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RADIAL SYMMETRY OF TOPOLOGICAL ONE-VORTEX SOLUTIONS IN THE MAXWELL-CHERN-SIMONS-HIGGS MODEL

JONGMIN HAN

ABSTRACT. In this paper we show the radial symmetry of topological one-vortex solutions in the Maxwell-Chern-Simons-Higgs Model.

1. Introduction

The self-duality is an important notion in various field theories in the sense that it allows a reduction of second order equations of motion to first order equations which are simpler to analyze and correspond to the minimization of energy. The classical Abelian-Higgs Model with the Maxwell term gives the phenomenological descriptions on superconductivity at low temperature, and it admits the self-dual structure [7]. On the other hand, for the high temperature superconductivity, we need to consider charged vortices which are obtained by adding the Chern-Simons term into the action. In [5, 6], the authors consider Chern-Simons-Higgs Model which contains the Chern-Simons term but exclude the Maxwell term. This model saturates the self-dual structure with the 6th order potential.

A natural question is whether there is any self-dual system containing both the Maxwell term and the Chern-Simons term. The Maxwell-Chern-Simons-Higgs Model(MCSH) was proposed in [8] for the purpose of unifying the Abelian-Higgs Model and the Chern-Simons-Higgs Model. In addition to allowing both the Maxwell term and the Chern-Simons term in MCSH, the authors introduce a neutral scalar field in

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order to get the self-dual structure. The static energy functional of MCSH is given by

$$\begin{aligned}\mathcal{E}(\phi, A, N) &= \int_{\mathbb{R}^2} |D_A\phi|^2 + \frac{1}{2}|F_A|^2 + q^2|\phi|^2 A_0^2 + \frac{1}{2}|\nabla A_0|^2 \\ &\quad + q^2|\phi|^2 N^2 + \frac{1}{2}|\nabla N|^2 + \frac{1}{2}(q|\phi|^2 + \kappa N - q)^2 \end{aligned} \tag{1.1}$$

with the following Gauss constraint equation

$$-\Delta A_0 + 2q^2|\phi|^2 A_0 = -\kappa F_A. \tag{1.2}$$

Here $i = \sqrt{-1}$, $A_0 : \mathbb{R}^2 \to \mathbb{R}$, and

$$\begin{cases} q > 0 & : \quad \text{the charge of the electron,} \\ \kappa > 0 & : \quad \text{the Chern-Simons coupling constant,} \\ \phi : \mathbb{R}^2 \to \mathbb{C} & : \quad \text{the complex Higgs field,} \\ A = (A_1, A_2) & : \quad \text{the coupled gauge potential,} \\ D_A\phi = \nabla\phi - iqA\phi, & : \quad \text{the covariant derivative,} \\ F_A = \partial_1 A_2 - \partial_2 A_1 & : \quad \text{the magnetic field,} \\ N : \mathbb{R}^2 \to \mathbb{R} & : \quad \text{the neutral scalar field.} \end{cases}$$

It is easy to check that the functional $\mathcal{E}(\phi, A, N)$ is invariant under the gauge transformation

$$(\phi, A, N) \to (e^{i\chi}\phi, A + \nabla\chi, N),$$

for $\chi : \mathbb{R}^2 \to \mathbb{R}$.

Using(1.2) and integrating by parts, we obtain

$$\begin{aligned}&\mathcal{E}(\phi, A, N) \\ =\ & \int_{\mathbb{R}^2} \Big(|D_1\phi \pm iD_2\phi|^2 + q^2|\phi|^2|A_0 \pm N|^2 + \frac{1}{2}|\nabla A_0 \pm \nabla N|^2 \\ & + \frac{1}{2}|F_A \pm (q|\phi|^2 + \kappa N - q)|^2\Big) dx \pm \Phi,\end{aligned}$$

where

$$\Phi = q\int_{\mathbb{R}^2} F_A dx.$$

If Φ is positive (negative), then choose the upper (lower) sign. This yields the lower bound of the energy functional:

$$\mathcal{E}(\phi, A, N) \geq |\Phi|,$$

which is saturated by the following system of self-dual equations

$$D_1\phi \pm iD_2\phi = 0, \tag{1.3}$$

$$A_0 \pm N = 0, \tag{1.4}$$

$$F_A \pm (q|\phi|^2 + \kappa N - q) = 0. \tag{1.5}$$

The boundary conditions are given by the finite energy condition of (1.1): as $|x| \to \infty$, either

$$|\phi| \to 1 \qquad \text{and} \qquad N \to 0,$$

or

$$|\phi| \to 0 \qquad \text{and} \qquad N \to \frac{q}{\kappa}.$$

The former is called topological, while the latter nontopological.

