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CONVERGENCE THEOREMS OF THE ITERATIVE SEQUENCES FOR NONEXPANSIVE MAPPINGS

JUNG IM KANG, YEOL JE CHO AND HAIYUN ZHOU

ABSTRACT. In this paper, we will prove the following: Let D be a nonempty subset of a normed linear space X and $T : D \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in D and $\{t_n\}$, $\{s_n\}$ be real sequences such that

(i) $0 \leq t_n \leq t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) (a) $0 \leq s_n \leq 1$, $s_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} t_n s_n < \infty$ or (b) $s_n = s$ for all $n \geq 1$ and $s \in [0, 1)$,
(iii) $x_{n+1} = (1-t_n)x_n + t_n T(s_n T x_n + (1-s_n)x_n)$ for all $n \geq 1$.

Then, if the sequence $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This result improves and complements a result of Deng [2]. Furthermore, we will show that certain conditions on D, X and T guarantee the weak and strong convergence of the Ishikawa iterative sequence to a fixed point of T.

Let X be a real normed linear space and D a nonempty subset of X. Let $T : D \to X$ be *nonexpansive*, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in D$.

In 1976, Ishikawa [4] proved the following interesting result:

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THEOREM I. *Let D be a nonempty subset of a normed linear space X and $T : D \to X$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{t_n\}$ be a sequence in D and a sequence of real numbers, respectively, such that*

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $x_{n+1} = (1 - t_n)x_n + t_n T x_n$ *for all* $n \ge 1$.

If $\{x_n\}$ *is bounded, then*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

In 1996, Deng [2, Theorem 1] extended Theorem I to the Ishikawa iterative sequence by proving the following:

THEOREM D. *Let D be a nonempty subset of a normed linear space X and $T : D \to X$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{t_n\}$, $\{s_n\}$ be a sequence in D and sequences of real numbers, respectively, such that*

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $0 \le s_n \le 1$ *and* $\sum_{n=1}^{\infty} s_n < \infty$,
(iii) $x_{n+1} = (1 - t_n)x_n + t_n T(s_n T x_n + (1 - s_n)x_n)$ *for all* $n \ge 1$.

If $\{x_n\}$ *is bounded, then*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

One question arises naturally: Is Theorem D with the conditions (ii) in Abstract above $\sum_{n=1}^{\infty} t_n s_n < \infty$ and $s_n \to 0$ as $n \to \infty$ or $s_n = s$ for all $n \ge 1$ and $s \in [0, 1)$ instead of the condition (ii) in Theorem D true?

It is our purpose in this paper to solve the above problem by proving the following result:

THEOREM 1. *Let D be a nonempty subset of a normed linear space X and $T : D \to X$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{t_n\}$, $\{s_n\}$ be a sequence in D and sequences of real numbers, respectively, such that*

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) (a) $0 \le s_n \le 1$, $\lim_{n\to\infty} s_n = 0$ *and* $\sum_{n=1}^{\infty} t_n s_n < \infty$ *or* (b) $s_n = s$ *for all* $n \ge 1$ *and* $s \in [0, 1)$,
(iii) $x_{n+1} = (1 - t_n)x_n + t_n T(s_n T x_n + (1 - s_n)x_n)$ *for all* $n \ge 1$.

If $\{x_n\}$ *is bounded, then*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

In order to prove Theorem 1, we shall need several important lemmas.

LEMMA 2. ([1]) *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists. In particular, if* $\{a_n\}$ *has a subsequence converging to zero, then* $\lim_{n\to\infty} a_n = 0$.

LEMMA 3. *Let* X *be a normed linear space and* D *be a nonempty subset of* X. *Let* $T : D \to X$ *be a nonexpansive mapping. Let* $\{x_n\}$ *be a sequence in* D *satisfying the condition* (iii) *in Theorem 1. Then we have*

$$\|x_{n+1} - Tx_{n+1}\| \leq (1 + 2t_n s_n)\|x_n - Tx_n\|$$

for all $n \geq 1$.

PROOF. Observe first that

$$\begin{aligned} \|x_n - Ty_n\| &\leq \|x_n - Tx_n\| + \|Tx_n - Ty_n\| \\ &\leq \|x_n - Tx_n\| + \|x_n - y_n\| \\ &= \|x_n - Tx_n\| + s_n\|x_n - Tx_n\| \\ &= (1+s_n)\|x_n - Tx_n\| \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x_n\| &= t_n\|x_n - Ty_n\| \\ &\leq t_n(1+s_n)\|x_n - Tx_n\|. \end{aligned}$$

