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THE STABILITY OF A MIXED TYPE FUNCTIONAL INEQUALITY WITH THE FIXED POINT ALTERNATIVE

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Abstract. In this note, by using the fixed point alternative, we investigate the modified Hyers-Ulam-Rassias stability for the following mixed type functional inequality which is either cubic or quadratic:

$$\|8f(x-3y)+24f(x+y)+f(8y) \\ -8[f(x+3y)+3f(x-y)+2f(2y)]\| \le \varphi(x,y).$$

1. Introduction

Under what condition does there exist a homomorphism near an approximately homomorphism between a group and a metric group ? This is called the stability problem of functional equations which was first raised by S. M. Ulam [27] in 1940. In the next year, D. H. Hyers [7] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias in [19]. The terminology Hyers-Ulam-Rassias stability originates from this historical background. Since then, a great deal of work has been done by a number of authors (for instance, [2, 3, 5, 6, 8, 9, 10, 18, 20, 21, 22, 23, 24, 25]).

In particular, one of the important functional equations studied is the following functional equation [1, 4, 12, 13, 14]:

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

The quadratic function $f(x)=qx^2$ is a solution of this functional equation, and so one usually call the above functional equation quadratic.

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The Hyers-Ulam stability problem for the quadratic functional inequality

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \delta$$

was first proved by F. Skof [26] for a function $f: X \to Y$, where X is a normed space and Y a Banach space. In [4], S. Czerwik generalized the Hyers-Ulam stability of the quadratic functional inequality.

On the other hand, consider the functional equation (see [11] and cf. [17])

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1}$$

The equation (1) is satisfied by the cubic function $f(x)=cx^3$ and hence, for convenience in this note, we promise that the equation (1) is called a cubic functional equation and that every solution of the equation (1) is said to be a cubic function.

The stability result for the cubic functional inequality

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \rho(x,y)$$

was obtained by K.-W. Jun and H.-M. Kim [11], where f is a function from the normed space X to the Banach space Y.

Now, let us introduce the following functional equation:

$$8f(x-3y)+24f(x+y)+f(8y)=8[f(x+3y)+3f(x-y)+2f(2y)]. \tag{2}$$

It is easy to see that all the real-valued functions $f: \mathbb{R} \to \mathbb{R}$ of mixed type, i.e., either $f(x)=cx^3$ or $f(x)=qx^2$, satisfy the functional equation (2). Our main goal in this note is to investigate the modified Hyers-Ulam-Rassias stability problem for the following mixed type functional inequality for a function $f: X \to Y$, where X is a normed space and Y a Banach space:

$$\begin{aligned} &\|8f(x-3y)+24f(x+y)+f(8y) \\ &-8[f(x+3y)+3f(x-y)+2f(2y)]\| \le \varphi(x,y) \end{aligned} \tag{3}$$

by using the fixed point alternative [15, 16].

2. Stability of inequality (3)

For explicitly later use, we first state the following theorem:

THEOREM 1. (*The alternative of fixed point* [15]) *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T: \Omega \to \Omega$ *with Lipschitz constant* Λ. *Then, for each* *given* $x \in \Omega$, *either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or

There exists a natural number n_0 *such that*

- $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geq n_0$;
- *the sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T;
- y^* *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-\Lambda} d(y, Ty)$ *for all* $y \in \Delta$.

Now let us prove the modified Hyers-Ulam-Rassias stability for the functional inequality (3) by using the fixed point alternative as in [16].

From now on, let X be a real vector space and Y be a real Banach space. Given a function $f : X \to Y$, we set

$$\begin{aligned} Df(x,y) := &\, 8f(x-3y) + 24f(x+y) + f(8y) \\ &-8[f(x+3y) + 3f(x-y) + 2f(2y)] \end{aligned}$$

for all $x, y \in X$.

Let $\varphi : X \times X \to [0, \infty)$ be a given function. Let $\psi : X \to [0, \infty)$ be the function defined by

$$\psi(x) = \frac{1}{2}\Big[\varphi\Big(\frac{x}{8}, \frac{x}{8}\Big) + \varphi\Big(-\frac{x}{8}, -\frac{x}{8}\Big)\Big]$$

for all $x \in X$ such that there exists a constant $L < 1$ satisfying the inequality

$$\psi(x) \leq L \cdot \lambda_i^3 \cdot \psi\Big(\frac{x}{\lambda_i}\Big) \tag{4}$$

for all $x \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$. Furthermore, assume that the identity

$$\lim_{n \to \infty} \frac{\varphi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{3n}} = 0 \tag{5}$$

holds for all $x, y \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$.