Let us take the upper signs in (1.3)-(1.5). To examine the self-dual equations further, we use the classical Jaffe-Taubes arguments [7]. In fact, the equation (1.3) implies that ϕ is holomorphic up to a nonvanishing multiple factor and has exactly d zeros allowing multiplicities. Thus we may assume that ϕ takes the form

$$\phi(z) = \exp\Big(\frac{1}{2}u(x) + i\sum_{j=1}^{k} n_j \arg(x - p_j) \Big), \tag{1.6}$$

where the points $p_1, \cdots, p_k$, called the vortex points, are the distinct zeros of ϕ with multiplicities $n_1, \cdots, n_k$, respectively. Clearly, $n_1 + \cdots + n_k = d$. We observe that the arbitrary choice on the imaginary part of ϕ merely reflects the gauge invariance of (1.2)-(1.5). Now the equations (1.2), (1.4), and (1.5) are transformed into

$$\Delta u = 2q^2(e^u - 1) - 2\kappa q A_0 + 4\pi \sum_{j=1}^{k} n_j \delta_{p_j}, \tag{1.7}$$

$$\Delta A_0 = -\kappa q(e^u - 1) + (\kappa^2 + 2q^2 e^u)A_0. \tag{1.8}$$

The boundary conditions are rewritten as

$$\text{topological} : u \to 0 \quad \text{and} \quad N = -A_0 \to 0, \tag{1.9}$$

$$\text{nontopological} : u \to -\infty \quad \text{and} \quad N = -A_0 \to \frac{q}{\kappa}, \tag{1.10}$$

as $|x| \to \infty$. Conversely, once we find a solution (u, A_0) of (1.7) and (1.8), we may recover A and N from (1.3) and (1.4) by the formula

$$qA_1 + iqA_2 = -2i\overline{\partial} \ln \phi, \qquad N = -A_0$$

where $\overline{\partial} = (\partial_1 + i\partial_2)/2$.

The existence of topological solutions for (1.7), (1.8), and (1.9) was proved in [2], while the nontopological solutions for (1.7), (1.8), and (1.10) in [1]. On the other hand one can consider the self-dual equations (1.2)-(1.5) on the Hooft type periodic domain, of which solutions are called condensate solutions. For the results of the condensate solutions, which satisfy (1.7) and (1.8) on a periodic domain, refer to [3, 9, 10].

In this paper we are interested in the topological solutions when all the vortex points are equal to a point, say, $p = 0$. For this case, let us rewrite (1.7) as

$$\Delta u = 2q^2(e^u - 1) - 2\kappa q A_0 + 4\pi d\delta_0. \tag{1.11}$$

Although it is expected that every solution of (1.8) and (1.11) with (1.9) is radial, it has not been proved rigorously yet. The purpose of this paper is to give a mathematical proof for it. We establish

THEOREM 1.1. *Every solution of* (1.8), (1.9), *and* (1.11) *is radially symmetric about the origin.*

We provide the proof of Theorem 1.1 in the next section. We close this section with a remark. In [4] topological solutions was studied in a unified framework for several self-dual Chern-Simons models which reduce to an elliptic equation by the Jaffe-Taubes argument. In particular, the radial symmetry of topological one-vortex solutions was proved for the following equation:

$$\Delta u = f(e^u) + 4\pi d\delta_0 \qquad \text{in } \mathbb{R}^2$$
$$u \to 0 \qquad \text{as } |x| \to \infty,$$

where $f : [0, \infty) \to \mathbb{R}$ is a smooth function satisfying $f(1) = 0$, $f'(1) > 0$,

$$f(t) < 0 \text{ on } (0,1), \qquad \text{and} \qquad f(t) > 0 \text{ on } (1,\infty).$$

Theorem 1.1 shows that the same conclusion holds for the self-dual equations (1.8), (1.9), and (1.11) which consist of *a system of elliptic equations.*

2. Proof of main theorem

This section is devoted to the proof of Theorem 1.1. We begin with the following lemma.

LEMMA 2.1. *If* (u, A_0) *is a solution of* (1.8), (1.9), *and* (1.11), *then* $u < 0$ *in* $\mathbb{R}^2\backslash\{0\}$ *and* $A_0 < 0$ *in* $\mathbb{R}^2$.

PROOF. Although this lemma is well known in [3], we provide a proof for the sake of completeness.