Thus it follows that

$$\begin{aligned} &\|x_{n+1} - Tx_{n+1}\| \\ &\leq (1-t_n)\|x_n - Tx_{n+1}\| + t_n\|Ty_n - Tx_{n+1}\| \\ &\leq (1-t_n)(\|x_n - Tx_n\| + \|Tx_n - Tx_{n+1}\|) + t_n\|y_n - x_{n+1}\| \\ &\leq (1-t_n)\|x_n - Tx_n\| + (1-t_n)\|x_n - x_{n+1}\| \\ &\quad + t_n\|y_n - x_n\| + t_n\|x_n - x_{n+1}\| \\ &\leq (1 - t_n + t_n s_n)\|x_n - Tx_n\| + t_n(1+s_n)\|x_n - Tx_n\| \\ &= (1+2t_n s_n)\|x_n - Tx_n\|. \end{aligned}$$

This completes the proof. □

Recall that a metric space (X, d) is said to be *of hyperbolic type* if X contains a family L of metric segments such that

(a) each two points $x,\ y \in X$ are endpoints of exactly one member segment of L,

(b) if $p,\ x,\ y \in X$ and $m \in \mathrm{seg}[x, y]$ satisfies $d(x, m) = \alpha d(x, y)$ for $\alpha \in [0, 1]$, then

$$d(p, m) \leq (1-\alpha)d(p, x) + \alpha d(p, y).$$

This class includes all normed linear spaces as well as all spaces with the metric of hyperbolic type (see [3]).

LEMMA 4. ([3]) *Let (X, d) be of hyperbolic type and $\{t_n\}$ be a sequence in $[0, 1)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that, for all $n \geq 1$,*

(i) $x_{n+1} \in \mathrm{seg}[x_n, y_n]$ *with* $d(x_n, x_{n+1}) = t_n d(x_n, y_n)$,

(ii) $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$,

(iii) $d(y_{i+n}, x_i) \leq M < \infty$ *for all* $i,\ n \geq 1$,

(iv) $t_n \leq t < 1$,

(v) $\sum_{n=1}^{\infty} t_n = \infty$.

Then we have

$$\lim_{n\to\infty} d(y_n, x_n) = 0.$$

THE PROOF OF THEOREM 1. We assume first that $\sum_{n=1}^{\infty} t_n s_n < \infty$ and $s_n \to 0$ as $n \to \infty$. By Lemmas 2 and 3, we see that $\|x_n - Tx_n\|$ exists, say d. Setting $a_n = Tx_n - x_n$, then we have $\|a_n\| \to d$ as $n \to \infty$. Without of loss of generality, we may assume that $t_n > 0$ for all $n \geq 1$. Otherwise, consider a subsequence $\{t_j\}$ of $\{t_n\}$. Setting

$$b_n = t_n{}^{-1}(Tx_{n+1} - Tx_n) + Tx_n - Ty_n,$$

we have $a_{n+1} = (1-t_n)a_n + t_n b_n$. Following the proof lines of Deng [2, Theorem 1], we can get the following conclusions:

(1) $\limsup_{n\to\infty} \|b_n\| \leq d$,

(2) $\|\sum_{i=1}^{n} t_i b_i\| \leq \|x_{n+1} - x_1\| + \sum_{i=1}^{n} t_i s_i \|Tx_i - x_i\|$ is bounded.

Thus the conclusion of Theorem 1 follows exactly from Deng [2, Lemma 2].

Next, we assume that $s_n = s$ for all $n \geq 1$ and $s \in [0,1)$. In order to end the proof of Theorem 1, we only need to verify the conditions (i)~(v) in Lemma 4.

The condition (i) is obvious and the conditions (iv), (v) are natural. Now, we prove only the conditions (ii), (iii).

Setting $z_n = (1-s)x_n + sTx_n$ and $y_n = Tz_n$, then we have

$$\begin{aligned} d(y_{n+1}, y_n) &= \|y_{n+1} - y_n\| \\ &= \|Tz_{n+1} - Tz_n\| \\ &\leq \|z_{n+1} - z_n\| \\ &= \|(1-s)x_{n+1} + sTx_{n+1} - (1-s)x_n - sTx_n\| \\ &\leq (1-s)\|x_{n+1} - x_n\| + s\|Tx_{n+1} - Tx_n\| \\ &\leq (1-s)\|x_{n+1} - x_n\| + s\|x_{n+1} - x_n\| \\ &= \|x_{n+1} - x_n\| \\ &= d(x_{n+1}, x_n), \end{aligned}$$

which verifies the condition (ii).

Next, we have the following.

$$\begin{aligned} d(y_{i+n}, x_i) &= \|y_{i+n} - x_i\| \\ &= \|Tz_{i+n} - x_i\| \\ &\leq \|Tz_{i+n} - Tx_{i+n}\| + \|Tx_{i+n} - x_i\| \\ &\leq \|z_{i+n} - x_{i+n}\| + \|Tx_{i+n} - x_i\| \\ &\leq s\|x_{i+n} - Tx_{i+n}\| + \|Tx_{i+n} - x_i\| \\ &\leq M < \infty, \end{aligned}$$

which verifies the condition (iii).