Similarly, we define a function $\phi : X \to [0, \infty)$ by

$$\phi(x) = \frac{1}{2}\Big[\varphi\Big(0, \frac{x}{8}\Big) + \varphi\Big(0, -\frac{x}{8}\Big)\Big]$$

for all $x \in X$, and suppose that there exists a constant $M < 1$ satisfying the inequality

$$\phi(x) \leq M \cdot \mu_j^2 \cdot \phi\Big(\frac{x}{\mu_j}\Big) \tag{6}$$

for all $x \in X$, where $\mu_j = 4$ if $j = 0$ and $\mu_j = \frac{1}{4}$ if $j = 1$. Moreover, the identity

$$\lim_{n\to\infty} \frac{\varphi(\mu_j^n x, \mu_j^n y)}{\mu_j^{2n}} = 0 \tag{7}$$

holds for all $x, y \in X$, where $\mu_j = 4$ if $j = 0$ and $\mu_j = \frac{1}{4}$ if $j = 1$.

THEOREM 2. *Suppose that a function $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{8}$$

for all $x, y \in X$ and $f(0) = 0$. If we take the conditions (4), (5), (6) and (7), then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - (C(x) + Q(x))\| \le \frac{L^{1-i}}{1-L}\psi(x) + \frac{M^{1-j}}{1-M}\phi(x), \tag{9}$$

$$\left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \le \frac{L^{1-i}}{1-L}\psi(x) \tag{10}$$

and

$$\left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\| \le \frac{M^{1-j}}{1-M}\phi(x) \tag{11}$$

for all $x \in X$, where $i, j = 0, 1$.

The functions C and Q are given by

$$C(x) = \lim_{n\to\infty} \frac{f(\lambda_i^n x) - f(-\lambda_i^n x)}{2 \cdot \lambda_i^{3n}} \text{ and } Q(x) = \lim_{n\to\infty} \frac{f(\mu_j^n x) + f(-\mu_j^n x)}{2 \cdot \mu_j^{2n}}$$

for all $x \in X$, respectively, where $i, j = 0, 1$.

PROOF. Consider the set

$$\Omega := \{k : k : X \to Y,\ k(0) = 0\}$$

and introduce the generalized metric on Ω:

$$d_\psi(k,l) = \inf\{K > 0 : \|k(x) - l(x)\| \le K\psi(x) \text{ for all } x \in X\}.$$

It is easy to see that (Ω, d_ψ) is complete.

Suppose first that we take the conditions (4) and (5). If we define a function $T : \Omega \to \Omega$ by

$$Tk(x) = \frac{1}{\lambda_i^3}\, k(\lambda_i x)$$

for all $x \in X$, then we obtain from (4) that for all $k, l \in \Omega$,

$$\begin{aligned} d_\psi(k,l) < K \quad & \Longrightarrow \|k(x) - l(x)\| \le K\psi(x), \ x \in X \\ & \Longrightarrow \left\| \frac{1}{\lambda_i^3}\, k(\lambda_i x) - \frac{1}{\lambda_i^3}\, l(\lambda_i x) \right\| \le \frac{1}{\lambda_i^3}\, K\psi(\lambda_i x), \ x \in X \\ & \Longrightarrow \left\| \frac{1}{\lambda_i^3}\, k(\lambda_i x) - \frac{1}{\lambda_i^3}\, l(\lambda_i x) \right\| \le LK\psi(x), \ x \in X \\ & \Longrightarrow d_\psi(Tk, Tl) \le LK. \end{aligned}$$

Hence we see that

$$d_\psi(Tk, Tl) \le L d_\psi(k, l)$$

for all $k, l \in \Omega$, that is, T is a strictly contractive self-mapping of Ω with the Lipschitz constant L.

Let $g : X \to Y$ be the function defined by $g(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then we have $g(0) = 0$, $g(-x) = -g(x)$ and

$$\begin{aligned} \|Dg(x,y)\| &= \|8g(x-3y) + 24g(x+y) + g(8y) \\ &\qquad -8[g(x+3y) + 3g(x-y) + 2g(2y)]\| \\ &\le \frac{1}{2}[\varphi(x,y) + \varphi(-x,-y)] \end{aligned} \tag{12}$$

for all $x, y \in X$.