Let y be a maximum point of A_0. Suppose that $A_0(y) > 0$. Applying the maximum principle to (1.8), we see that

$$A_0(y) \le \frac{\kappa q}{\kappa^2 + 2q^2 e^{u(y)}}(e^{u(y)} - 1).$$

In particular, $y \neq 0$ and $u(y) > 0$. Let z be a maximum point of u. Again, it comes from the maximum principle applied to (1.11) that

$$A_0(z) \ge \frac{q}{\kappa}(e^{u(z)} - 1).$$

Consequently,

$$\begin{aligned} \frac{q}{\kappa}(e^{u(y)} - 1) &\le \frac{q}{\kappa}(e^{u(z)} - 1) \\ &\le A_0(z) \le A_0(y) \\ &\le \frac{\kappa q}{\kappa^2 + 2q^2 e^{u(y)}}(e^{u(y)} - 1). \end{aligned}$$

Since $u(y) > 0$, this gives a contradiction. Therefore we proved that $A_0 \le 0$.

On the other hand, since $A_0 \le 0$, the equation (1.11) reads

$$\Delta u \ge 2q^2(e^u - 1) + 4\pi d\delta_0$$

in the sense of distribution. Then the strong maximum principle implies that $u < 0$ in $\mathbb{R}^2 \backslash \{0\}$. Let us rewrite (1.8) as

$$\begin{aligned} \Delta A_0 - (\kappa^2 + 2q^2 e^u)A_0 &= -\kappa q(e^u - 1) \ge 0 \qquad \text{in } \mathbb{R}^2, \\ A_0 \to 0 \quad &\text{as} \quad |x| \to \infty. \end{aligned}$$

Then, similarly by the strong maximum principle, we find that $A_0 < 0$ in $\mathbb{R}^2$. □

For the proof of Theorem 1.1, we use the method of moving planes. To this aim, for $\lambda < 0$ and a solution (u, A_0) of (1.8), (1.9), and (1.11), let

$$\begin{aligned} \Sigma_\lambda &= \{x \in \mathbb{R}^2 | x_1 < \lambda\}, \\ \Gamma_\lambda &= \partial\Sigma_\lambda, \\ x_\lambda &= (2\lambda, 0), \\ u_\lambda(x_1, x_2) &= u(2\lambda - x_1, x_2) && \text{for } x \in \tilde{\Sigma}_\lambda = \Sigma_\lambda \backslash \{x_\lambda\}, \\ A_{0,\lambda}(x_1, x_2) &= A_0(2\lambda - x_1, x_2) && \text{for } x \in \Sigma_\lambda, \\ v_\lambda(x) &= u_\lambda(x) - u(x) && \text{for } x \in \tilde{\Sigma}_\lambda, \\ w_\lambda(x) &= A_{0,\lambda}(x) - A_0(x) && \text{for } x \in \Sigma_\lambda. \end{aligned}$$

LEMMA 2.2. *There exists a number $R_0 > 0$ such that $v_\lambda < 0$ in $\tilde{\Sigma}_\lambda$ and $w_\lambda < 0$ in Σ_λ for all $\lambda < -R_0$.*

PROOF. Fix $\eta \in (0,1)$ so close to 1 that

$$q^2\eta^2 > \kappa^2(1-\eta).$$

Then there exists a number $R_1 > 0$ such that

$$\ln \eta < u(x) < 0 \qquad \text{for} \qquad |x| \geq R_1. \tag{2.1}$$

We also choose $R_2 > 0$ satisfying

$$-\frac{\kappa}{2q}(1-\eta) < A_0(x) < 0 \qquad \text{for} \qquad |x| \geq R_2. \tag{2.2}$$

Set

$$m_1 = \max_{|x|\leq R_3} u(x), \qquad \text{and} \qquad m_2 = \max_{|x|\leq R_3} A_0(x),$$

where $R_3 = \max\{R_1, R_2\}$. Since $u, A \to 0$ as $|x| \to \infty$, there exists a number $R_4 > R_3$ verifying

$$m_1 < u(x) < 0, \qquad m_2 < A_0(x) < 0 \qquad \forall\ |x| \geq R_4.$$

Set $R_0 = 2R_4$ and let $\lambda < -R_0$ be given. Then it is obvious that

$$\begin{aligned} v_\lambda &\leq 0 \qquad \text{on } B_{R_3}(x_\lambda)\backslash\{x_\lambda\}, \\ w_\lambda &\leq 0 \qquad \text{on } B_{R_3}(x_\lambda). \end{aligned}$$