By Lemma 4, we assert that $\lim_{n\to\infty} \|Tz_n - x_n\| = 0$. Observe that

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - Tz_n\| + \|Tz_n - x_n\| \\ &\leq \|x_n - z_n\| + \|Tz_n - x_n\| \\ &= s\|x_n - Tx_n\| + \|Tz_n - x_n\|, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

As immediate consequences of Theorem 1, we have the following.

COROLLARY 5. *Let X be a real normed linear space, D be a nonempty compact convex subset of X and $T : D \to D$ be a nonexpansive mapping. Let the sequence $\{x_n\}$ in D be defined as in Theorem 1. Then $\{x_n\}$ converges strongly to a fixed point of T.*

PROOF. Since D is a convex subset of X and $T : D \to D$ is a self-mapping, we see that the sequence $\{x_n\}$ is well-defined. From the compactness of D, we assert that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ as $j \to \infty$. Thus it follows from Theorem 1 that $Tx_{n_j} \to p$ as $j \to \infty$. By virtue of the continuity of T, we conclude that $Tp = p$, which means that $p \in F(T)$, where $F(T)$ denotes the set of fixed points of T. Since $T : D \to D$ is nonexpansive, we have

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for all $n \geq 1$ and $q \in F(T)$ (see [6]), which shows that $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(T)$. Consequently, we have $x_n \to p$ as $n \to \infty$. This completes the proof. □

Recall that a mapping $T : D \to D$ with the fixed point set $F(T)$ satisfies *Condition* (A) if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\|x - Tx\| \geq \phi(d(x, F(T)))$$

for all $x \in D$ (see [5]).

COROLLARY 6. *Let X be a real Banach space, D be a nonempty closed convex subset of X and $T : D \to D$ be a nonexpansive mapping with the nonempty fixed point set $F(T)$. Let the sequence $\{x_n\}$ in D be as in Theorem 1. If the mapping T satisfies Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

PROOF. It follows from Theorem 1 and Condition (A) that

$$\phi(d(x_n, F(T))) \to 0$$

as $n \to \infty$. Since $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing, we assert that $d(x_n, F(T)) \to 0$ as $n \to \infty$. Observe that

$$\|x_{n+m} - q\| \leq \|x_n - q\|$$

for all n, $m \geq 1$ and $q \in F(T)$, which implies that

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq 2\|x_n - q\|\end{aligned}$$

and hence

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F(T)),$$

which shows that $\{x_n\}$ is a Cauchy sequence in D. Since D is complete, we may assume that $x_n \to p$ as $n \to \infty$. Therefore, $d(p, F(T)) = 0$ and $p \in F(T)$ because of the closedness of $F(T)$. This completes the proof.□

REMARK 1. In view of Theorem 1, we can also establish several weak and strong convergence theorems similar to Theorems 2∼6 of Deng [2]. We omit to prove them because of similarity of the proof lines.

REMARK 2. For the parameters of our theorems, one can make the following choices: If we put $t_n = \frac{1}{n+1}$ and $s_n = \frac{1}{n}$ or $s_n = s \in [0, 1)$ for all $n \geq 1$. Then these parameters satisfy all requirements of our results, however, they do not satisfy the requirements of Theorem D and the results of Tan and Xu [6].

Recall that a Banach space X satisfies *Opial's condition* if the condition $x_n \to x_0$ weakly implies

$$\limsup_{n\to\infty} \|x_n - x_0\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \neq x_0$.

THEOREM 7. *Let X be a Banach space which satisfies Opial's condition, D be weakly compact and T, $\{x_n\}$ be as in Theorem 1. Then $\{x_n\}$ converges weakly to a fixed point of T.*

PROOF. By the weak compactness of D, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $p \in D$. With the standard proof, we can show that $p = Tp$. Suppose that $\{x_n\}$ does not converge weakly to the point p. Then there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \neq q$ such that $x_{n_j} \to q$ weakly and $q = Tq$. Since $T : D \to X$ is nonexpansive, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

for all $n \geq 1$ and $p \in F(T)$ (see [6]). Thus, by Opial's condition of X,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\ &= \lim_{j\to\infty} \|x_{n_j} - q\| < \lim_{j\to\infty} \|x_{n_j} - p\| \\ &= \lim_{n\to\infty} \|x_n - p\|,\end{aligned}$$

which is a contradiction. This completes the proof. □

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Jung Im Kang and Yeol Je Cho
Department of Mathematics
the Research Institute of Natural Sciences
Gyeongsang National University
Chinju 660-701, Korea
E-mail: jungim-kang@hanmail.net
yjcho@gsnu.ac.kr

Haiyun Zhou
Department of Mathematics
Shijiazhuang Mechanical Engineering College
Shijiazhuang 050003, P. R. China
E-mail: witman66@yahoo.com.cn