Putting $y := x$ in (12) yields

$$\|g(8x) - 8g(4x)\| \le \frac{1}{2}[\varphi(x,x) + \varphi(-x,-x)], \tag{13}$$

which, by setting $x := \frac{x}{4}$ in (13) and using (4) with the case $i = 0$, gives

$$\left\| g(x) - \frac{g(2x)}{2^3} \right\| \le \frac{1}{2^3}\psi(2x) \le L\psi(x)$$

for all $x \in X$, that is, $d_\psi(g, Tg) \le L < \infty$.

If we substitute $x := \frac{x}{8}$ in (13) and use (4) with the case $i = 1$, then we see that

$$\left\| g(x) - 2^3 g\Big(\frac{x}{2}\Big) \right\| \le \psi(x)$$

for all $x \in X$, that is, $d_\psi(g, Tg) \le 1 < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point $C : X \to Y$ of T in Ω, i.e., $C(2x) = 8C(x)$ holds for all $x \in X$ such that

$$C(x) = \lim_{n\to\infty} \frac{g(\lambda_i^n x)}{\lambda_i^{3n}} \tag{14}$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n g, C) = 0$.

To show that the function C is cubic, let us replace x and y by $\lambda_i^n x$ and $\lambda_i^n y$ in (12), respectively and divide by λ_i^{3n}. Then it follows from (5) and (14) that

$$\begin{aligned}\|DC(x,y)\| &= \lim_{n\to\infty} \frac{\|Dg(\lambda_i^n x, \lambda_i^n y)\|}{\lambda_i^{3n}} \\ &\le \frac{1}{2} \lim_{n\to\infty} \frac{\varphi(\lambda_i^n x, \lambda_i^n y) + \varphi(-\lambda_i^n x, -\lambda_i^n y)}{\lambda_i^{3n}} = 0\end{aligned}$$

for all $x, y \in X$, namely, C satisfies the functional equation (2). Since the identity $C(2x) = 8C(x)$ holds for all $x \in X$, the equation (2) is reduced to the form

(15) $\quad C(x+3y) + 3C(x-y) = C(x-3y) + 3C(x+y) + 48C(y)$

for all $x, y \in X$. Let us replace x by $-x$ in (14). Then it follows from the oddness of g that C is odd, and hence interchanging x and y in (15) yields

(16) $\quad C(3x+y) + C(3x-y) = 3C(x+y) + 3C(x-y) + 48C(x).$

If we put $y := -x + y$ and $y := -x - y$ in (16), respectively and compare the results, then we obtain

(17) $\quad C(4x+y) + C(4x-y) = 2C(2x+y) + 2C(2x-y) + 96C(x).$

Finally, replacing y by $2y$ in (17) and using $C(2x) = 8C(x)$, we see that C satisfies the functional equation (1). Therefore C is cubic.

According to the fixed point alternative, since C is the *unique* fixed point of T in the set $\Delta = \{k \in \Omega : d_\psi(g,k) < \infty\}$, C is the unique function such that

$$\|g(x) - C(x)\| \le K\psi(x)$$

for all $x \in X$ and some constant $K > 0$. Again using the fixed point alternative, we have

$$d_\psi(g, C) \le \frac{1}{1-L} d_\psi(g, Tg),$$

and so we obtain the inequality

$$d_\psi(g, C) \le \frac{L^{1-i}}{1-L}$$

which yields the inequality (10), where $i = 0, 1$.

As in the previous case, by introducing the following generalized metric on Ω:

$$d_\phi(k, l) = \inf\{R > 0 : \|k(x) - l(x)\| \le R\phi(x) \text{ for all } x \in X\},$$

we also see that (Ω, d_ϕ) is complete.

Assume now that we take the conditions (6) and (7). Defining a function $S : \Omega \to \Omega$ by

$$Sk(x) = \frac{1}{\mu_j^2}\, k(\mu_j x)$$

for all $x \in X$, we obtain from (6) that for all $k, l \in \Omega$,

$$\begin{aligned} d_\phi(k,l) < R \quad &\Longrightarrow \|k(x) - l(x)\| \le R\phi(x),\ x \in X \\ &\Longrightarrow \left\| \frac{1}{\mu_j^2}\, k(\mu_j x) - \frac{1}{\mu_j^2}\, l(\mu_j x) \right\| \le \frac{1}{\mu_j^2}\, R\phi(\mu_j x),\ x \in X \\ &\Longrightarrow \left\| \frac{1}{\mu_j^2}\, k(\mu_j x) - \frac{1}{\mu_j^2}\, l(\mu_j x) \right\| \le MR\phi(x),\ x \in X \\ &\Longrightarrow d_\phi(Sk, Sl) \le MR. \end{aligned}$$

Hence we see that

$$d_\phi(Sk, Sl) \le M d_\phi(k,l)$$

for all $k, l \in \Omega$, that is, S is a strictly contractive self-mapping of Ω with the Lipschitz constant M.