We claim that $w_\lambda \leq 0$ on Σ_λ. Assume the contrary and let y be a maximum point of w_λ on Σ_λ with $w_\lambda(y) > 0$. Then $y \in \Sigma_\lambda \backslash B_{R_3}(x_\lambda)$. A simple computation yields that on $\Sigma_\lambda \backslash B_{R_3}(x_\lambda)$,

$$\Delta v_\lambda = 2q^2 e^{\zeta} v_\lambda - 2\kappa q w_\lambda, \tag{2.3}$$

$$\Delta w_\lambda = (\kappa^2 + 2q^2 e^{u_\lambda}) w_\lambda - (\kappa q - 2q^2 A_0) e^{\zeta} v_\lambda, \tag{2.4}$$

where $\zeta(x)$ lies between $u_\lambda(x)$ and $u(x)$. As in Lemma 2.1, by (2.1) and (2.2), we are led to

$$w_\lambda(y) \leq \frac{\kappa q - 2q^2 A_0(y)}{\kappa^2 + 2q^2 e^{u_\lambda(y)}}\, e^{\zeta(y)} v_\lambda(y).$$

In particular, $v_\lambda(y) > 0$. Thus,

$$\begin{aligned} w_\lambda(y) &\leq \frac{\kappa q - 2q^2 A_0(y)}{\kappa^2 + 2q^2\eta}\, v_\lambda(y) \\ &\leq \frac{\kappa q + \kappa q(1-\eta)}{\kappa^2 + 2q^2\eta}\, v_\lambda(y). \end{aligned} \tag{2.5}$$

Let z be a maximum point of v_λ on $\tilde{\Sigma}_\lambda$. Then

$$v_\lambda(z) > v_\lambda(y) > 0.$$

Hence by (2.1) and (2.3)

$$w_\lambda(z) \geq \frac{q}{\kappa} e^{\zeta(z)} v_\lambda(z) \geq \frac{q}{\kappa} \eta \, v_\lambda(y). \tag{2.6}$$

Since $v_\lambda(y) > 0$, the equation (2.5) together with (2.6) implies that

$$\frac{q}{\kappa}\eta \leq \frac{\kappa q + \kappa q(1-\eta)}{\kappa^2 + 2q^2\eta},$$

namely,

$$q^2\eta^2 \leq \kappa^2(1-\eta).$$

This violates the choice of η. Hence $w_\lambda \leq 0$ on Σ_λ.

Now, on $\tilde{\Sigma}_\lambda \backslash B_{R_3}(x_\lambda)$, we get

$$\Delta v_\lambda - 2q^2 e^\zeta v_\lambda = -2\kappa q w_\lambda \geq 0.$$

Since $v_\lambda \leq 0$ on $\partial(\, \tilde{\Sigma}_\lambda \backslash B_{R_3}(x_\lambda)\,)$, it comes from the maximum principle that $v_\lambda \leq 0$ on $\tilde{\Sigma}_\lambda \backslash B_{R_3}(x_\lambda)$. This finishes the proof. □

In view of Lemma 2.2 we can define a number

$$\lambda_0 = \sup\{\lambda < 0 \mid v_\mu \leq 0 \text{ on } \tilde{\Sigma}_\mu, \;\; w_\mu \leq 0 \text{ on } \Sigma_\mu, \qquad \forall\, \mu < \lambda\}.$$

LEMMA 2.3. $\lambda_0 = 0$.

PROOF. Suppose $\lambda_0 \neq 0$. For $\lambda_0 < \lambda < 0$, let y_λ and z_λ be maximum points of w_λ and v_λ, respectively. Then either $w_\lambda(y_\lambda) > 0$ or $v_\lambda(z_\lambda) > 0$. Applying the maximum principle to (2.3) and (2.4), we observe that

$$\begin{aligned} w_\lambda(y_\lambda) &\leq \frac{\kappa q - 2q^2 A_0(y_\lambda)}{\kappa^2 + 2q^2 e^{u_\lambda(y_\lambda)}} e^{\zeta(y_\lambda)} v_\lambda(y_\lambda) \\ &\leq \frac{\kappa q - 2q^2 A_0(y_\lambda)}{\kappa^2 + 2q^2 e^{u_\lambda(y_\lambda)}} e^{\zeta(y_\lambda)} v_\lambda(z_\lambda) \\ &\leq \frac{\kappa q - 2q^2 A_0(y_\lambda)}{\kappa^2 + 2q^2 e^{u_\lambda(y_\lambda)}} e^{\zeta(y_\lambda)} \cdot \frac{\kappa}{q e^{\zeta(z_\lambda)}} w_\lambda(z_\lambda) \\ &\leq \frac{\kappa q - 2q^2 A_0(y_\lambda)}{\kappa^2 + 2q^2 e^{u_\lambda(y_\lambda)}} e^{\zeta(y_\lambda)} \cdot \frac{\kappa}{q e^{\zeta(z_\lambda)}} w_\lambda(y_\lambda). \end{aligned}$$

