, we have the following theorem.

THEOREM 4.22. *There exists a 2-rotational* $EDTS(v,1,\frac{v-1}{2},\frac{v-1}{2},0)$ *if and only if* $v \equiv 3$ *(mod 6),* $v \neq 3$.

Now, we can conclude the following theorem

THEOREM 4.23. *There exists a 2-rotational* $EDTS(v,\rho,\eta_1,\eta_2,\eta_3)$ *if and only if*

(i) $v \equiv 1$ *(mod 6),* $\rho = v$, *and* $\eta_1 = \eta_2 = \eta_3 = 0$, *or*
(ii) $v \equiv 5$ *(mod 6),* $\rho = \frac{v+1}{2}$, $\eta_1 = \frac{v-1}{2}$, *and* $\eta_2 = \eta_3 = 0$, *or*
(iii) $v \equiv 1$ *(mod 6),* $\rho = \frac{v+1}{2}$, $\eta_2 = \frac{v-1}{2}$, *and* $\eta_1 = \eta_3 = 0$, *or*
(iv) $v \equiv 5$ *(mod 6),* $\rho = \frac{v+1}{2}$, $\eta_3 = \frac{v-1}{2}$, *and* $\eta_1 = \eta_2 = 0$, *or*
(v) $v \equiv 1$ *(mod 6),* $\rho = 1$, $\eta_1 = v-1$, *and* $\eta_2 = \eta_3 = 0$, *or*
(vi) $v \equiv 5$ *(mod 6),* $\rho = 1$, $\eta_2 = v-1$, *and* $\eta_1 = \eta_3 = 0$, *or*
(vii) $v \equiv 1$ *(mod 6),* $\rho = 1$, $\eta_3 = v-1$, *and* $\eta_1 = \eta_2 = 0$, *or*
(viii) $v \equiv 1$ *(mod 6),* $\rho = 1$, $\eta_1 = \eta_3 = \frac{v-1}{2}$, *and* $\eta_2 = 0$, *or*
(ix) $v \equiv 3$ *(mod 6),* $v \neq 3$, $\rho = 1$, $\eta_1 = \eta_2 = \frac{v-1}{2}$, *and* $\eta_3 = 0$, *or*
(x) $v \equiv 3$ *(mod 6),* $v \neq 3$, $\rho = 1$, $\eta_2 = \eta_3 = \frac{v-1}{2}$, *and* $\eta_1 = 0$.

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