Let $h : X \to Y$ be the function defined by $h(x) = \frac{1}{2}\,[f(x) + f(-x)]$ for all $x \in X$. Then we have $h(0) = 0$, $h(-x) = h(x)$ and

$$\begin{aligned} \|Dh(x,y)\| &= \|8h(x-3y) + 24h(x+y) + h(8y) \\ &\qquad -8[h(x+3y) + 3h(x-y) + 2h(2y)]\| \\ &\le \frac{1}{2}\,[\varphi(x,y) + \varphi(-x,-y)] \end{aligned} \tag{18}$$

for all $x, y \in X$. By setting $x := 0$ in (18) and then letting $y := x$, we get

$$\|h(8x) - 16h(2x)\| \le \frac{1}{2}\,[\varphi(0,x) + \varphi(0,-x)]\,. \tag{19}$$

Replacing x by $\frac{x}{2}$ in (19) and then employing (6) with $j = 0$, we obtain

$$\left\| h(x) - \frac{h(4x)}{4^2} \right\| \le \frac{1}{4^2}\phi(4x) \le M\phi(x) \tag{20}$$

for all $x \in X$, that is, $d_\phi(h, Sh) \le M < \infty$.

If we substitute $x := \frac{x}{8}$ in (19) and use (6) with the case $j = 1$, then we see that

$$\left\| h(x) - 4^2 h\Big(\frac{x}{4}\Big) \right\| \le \phi(x)$$

for all $x \in X$, that is, $d_\phi(h, Sh) \le 1 < \infty$.

Now, by applying the fixed point alternative in both cases, we see that there exists a fixed point $Q : X \to Y$ of S in Ω, i.e., $Q(4x) = 16Q(x)$ holds for all $x \in X$ such that

$$Q(x) = \lim_{n\to\infty} \frac{h(\mu_j^n x)}{\mu_j^{2n}} \tag{21}$$

for all $x \in X$ since $\lim_{n\to\infty} d_\phi(S^n h, Q) = 0$.

To show that the function Q is quadratic, let us replace x and y by $\mu_j^n x$ and $\mu_j^n y$ in (18), respectively and divide by μ_j^{2n}. Then it follows from (7) and (21) that

$$\begin{aligned}\|DQ(x,y)\| &= \lim_{n\to\infty} \frac{\|Dh(\mu_j^n x, \mu_j^n y)\|}{\mu_j^{2n}} \\ &\le \frac{1}{2} \lim_{n\to\infty} \frac{\varphi(\mu_j^n x, \mu_j^n y) + \varphi(-\mu_j^n x, -\mu_j^n y)}{\mu_j^{2n}} = 0\end{aligned}$$

for all $x, y \in X$ which means that Q satisfies the functional equation (2). Since the identity $Q(4x) = 16Q(x)$ holds for all $x \in X$, the equation (2) is reduced to the form

$$Q(x+3y) + 3Q(x-y) = Q(x-3y) + 3Q(x+y) \tag{22}$$

for all $x, y \in X$. By (21), it is immediate that $Q(0) = 0$ and Q is even. Let us replace x by y in (22) and then put $y := \frac{y}{2}$. Then we get $Q(2y) = 4Q(y)$. If we set $x := 3y$ in (22) and use $Q(2y) = 4Q(y)$, then we have $Q(3y) = 9Q(y)$.

Substituting $x := x - y$ and $x := x + y$ in (22), respectively and then comparing the results, we obtain