Therefore, if one of $w_\lambda(y_\lambda)$ and $v_\lambda(z_\lambda)$ is positive, then so is the other. Consequently, we see that both $w_\lambda(y_\lambda) > 0$ and $v_\lambda(z_\lambda) > 0$. Let η, m_i, and R_j be the same numbers as in the proof of Lemma 2.2 with $i = 1, 2$ and $j = 0, \cdots, 4$. Set $r_0 = 3\max\{R_0, -\lambda_0\}$. Then it holds that either $|y_\lambda| \leq r_0$ or $|z_\lambda| \leq r_0$. Otherwise, since $w_\lambda(y_\lambda) > 0$ and $v_\lambda(z_\lambda) > 0$, the equations (2.5) and (2.6) would be satisfied by means of the replacement of y and z by y_λ and z_λ. This leads us to a contradiction as in the proof of Lemma 2.2.

Now for $\lambda_0 < \lambda < 0$, either $P_\lambda = \{y_\lambda : |y_\lambda| \le r_0\}$ or $Q_\lambda = \{z_\lambda : |z_\lambda| \le r_0\}$ is an infinite set. Suppose that P_λ is infinite. Then passing to a subsequence, we may assume that y_λ converges to a point y. Obviously, $y \in \Sigma_{\lambda_0} \cup \Gamma_{\lambda_0}$, $w_{\lambda_0}(y) \ge 0$, and $\nabla w_{\lambda_0}(y) = 0$. We see from (2.4) that

$$\Delta w_{\lambda_0} - (\kappa^2 + 2q^2 e^u) w_{\lambda_0} = -(\kappa q e^\zeta - 2q^2 e^\zeta A_{0,\lambda_0}) v_{\lambda_0} \ge 0 \qquad \text{on } \Sigma_{\lambda_0}.$$

Thus it follows from the strong maximum principle that w_{λ_0} cannot attain its maximum value in Σ_{λ_0}. This implies that $y \in \Gamma_{\lambda_0}$. But in this case it come from the Hopf Lemma that $(\partial w_{\lambda_0}/\partial x_1)(y) > 0$, which contradicts to the fact that $\nabla w_{\lambda_0}(y) = 0$.

If Q_λ is an infinite set, we come to a similar contradiction by virtue of (2.3). In the sequel, we have $\lambda_0 = 0$. □

Since $\lambda_0 = 0$, it is seen that for $x_1 < 0$

$$u(-x_1, x_2) \le u(x_1, x_2), \qquad \text{and} \qquad A_0(-x_1, x_2) \le A_0(x_1, x_2).$$

Now the standard argument of moving planes assures Theorem 1.1.

References

[1] D. Chae and O. Yu. Imanuvilov, *Non-Topological Multivortex Solutions to the Self-Dual Maxwell-Chern-Simons-Higgs Systems*, J. Func. Anal. **196** (2002), 87–118.

[2] D. Chae and N. Kim, *Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs system*, J. Diff. Eqns. **134** (1997), 154–182.

[3] ———, *Vortex condensates in the relativistic self-dual Maxwell-Chern-Simons-Higgs system*, RIM-GARC Preprint Ser. **97-50** (1997).

[4] J. Han, *Existence of topological multivortex solutions in the self-dual gauge theories*, Proc. Royal Soc. Edinburgh A **130** (2000), 1293–1309.

[5] J. Hong, Y. Kim, and P. Y. Pac, *Multivortex Solutions of the Abelian Chern-Simons-Higgs Theory*, Phys. Rev. Lett. **64** (1990), 2230–2233.

[6] R. Jackiw and E. J. Weinberg, *Self-dual Chern-Simons Vortices*, Phys. Rev. Lett. **64** (1990), 2234–2237.

[7] A. Jaffe and C.H. Taubes, Vortices and Monopoles, Birkhäuser, Boston, 1980.

[8] C. Lee, K. Lee, and H. Min, *Self-dual Maxwell-Chern-Simons solitons*, Phys. Lett. B **252** (1990), 79–83.

[9] T. Ricciardi, *Asymptotics for Maxwell-Chern-Simons multivortices*, Nonlinear Anal. TMA **50** (2002), 1093–1106.

[10] T. Ricciardi and G. Tarantello, *Vortices in the Maxwell-Chern-Simons theory*, Comm. Pure Appl. Math. **53** (2000), 811–851.

Department of Mathematics
Hankuk University of Foreign Studies
Kyounggi-do 449-791, Korea
E-mail: jmhan@hufs.ac.kr