$$Q(x+4y) + 2Q(x-2y) = Q(x-4y) + 2Q(x+2y). \tag{23}$$

Replacing x by $2x$ in (23) and using $Q(2y) = 4Q(y)$, we have

$$Q(x+2y) + 2Q(x-y) = Q(x-2y) + 2Q(x+y). \tag{24}$$

From the substitutions $x := x + y$ and $y := x - y$ in (24), we deduce

$$Q(3x-y) + 8Q(y) = Q(x-3y) + 8Q(x),$$

and replacing y by $-y$ gives

$$Q(3x+y) + 8Q(y) = Q(x+3y) + 8Q(x),$$

that is,

$$Q(3x+y) - Q(x+3y) = 8Q(x) - 8Q(y). \tag{25}$$

Setting $x+y$ instead of x in (24), we get

$$Q(x+3y)+2Q(x)=2Q(x+2y)+2Q(x-y) \tag{26}$$

and interchanging x and y in (26) yields

$$Q(3x+y)+2Q(y)=2Q(2x+y)+2Q(x-y). \tag{27}$$

If we subtract (27) from (26) and use (25), we obtain

$$Q(x+2y)+3Q(x)=Q(2x+y)+3Q(y) \tag{28}$$

which, by putting $y:=2y$ in (28) and using $Q(2y)=4Q(y)$, leads to

$$Q(x+4y)+3Q(x)=4Q(x+y)+12Q(y). \tag{29}$$

Interchanging x with y in (29) gives

$$Q(4x+y)+3Q(y)=4Q(x+y)+12Q(x), \tag{30}$$

and by replacing y by $-y$ in (30), we arrive at

$$Q(4x-y)+3Q(y)=4Q(x-y)+12Q(x). \tag{31}$$

Comparing (30) with (31), we have

$$Q(4x+y)+Q(4x-y)+6Q(y)=4Q(x+y)+4Q(x-y)+24Q(x). \tag{32}$$

Now utilizing the substitutions $x:=x+y$ and $y:=x-\frac{y}{2}$ in (28), we obtain

$$Q(3x)+3Q(x+y)=Q\left(3\left(x+\frac{y}{2}\right)\right)+3Q\left(x-\frac{y}{2}\right),$$

and letting $y:=-y$ in this relation yields

$$Q(3x)+3Q(x-y)=Q\left(3\left(x-\frac{y}{2}\right)\right)+3Q\left(x+\frac{y}{2}\right).$$

Since $Q(2x)=4Q(x)$ and $Q(3x)=9Q(x)$, we add the above two relations to obtain

$$Q(2x+y)+Q(2x-y)=Q(x+y)+Q(x-y)+6Q(x). \tag{33}$$

Replacing x by $2x$ in (33), we get

$$Q(4x+y)+Q(4x-y)=Q(2x+y)+Q(2x-y)+24Q(x)$$

which, by (33), gives

$$Q(4x+y)+Q(4x-y)=Q(x+y)+Q(x-y)+30Q(x). \tag{34}$$

By comparing (32) with (34), we conclude that

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$$

which implies that Q is quadratic.

In view of the fixed point alternative, since Q is the *unique* fixed point of S in the set $\Delta = \{k \in \Omega : d_\phi(h,k) < \infty\}$, Q is the unique function such that

$$\|h(x) - Q(x)\| \leq R\phi(x)$$

for all $x \in X$ and some constant $R > 0$. Finally, again using the fixed point alternative, we have

$$d_\phi(h,Q) \leq \frac{1}{1-M} d_\phi(h, Sh),$$

and so we obtain the inequality

$$d_\phi(h,Q) \leq \frac{M^{1-j}}{1-M}$$

which yields the inequality (11), where $j = 0, 1$.

Since we have $f(x) = g(x) + h(x)$ for all $x \in X$, we see that

$$\begin{aligned}\|f(x) - (C(x) + Q(x))\| &\leq \|g(x) - C(x)\| + \|h(x) - Q(x)\| \\ &\leq \frac{L^{1-i}}{1-L}\psi(x) + \frac{M^{1-j}}{1-M}\phi(x)\end{aligned}$$

for all $x \in X$, where $i, j = 0, 1$. We complete the proof of the theorem. □

From Theorem 2, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [19] of the functional inequality (3).

Let $p \neq 2, 3$ be any real number. For the convenience, set

$$\chi_1(p) := \frac{1}{2^{2p+2} \cdot (2^{p-3} - 1)}, \quad \chi_2(p) := \frac{1}{2^{3p-1} \cdot (2^{3-p} - 1)}$$

and

$$\chi_3(p) := \frac{1}{2^{p+4} \cdot (4^{p-2} - 1)}, \quad \chi_4(p) := \frac{1}{2^{3p} \cdot (4^{2-p} - 1)}.$$

COROLLARY 3. *Let X be a normed space and let $\varepsilon \geq 0$, $p \neq 2, 3$ be real numbers. Suppose that a function $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $f(0) = 0$. Then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - (C(x) + Q(x))\| \leq \chi(p)\varepsilon\|x\|^p,$$

$$\left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \leq \chi_t(p)\varepsilon\|x\|^p \quad (t = 1 \text{ or } 2)$$

and

$$\left\|\frac{f(x)+f(-x)}{2} - Q(x)\right\| \le \chi_t(p)\varepsilon\|x\|^p \quad (t=3 \ \text{or} \ 4),$$

for all $x \in X$, where

$$\chi(p) = \begin{cases} \chi_1(p)+\chi_3(p) & \text{if } p>3 \\ \chi_2(p)+\chi_3(p) & \text{if } 2<p<3 \\ \chi_2(p)+\chi_4(p) & \text{if } p<2. \end{cases}$$

The functions C and Q are given by

$$C(x) = \lim_{n\to\infty} \frac{f(\lambda_i^n x) - f(-\lambda_i^n x)}{2\cdot\lambda_i^{3n}} \quad \text{and} \quad Q(x) = \lim_{n\to\infty} \frac{f(\mu_j^n x) + f(-\mu_j^n x)}{2\cdot\mu_j^{2n}}$$

for all $x \in X$, respectively, where $i=0$ if $p<3$ and $i=1$ if $p>3$, $j=0$ if $p<2$ and $j=1$ if $p>2$.

PROOF. Let $\varphi(x,y) := \varepsilon(\|x\|^p + \|y\|^p)$ for all $x,y \in X$. Let $p \ne 3$. Then we have

$$\frac{\varphi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{3n}} = (\lambda_i^n)^{p-3}\varepsilon(\|x\|^p + \|y\|^p) \longrightarrow 0$$

as $n \to \infty$, that is, (5) is true.

Since the identity

$$\frac{1}{\lambda_i^3}\psi(\lambda_i x) = \lambda_i^{p-3}2^{1-3p}\varepsilon\|x\|^p = \lambda_i^{p-3}\psi(x)$$

holds for all $x \in X$, we see that the inequality (4) holds with either $L = 2^{p-3}$ $(p<3)$ or $L = \frac{1}{2^{p-3}}$ $(p>3)$.

On the other hand, letting $p \ne 2$, the relation

$$\frac{\varphi(\mu_j^n x, \mu_j^n y)}{\mu_j^{2n}} = (\mu_j^n)^{p-2}\varepsilon(\|x\|^p + \|y\|^p) \longrightarrow 0$$

as $n \to \infty$ holds for all $x,y \in X$, therefore (7) is true. Since the identity

$$\frac{1}{\mu_j^2}\phi(\mu_j x) = \mu_j^{p-2}2^{-3p}\varepsilon\|x\|^p = \mu_j^{p-2}\phi(x)$$

holds for all $x \in X$, we see that the inequality (6) holds with either $M = 4^{p-2}$ $(p<2)$ or $M = \frac{1}{4^{p-2}}$ $(p>2)$.

Hence we deduce that

$$\frac{L^{1-i}}{1-L}\psi(x)+\frac{M^{1-j}}{1-M}\phi(x) := \chi(p)\varepsilon\|x\|^p = \begin{cases} (\chi_1(p)+\chi_3(p))\varepsilon\|x\|^p & \text{if } p>3 \\ (\chi_2(p)+\chi_3(p))\varepsilon\|x\|^p & \text{if } 2<p<3 \\ (\chi_2(p)+\chi_4(p))\varepsilon\|x\|^p & \text{if } p<2 \end{cases}$$

for all $x \in X$ which completes the proof of the corollary. □

The following corollary is the Hyers-Ulam stability of the inequality (3) which is an immediate consequence of Corollary 3.

COROLLARY 4. *Let $\theta \geq 0$ be a real number. Suppose that a function $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y)\| \leq \theta$$

for all $x, y \in X$ and $f(0) = 0$. Then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x)-(C(x)+Q(x))\| \leq \frac{37}{210}\theta,$$

$$\left\|\frac{f(x)-f(-x)}{2}-C(x)\right\| \leq \frac{1}{7}\theta$$

and

$$\left\|\frac{f(x)+f(-x)}{2}-Q(x)\right\| \leq \frac{1}{30}\theta$$

for all $x \in X$.

The functions C and Q are given by

$$C(x)=\lim_{n\to\infty}\frac{f(\lambda_i^n x)-f(-\lambda_i^n x)}{2\cdot\lambda_i^{3n}} \text{ and } Q(x)=\lim_{n\to\infty}\frac{f(\mu_j^n x)+f(-\mu_j^n x)}{2\cdot\mu_j^{2n}}$$

for all $x \in X$, respectively, where $i, j = 0, 1$.

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